CHAPTER 3

Main Results

In this chapter, we present new stability conditions for neutral-type neural networks with interval non-differentiable and distributed time-varying delays based on Lyapunov stability theory and linear matrix inequality (LMI) techniques. In section 3.1, we derive new sufficient conditions for exponential stability of neutral-type neural networks with interval non-differentiable and distributed time-varying delays with matrix uncertainties. Then, we establish exponential stability of neutral-type neural networks with interval non-differentiable and distributed time-varying delays without matrix uncertainties in section 3.2. Some numerical simulations are given to illustrate the effectiveness of our theoretical results.

Consider the following uncertain neural networks with interval time-varying discrete and neutral delays:

$$\begin{aligned} \dot{u}(t) - A_2\dot{u}(t-\eta(t)) &= -(A+\Delta A(t))u(t) + (W_0+\Delta W_0(t))l(u(t)) \\ &\quad +(W_1+\Delta W_1(t))m(u(t-h(t)) \\ &\quad +(W_2+\Delta W_2(t))\int_{t-k(t)}^{t} r(u(s))ds + J \qquad (3.1) \end{aligned}$$

where $u(t) = [u_1(t)\ u_2(t) \dots u_n(t)]^T \in \mathbb{R}^n$ is the neural state vector. $A = \text{diag}\{a_1, a_2, \dots, a_n\} > 0$ is the state feedback coefficient matrix; $l(u(t)) = [l_1(u_1(t))\ l_2(u_2(t)) \dots l_n(u_n(t))]^T$, $m(u(t)) = [m_1(u_1(t))\ m_2(u_2(t)) \dots m_n(u_n(t))]^T$, and $r(u(t)) = [r_1(u_1(t))\ r_2(u_2(t)) \dots r_n(u_n(t))]^T$ are the activation functions. W_0 is the connection weight matrix and W_1, W_2 are the delayed connection weight matrices; $J = [J_1\ J_2 \dots J_n]^T$ represents the external inputs.

Assumption 3.1.1 The activation functions $l(.), m(.), r(.)$ are bounded and satisfy the Lipschitz condition,i.e.,

$$\begin{aligned} &|l_i(\xi_1) - l_i(\xi_2)| \leq d_i|\xi_1 - \xi_2|, \\ &|m_i(\xi_1) - m_i(\xi_2)| \leq e_i|\xi_1 - \xi_2|, \\ &|r_i(\xi_1) - r_i(\xi_2)| \leq j_i|\xi_1 - \xi_2|, \\ &d_i, e_i, j_i > 0\ , i = 1, 2, ..., n, \forall \xi_1, \xi_2 \in R. \end{aligned}$$

From Assumption 3.1.1, we obtained that there exists at least one equilibrium point for system (3.1) [14]. For the sake of simplicity in the exponential stability of system (3.1), we make the transformation $x(\cdot) = u(\cdot) - u^*$, then we have

$$\begin{aligned} \dot{x}(t) &= -(A+\Delta A(t))x(t) + (W_0+\Delta W_0(t))f(x(t)) + (W_1+\Delta W_1(t))g(x(t-h(t)) \\ &\quad +(W_2+\Delta W_2(t))\int_{t-k(t)}^{t} h(x(s))ds + A_2\dot{x}(t-\eta(t)) \end{aligned} \tag{3.2}$$

$x(\theta) = \phi(\theta), \dot{x}(\theta) = \varphi(\theta), \theta \in [-d, 0], d = \max\{h_2, k, \eta_2\}$,

where $x(t) \in R^n$ is state vector of the transformed system; n is the number of neurons. Let t be defined $x_t(\theta) = x(t+\theta), \forall\theta \in [-d, 0]$, and $A = \text{diag}(\bar{a_1}, \bar{a_2}, ..., \bar{a_n})$, $\bar{a_i} > 0$ represent the self-feedback term and $W_0,\ W_1,\ W_2$ denote the connection weight matrix, the discretely delayed connection weight matrix and distributively delayed connection weight matrix, respectively.

