CHAPTER 6

Conclusion

The main results obtained in this thesis have been presented in this chapter. In Chapter 3, we characterized Cayley digraphs of completely simple semigroups which are Cayley digraphs of a Rees matrix semigroup. Moreover, we have the conditions for Cayley digraphs of Rees matrix semigroups with a one-element connection set are isomorphic and connected as follows:

(1) A digraph (V, E) is a Cayley digraph of a completely simple semigroup if and only if the following conditions hold:

  (i) (V, E) is the disjoint union of n isomorphic subdigraphs $(V_1, E_1), (V_2, E_2), ...,$ (V_n, E_n) for some $n \in \mathbb{N}$;

  (ii) (V_i, E_i) has n subdigraphs $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), ..., (V_{in}, E_{in})$ such that $(V_i, E_i) = \oplus_{j=1}^{n}(V_{ij}, E_{ij})$ with $V_i = V_{ij}$ for every $j \in \{1, 2, .., n\}$;

  (iii) (V_{ij}, E_{ij}) contains m disjoint strong subdigraphs $(V_{ij}^1, E_{ij}^1), (V_{ij}^2, E_{ij}^2), ...,$ (V_{ij}^m, E_{ij}^m) such that $V_{ij} = \cup_{\alpha=1}^{m} V_{ij}^{\alpha}$;

  (iv) there exist a group G and a family of digraph isomorphisms $\{f_{ij}^{\alpha}\}_{\alpha=1}^{m}$ such that $f_{ij}^{\alpha} : (V_{ij}^{\alpha}, E_{ij}^{\alpha}) \to Cay(G, a_{ij}^{\alpha} A_{ij}^{\alpha})$ for some $a_{ij}^{\alpha} \in G$, $A_{ij}^{\alpha} \subseteq G$ with $A_{kj}^{\alpha} = A_{tj}^{\alpha}$, $a_{kj}^{\alpha} = a_{tj}^{\alpha}$ for all $k, t \in \{1, 2, ..., n\}$;

  (v) for each $u \in V_{ij}^{\alpha}, v \in V_{ij}^{\beta}$, $(u, v) \in E$ if and only if $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u) a_{ij}^{\alpha} a$ for some $a \in A_{ij}^{\beta}$

(2) Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$,$a = (g, j, \beta), b = (g', i, \lambda) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $|\langle p_{\beta j} g \rangle| = |\langle p_{\lambda i} g' \rangle|$.

(3) Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $a = (g, j, \beta) \in S$. Then $Cay(S, \{a\})$ is connected if and only if $G = \langle p_{\beta j} g \rangle$ and $|I| = 1$.

Open problem: For any nonempty subsets A and B of a Rees matrix semigroup S, what are the necessary and sufficient conditions for two Cayley digraphs $Cay(S, A)$ and $Cay(S, B)$ to be isomorphic to each other?

In Chapter 4, we have the condition for Cayley digraphs of a rectangular group to be isomorphic to each other. Moreover, we have isomorphism condition for Cayley digraphs of a right group and Cayley digraphs of a rectangular band as follows:

(1) Let $A, B \subseteq R_n$. Then $Cay(R_n, A) \cong Cay(R_n, B)$ if and only if $|A| = |B|$.

(2) Let $S = L_m \times R_n$ be a rectangular band and $A, B \subseteq S$. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $|p_2(A)| = |p_2(B)|$.

(3) Let $S = G \times R_n$ be a right group, and let $(g, \lambda), (h, \beta) \in S$, where $g, h \in G$ and $\lambda, \beta \in R_n$. Then $Cay(S, \{(g, \lambda)\}) \cong Cay(S, \{(h, \beta)\})$ if and only if $|\langle g\rangle| = |\langle h\rangle|$.

(4) Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r | r \in p_2(A)\}$. $B_r, \hat{B}_r$ and $\hat{B}$ are defined similarly. If $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$, then $\left|\hat{A}\right| = \left|\hat{B}\right|$ and $\left|\langle p_1(A)\rangle\right| = \left|\langle p_1(B)\rangle\right|$.

(5) Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r | r \in p_2(A)\}$. $B_r, \hat{B}_r$ and $\hat{B}$ are defined similarly. Then $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ if the following conditions hold:

   (i) $\left|\hat{A}\right| = \left|\hat{B}\right|$ and $\left|\langle p_1(A)\rangle\right| = \left|\langle p_1(B)\rangle\right|$;

   (ii) There exists a bijection $f : \hat{A} \to \hat{B}$ such that $\left|\hat{A}_r\right| = \left|f(\hat{A}_r)\right|$ for all $\hat{A}_r \in \hat{A}$;

   (iii) For each $\hat{A}_r \in \hat{A}$, there exists a bijection $\varphi_r : \hat{A}_r \to f(\hat{A}_r)$ such that $|\langle p_1(a)\rangle| = |\langle p_1(\varphi_r(a))\rangle|$ for all $a \in \hat{A}_r$.

(6) Let $S = G \times R_n$ be a right group, A and B be nonempty subsets of S. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.

(7) Let $S = G \times L_m \times R_n$ be a rectangular group, A, B nonempty subsets of S. Let $S' = G \times R_n$. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(S', A') \cong Cay(S', B')$, where $A' = \{(g, r) \mid (g, l, r) \in A\}$ and $B' = \{(g', r') \mid (g', l', r') \in B\}$.

Open problem: For any nonempty subsets A and B of a right group S, what are the necessary conditions for two Cayley digraphs $Cay(\langle A\rangle, A)$ and $Cay(\langle B\rangle, B)$ to be isomorphic to each other?

In Chapter 5, we described Cayley digraphs of Brandt semigroups relative to the equivalence classes of Green's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$, respectively. We gave necessary and sufficient conditions for two Cayley digraphs of a Brandt semigroup relative to $\mathcal{H}$-classes to be isomorphic to each other. Moreover, we have necessary conditions for Cayley digraphs $Cay(S, A)$ and $Cay(S, B)$ are isomorphic as follows:

(1) Let $S = B(G, I)$ be a Brandt semigroup, $i, j \in I$, and $S_{\_i}, S_{\_j}$ be $\mathcal{L}$-classes of S. Then $Cay(S, S_{\_i}) \cong Cay(S, S_{\_j})$.

(2) Let $S = B(G, I)$ be a Brandt semigroup, $i, j \in I$, and $S_{i\_}, S_{j\_}$ be $\mathcal{R}$-classes of S. Then $Cay(S, S_{i\_}) \cong Cay(S, S_{j\_})$.

(3) Let $S = B(G, I)$ be a Brandt semigroup, $i, j, l, m \in I$ and S_{ij}, S_{lm} be $\mathcal{H}$-classes of S. Then $Cay(S, S_{ij}) \cong Cay(S, S_{lm})$ if and only if one of the following conditions holds:

  (i) If $i = j$ then $l = m$;

  (ii) If $i \neq j$ then $l \neq m$.

(4) Let $S = B(G, I)$ be a Brandt semigroup, A, B nonempty subsets of S, $L_A = \{j|(i, g, j) \in A\}$, $L_B = \{j|(i, g, j) \in B\}$, $R_A = \{i|(i, g, j) \in A\}$, and $R_B = \{i|(i, g, j) \in B\}$. If $Cay(S, A) \cong Cay(S, B)$ then $|L_A| = |L_B|$ and $|R_A| = |R_B|$.

Open problem: For any nonempty subsets A and B of a Brandt semigroup S, what are the sufficient conditions for two Cayley digraphs $Cay(S, A)$ and $Cay(S, B)$ to be isomorphic to each other?