CHAPTER 2

Preliminaries

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters.

2.1 Notations

The following notations that will be used in this thesis :

$\mathbb{R}^n$ – the n dimensional Euclidean space,

$\mathbb{R}^{n\times n}$ – the set of all $n \times n$ real matrices,

$\|x\|$ – the Euclidean norm of vector x,

diag$\{\cdot\}$ – the block diagonal matrix,

I – the identity matrix,

A^T – the transpose of matrix A,

A^{-1} – the inverse of matrix A,

$A > 0, A \geq 0, A < 0, A \leq 0$ – means that A is symmetric positive definite, positive semi-definite, negative definite and negative semi-definite; *respectively*,

$\lambda(A)$ – the set of all eigenvalues of matrix A,

$\lambda_{max}(A)$ – maximum eigenvalue of matrix A,

$\lambda_{min}(A)$ – minimum eigenvalue of matrix A,

$C_h = C([-h, 0], R^n), h > 0$ – denotes the Banach space of continuous functions, mapping the interval $[-h, 0]$ into R^n , with the topology of uniform convergence,

$\|x_t\| \in C_h$ defined $x_t = x(t+\theta), -h \leq \theta \leq 0$ and $\|x_t\|_{C_h} = \sup\limits_{-h\leq\theta\leq 0} \|x(t+\theta)\|$,

$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ – $*$ represents the symmetric form of matrix, namely $* = B^T$.

2.2 Lyapunov Function

Consider the system described by

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

where $x \in R^n, x_i = x_i(t), f = (f_1, f_2, ..., f_n)$ and $f_i = f_i(t, x_1, x_2, ..., x_n)$ for $i = 1, 2, ..., n$

Definition 2.2.1 (Lyapunov Function [1]) Let D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$ $V : D \subseteq \mathbb{R}^n \to \mathbb{R}$, We say that $V(x)$ is a Lyapunov function of system (2.1) if the following conditions hold :

1. $V(x)$ is continuous on $D \subseteq \mathbb{R}^n$.
2. $V(x)$ is positive definite such that $V(\bar{0}) = 0$ and $V(x) > 0$ for $x \neq \bar{0}$.
3. the derivative of V with respect to (2.1) is negative semidefinite (i.e. $\dot{V}(0) = 0$, and for all x in $\|x\| \leq k$, $\dot{V}(x) \leq 0$).

2.3 Stability

Definition 2.3.1 [8] A point $\bar{x}$ is called an *equilibrium point* of equation (2.1) if $f(t, \bar{x}) = 0$ for all $t \geq t_0$. For all purposes of the stability theory we can assume that $\bar{0}$ is an equilibrium of (2.1).

Definition 2.3.2 [8] The equilibrium point $\bar{x}$ of equation (2.1) is called *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0 \geq 0$.

Definition 2.3.3 [8] The equilibrium point $\bar{x}$ of equation (2.1) is called *unstable* if it is not stable.

Definition 2.3.4 [8] The equilibrium point $\bar{x}$ of equation (2.1) is called *asymptotically stable* (denoted A.S.) if it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.

Definition 2.3.5 [8] The equilibrium point $\bar{x}$ of equation (2.1) is called *uniformly asymptotically stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ and $\|x(t)\| \to 0$ as $t \to \infty$ for all $t \geq t_0 \geq 0$.

Theorem 2.3.1 [1] The equilibrium point $\bar{x}$ of equation (2.1) is stable if there exists a Lyapunov function for system (2.1). Moreover, if there exists a Lyapunov function whose derivative is negative definite, then the equilibrium point $\bar{x}$ is A.S.

Definition 2.3.6 [7] The operator D is said to be *stable* if solution $\bar{x} = 0$ of the homogeneous difference equation $D(x_t) = 0, \quad t \geq 0$ is stable where $D : C_h \to R^n$.

Definition 2.3.7 [7] Suppose $f : C_h \to R^n, D : C_h \to R^n$ are given continuous functions. The relation

$$\frac{d}{dt}D(t, x_t) = f(t, x_t),$$

is called the neutral differential equation. The function D will be called the operator for the neutral differential equation.

Theorem 2.3.2 [7] Suppose D is stable, $f : C_h \to R^n$ and suppose $u(s), v(s)$ and $w(s)$ are continuous, nonnegative and nondecreasing with $u(s), v(s) > 0$ for $s \neq 0$ and $u(0) = v(0) = 0$. If there is a continuous function $V : C_h \to R^n$ such that

$$\begin{aligned} u(\|D(x_t)\|) \leq V(x_t) &\leq v(\|x_t\|_{C_h}), \\ \dot{V}(x_t) &\leq -w(\|D(x_t)\|). \end{aligned}$$

If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ of the neutral differential equation is uniformly asymptotically stable. The same conclusion holds if the upper bound on $\dot{V}(x_t)$ is given by $-w(\|x(t)\|)$.

Definition 2.3.8 [5] The equilibrium point $\bar{x}$ of equation (2.1) is called *exponentially stable* if $\|\bar{x}(t, \phi)\| \leq \gamma e^{-\alpha t}\|\phi\|, \forall t \geq 0$, $\gamma > 0$ is a convergence coefficient and $\alpha > 0$ is a convergence rate .

Lemma 2.3.1 (Schur Complement [3]) Given constant symmetric matrices Q, S and $R \in \mathbb{R}^{n \times n}$ where $R(x) < 0$, $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$ we have

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \Leftrightarrow Q(x) - S(x)R^{-1}(x)S^T(x) < 0.$$

Lemma 2.3.2 [7] Suppose $\lambda_{min}(Q)$ is the minimum eigenvalue of matrix Q and $\lambda_{max}(Q)$ is the maximum eigenvalue of matrix Q. The following inequalities hold:

$$\lambda_{min}(Q)x^T x \leq x^T Q x \leq \lambda_{max}(Q)x^T x,$$

for symmetric matrix $Q \in R^{n \times n}$ for all $x \in R^n$.

Lemma 2.3.3 [8] For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the concerned integration are well defined. Then the following inequality holds

$$(\int_0^\gamma \omega(s)ds)^T M(\int_0^\gamma \omega(s)ds) \leq \gamma(\int_0^\gamma \omega^T(s)M\omega(s)ds).$$

Lemma 2.3.4 (Cauchy Inequality [8]) For any symmetric positive definite matrix $N \in \mathbb{R}^{n\times n}$ and $x,\ y \in \mathbb{R}^n$, the following inequalities hold

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

2.4 Types of Matrix

Definition 2.4.1 (Symmetric Matrix [3]) A real $n \times n$ matrix A is called *symmetric* if

$$A^T = A.$$

Definition 2.4.2 (Positive Definite Matrix [3])A real $n \times n$ matrix A is called *positive definite* if

$$x^T A x > 0$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *positive semidefinite* if

$$x^T A x \geq 0.$$

Definition 2.4.3 (Negative Definite Matrix [3]) A real $n \times n$ matrix A is called *negative definite* if

$$x^T A x < 0$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *negative semidefinite* if

$$x^T A x \leq 0.$$

The follows result are well known

Lemma 2.4.1 [3] A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are nonnegative (positive).

Lemma 2.4.2 [3] A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are nonpositive (negative).