$$\begin{aligned} f(x(t)) &= [f_1^T(x_1(t)), f_2^T(x_2(t)), ..., f_n^T(x_n(t))]^T, \\ g(x(t)) &= [g_1^T(x_1(t)), g_2^T(x_2(t)), ..., g_n^T(x_n(t))]^T, \\ h(x(t)) &= [h_1^T(x_1(t)), h_2^T(x_2(t)), ..., h_n^T(x_n(t))]^T, \\ f_i(x_i(t)) &= l_i(x_i(t)+u^*) - l_i(u^*), \quad i = 1, 2, ..., n, \\ g_i(x_i(t)) &= m_i(x_i(t)+u^*) - m_i(u^*), \quad i = 1, 2, ..., n, \\ h_i(x_i(t)) &= r_i(x_i(t)+u^*) - r_i(u^*), \quad i = 1, 2, ..., n, \end{aligned}$$

According to Assumption 3.1.1, function $f(.), g(.), h(.), i = 1, 2, ..., n$ satisfy the following condition:

$$\begin{aligned} &|f_i(x_i)| \le a_i|x_i|, \\ &|g_i(x_i)| \le b_i|x_i|, \\ &|h_i(x_i)| \le c_i|x_i|, \\ &a_i, b_i, c_i > 0, \forall x_i \neq 0, i = 1, 2, ..., n. \end{aligned} \tag{3.3}$$

The time-varying delay functions $h(t), k(t), \eta(t)$ satisfy the conditions

$$\begin{aligned} 0 \le h_1 \le h(t) &\le h_2, \\ 0 \le k(t) &\le k, \\ 0 \le \eta_1 \le \eta(t) &\le \eta_2. \end{aligned}$$

It is worth noting that time delays are assumed to be a continuous function belonging to a given interval. The state and neutral delay functions are bounded below but not restricted to be zero. Both time-varying state and distributed delays are not necessarily differentiable. The neutral delay function $\eta(t)$ is a differentiable bounded function. The initial function $\phi(t) \in C^1([-d,0], R^n)$, with the norm

$$\|\phi\| = \sup_{-d\le t\le 0} \sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

The uncertainties satisfy the following condition:

$$[\Delta A(t)\ \Delta W_0(t)\ \Delta W_1(t)\ \Delta W_2(t)] = BF(t)[E_a\ E_0\ E_1\ E_2] \tag{3.4}$$

where $E_i,\ i = a, 0, 1, 2$ are given constant matrices with appropriate - dimensions, $F(t)$ are unknown real matrices with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \le I, \quad \forall t \ge 0.$$

3.1 Exponential stability of neutral-type neural networks with interval non-differentiable and distributed time-varying delays with uncertainties

Throughout this section, we assume that all eigenvalues of matrix A_2 are inside the unit circle. Rewrite system (3.2) in the following descriptor system:

$$\begin{aligned} y(t) &= \dot{x}(t) - A_2\dot{x}(t-\eta(t)) \qquad (3.5)\\ y(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t)) + W_2 \int_{t-k(t)}^{t} h(x(s))ds + Bp(t)\\ p(t) &= F(t)(E_a x(t) + E_0 f(x(t)) + E_1 g(x(t-h(t))) + E_2 \int_{t-k(t)}^{t} h(x(s))ds). \end{aligned}$$

Theorem 3.1.1 *Given $\alpha > 0$. The system (3.5) is $\alpha-$exponentially stable if there exist $\epsilon > 0$, symmetric positive definite matrices $P_{23}, S, D, Q, R, U, T, M, N, D_2$ and matrices $P_{1a}, P_{2a}, P_i, i = 1, 2, ..., 22$ of appropriate dimensions such that the following LMIs hold :*

$$\begin{aligned}\mathcal{M}_1 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ & \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathcal{M}_2 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ & \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
\mathcal{M}_3 = & \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
\mathcal{M}_4 = & \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & I & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & I & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.9}
$$

$$
\mathcal{M} = \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} & \phi_{17} & \phi_{18} & \phi_{19} & \phi_{1a} & \phi_{1b} & \phi_{1c} \\
* & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{28} & \phi_{29} & \phi_{2a} & \phi_{2b} & \phi_{2c} \\
* & * & \phi_{33} & \phi_{34} & \phi_{35} & \phi_{36} & \phi_{37} & \phi_{38} & \phi_{39} & \phi_{3a} & \phi_{3b} & \phi_{3c} \\
* & * & * & \phi_{44} & 0 & 0 & \phi_{47} & 0 & 0 & \phi_{4a} & \phi_{4b} & 0 \\
* & * & * & * & \phi_{55} & 0 & \phi_{57} & 0 & 0 & \phi_{5a} & \phi_{5b} & 0 \\
* & * & * & * & * & \phi_{66} & \phi_{67} & 0 & 0 & \phi_{6a} & \phi_{6b} & \phi_{6c} \\
* & * & * & * & * & * & \phi_{77} & \phi_{78} & \phi_{79} & \phi_{7a} & \phi_{7b} & \phi_{7c} \\
* & * & * & * & * & * & * & \phi_{88} & 0 & \phi_{8a} & \phi_{8b} & 0 \\
* & * & * & * & * & * & * & * & \phi_{99} & \phi_{9a} & \phi_{9b} & 0 \\
* & * & * & * & * & * & * & * & * & \phi_{10a} & \phi_{10b} & \phi_{10c} \\
* & * & * & * & * & * & * & * & * & * & \phi_{11b} & \phi_{11c} \\
* & * & * & * & * & * & * & * & * & * & * & \phi_{12c}
\end{bmatrix},
$$

where

$$
\begin{aligned}
\phi_{11} &= -P_{12}^T - P_{12}, \\
\phi_{12} &= -P_1^T - P_{13}, \\
\phi_{13} &= -P_1^T A + P_1^T W_0 F + P_{12}^T - P_{14}, \\
\phi_{14} &= -P_{15}, \\
\phi_{15} &= -P_{16}, \\
\phi_{16} &= -P_{17}, \\
\phi_{17} &= P_1^T W_1 G - P_{18}, \\
\phi_{18} &= -P_{19}, \\
\phi_{19} &= -P_{20}, \\
\phi_{1a} &= -P_{12}^T A_2 - P_{21}, \\
\phi_{1b} &= P_1^T W_2 - P_{22}, \\
\phi_{1c} &= -P_{2a}^T + P_1^T B,
\end{aligned}
$$

$$
\begin{aligned}
\phi_{22} &= -P_2^T - P_2, \\
\phi_{23} &= -P_2^T A + P_2^T W_0 F + P_{13}^T - P_3, \\
\phi_{24} &= -P_4, \\
\phi_{25} &= -P_5, \\
\phi_{26} &= -P_6, \\
\phi_{27} &= P_2^T W_1 G - P_7, \\
\phi_{28} &= -P_8, \\
\phi_{29} &= -P_9, \\
\phi_{2a} &= -P_{13}^T A_2 - P_{10}, \\
\phi_{2b} &= P_2^T W_2 - P_{11}, \\
\phi_{2c} &= -P_{1a} + P_2^T B, \\
\phi_{33} &= -A^T P_3 - P_3^T A + F^T W_0^T P_3 + P_3^T W_0 F + P_{14}^T + P_{14} + \alpha P_{23}^T + \alpha P_{23} \\
&\quad + Q + Q^T - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + T + T^T - e^{-2\alpha \eta_1} M - e^{-2\alpha \eta_2} M \\
&\quad + K^2 H D_2 H + K^2 H^T D_2^T H^T + A^T A + F W_0^T W_0 F + 2F^2 + A^T W_0 W_0^T A \\
&\quad + A^T W_1 W_1^T A + \epsilon E_a^T E_a + \epsilon E_a^T E_0 F + \epsilon F^T E_0^T E_a + \epsilon F^T E_0^T E_0 F, \\
\phi_{34} &= -A^T P_4 + F^T W_0^T P_4 + P_{15} + e^{-2\alpha h_1} R, \\
\phi_{35} &= -A^T P_5 + F^T W_0^T P_5 + P_{16} + e^{-2\alpha h_2} R, \\
\phi_{36} &= P_{23}^T - A^T P_6 + F^T W_0^T P_6 + P_{17} + A^T - A^T D^T + F^T W_0^T D^T, \\
\phi_{37} &= P_3^T W_1 G - A^T P_7 + F^T W_0^T P_7 + P_{18} + F^T W_0^T W_1 G + \epsilon E_a^T E_1 G \\
&\quad + \epsilon F^T E_0^T E_1 G, \\
\phi_{38} &= -A^T P_8 + F^T W_0^T P_8 + P_{19} + e^{-2\alpha \eta_1} M, \\
\phi_{39} &= -A^T P_9 + F^T W_0^T P_9 + P_{20} + e^{-2\alpha \eta_2} M, \\
\phi_{3a} &= -P_{14}^T A_2 - A^T P_{10} + F^T W_0^T P_{10} + P_{21}, \\
\phi_{3b} &= P_3^T W_2 - A^T P_{11} + F^T W_0^T P_{11} + P_{22} - A^T W_2 + F W_0^T W_2 + \epsilon E_a^T E_2 \\
&\quad + \epsilon F^T E_0^T E_2, \\
\phi_{3c} &= -A^T P_{1a} + F^T W_0^T P_{1a} + P_{2a} + P_3^T B, \\
\phi_{44} &= -e^{-2\alpha h_1} Q - e^{-2\alpha h_1} R - e^{-2\alpha h_2} U, \\
\phi_{47} &= P_4^T W_1 G + e^{-2\alpha h_2} U, \\
\phi_{4a} &= -P_{15}^T A_2, \\
\phi_{4b} &= P_4^T W_2,
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4c} &= P_4^T B, \\
\phi_{55} &= -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \\
\phi_{57} &= P_5^T W_1 G + e^{-2\alpha h_2}U, \\
\phi_{5a} &= -P_{16}^T A_2, \\
\phi_{5b} &= P_5^T W_2, \\
\phi_{5c} &= P_5^T B, \\
\phi_{66} &= h_1^2 R + h_2^2 R + (h_1 - h_2)^2 U + \eta_1^2 M + \eta_2^2 M + (\eta_1 - \eta_2)^2 N - 2D + I + D^2 \\
&\quad + W_0 W_0^T + W_1 W_1^T, \\
\phi_{67} &= P_6^T W_1 G + D W_1 G, \\
\phi_{6a} &= -P_{17}^T A_2, \\
\phi_{6b} &= P_6^T W_2 - W_2 + D W_2, \\
\phi_{6c} &= DB + P_6^T B, \\
\phi_{77} &= P_7^T W_1 G + G^T W_1^T P_7 - e^{-2\alpha h_2}U - e^{-2\alpha h_2}U^T + G W_1^T W_1 G + \epsilon G^T E_1^T E_1 G \\
&\quad + 2G^2, \\
\phi_{78} &= G^T W_1^T P_8, \\
\phi_{79} &= G^T W_1^T P_9, \\
\phi_{7a} &= -P_{18}^T A_2 + G^T W_1^T P_{10}, \\
\phi_{7b} &= P_7^T W_2 + G^T W_1^T P_{11} + G W_1^T W_2 + \epsilon G^T E_1^T E_2, \\
\phi_{7c} &= G^T W_1^T P_{1a} + P_7^T B, \\
\phi_{88} &= -e^{-2\alpha \eta_1}T - e^{-2\alpha \eta_1}M - e^{-2\alpha \eta_2}N, \\
\phi_{8a} &= -P_{19}^T A_2 + e^{-2\alpha \eta_2}N, \\
\phi_{8b} &= P_8^T W_2, \\
\phi_{8c} &= P_8^T B, \\
\phi_{99} &= -e^{-2\alpha \eta_2}T - e^{-2\alpha \eta_2}M - e^{-2\alpha \eta_2}N, \\
\phi_{9a} &= -P_{20}^T A_2 + e^{-2\alpha \eta_2}N, \\
\phi_{9b} &= P_9^T W_2, \\
\phi_{9c} &= P_9^T B, \\
\phi_{10a} &= -P_{21}^T A_2 - A_2^T P_{21} - e^{-2\alpha \eta_2}N - e^{-2\alpha \eta_2}N^T, \\
\phi_{10b} &= P_{10}^T W_2 - A_2^T P_{22}, \\
\phi_{10c} &= -A_2^T P_{2a} + P_{10}^T B,
\end{aligned}
$$

$$\begin{aligned}
\phi_{11b} &= P_{11}^T W_2 + W_2^T P_{11} - e^{-2\alpha k} D_2 - e^{-2\alpha k} D_2^T + W_2^T W_2 + \epsilon E_2^T E_2,\\
\phi_{11c} &= W_2^T P_{1a} + P_{11}^T B,\\
\phi_{12c} &= B^T P_{1a} - \epsilon I.
\end{aligned}$$

The solution $x(t)$ of the system satisfies,

$$\|x(t)\| \le \sqrt{\frac{a\|\phi\|^2 + b\|M_1\|^2}{\lambda_{\min}(P_{23})}}.e^{-\alpha t}, \tag{3.10}$$

where $a = \lambda_{\max}(P_{23}) + 2h_2^2\lambda_{\max}(R)\frac{1-e^{-2\alpha h_2}}{2\alpha} + h_2^2\lambda_{\max}(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}$
$+ 2\eta_2^2\lambda_{\max}(M)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + \eta_2^2\lambda_{\max}(N)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + k^2\lambda_{\max}(HD_2H)\frac{1-e^{-2\alpha k}}{2\alpha}$,
$b = 2\lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha} + 2\lambda_{\max}(T)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}$ and $\|M_1\| = \sup_{-d\le s\le 0}\|x(s)\|$.

Proof. We consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{12} V_i, \tag{3.11}$$

where

$$\begin{aligned}
V_1 &= e^{2\alpha t}\zeta^T(t)E^T P\zeta(t) = e^{2\alpha t}x^T(t)P_{23}x(t),\\
V_2 &= \int_{t-h_1}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_3 &= \int_{t-h_2}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_4 &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_5 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_6 &= (h_2 - h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds,\\
V_7 &= \int_{t-\eta_1}^{t} e^{2\alpha s}x^T(s)Tx(s)\,ds,\\
V_8 &= \int_{t-\eta_2}^{t} e^{2\alpha s}x^T(s)Tx(s)\,ds,\\
V_9 &= \eta_1\int_{-\eta_1}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds,\\
V_{10} &= \eta_2\int_{-\eta_2}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds,\\
V_{11} &= (\eta_2 - \eta_1)\int_{-\eta_2}^{-\eta_1}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)N\dot{x}(\tau)\,d\tau\,ds,\\
V_{12} &= k\int_{-k}^{0}\int_{t+s}^{t} e^{2\alpha\tau}h^T(x(\tau))D_2h(x(\tau))\,d\tau\,ds,
\end{aligned}$$

where $\zeta(t) = [\Delta^T(t) \;\; y^T(t) \;\; x^T(t) \;\; x^T(t-h_1) \;\; x^T(t-h_2) \;\; \dot{x}^T(t) \;\; x^T(t-h(t))$
$x^T(t-\eta_1) \;\; x^T(t-\eta_2) \;\; x^T(t-\eta(t)) \;\; (\int_{t-k(t)}^{t} h(x(s))\, ds)^T \;\; p^T(t)]^T$,
$\Delta(t) = x(t) - A_2 x(t-\eta(t))$ and

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & P_{1a} \\ P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} & P_{20} & P_{21} & P_{22} & P_{2a} \\ 0 & 0 & P_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking the derivative of $V(x_t)$ along any trajectory of solution of system (3.5), we have

$$\begin{aligned} \dot{V}_1 &= 2\alpha e^{2\alpha t} x^T(t) P_{23} x(t) + 2e^{2\alpha t} x^T(t) P_{23} \dot{x}(t) \\ &= 2\alpha e^{2\alpha t} x^T(t) P_{23} x(t) + 2e^{2\alpha t} \zeta^T(t) P^T \begin{bmatrix} 0 \\ 0 \\ \dot{x}(t) \end{bmatrix}. \end{aligned}$$

From (3.5), we have

$$\begin{aligned} \dot{V}_1 = \;& 2\alpha e^{2\alpha t} x^T(t) P_{23} x(t) + 2e^{2\alpha t} \zeta^T(t) P^T \\ & \times \begin{bmatrix} -y(t) - Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) \\ +W_2 \int_{t-k(t)}^{t} h(x(s)) ds + Bp(t), \\ -\Delta(t) + x(t) - A_2 x(t-\eta(t)), \\ \dot{x}(t) \end{bmatrix} \end{aligned}$$

Using condition (3.3) we have,

$$\begin{aligned} \dot{V}_1 \leq & 2e^{2\alpha t}\{\alpha x^T(t) P_{23} x(t) - \Delta^T(t) P_1^T y(t) - y^T(t) P_2^T y(t) - x^T(t) P_3^T y(t) \\ & - x^T(t-h_2) P_5^T y(t) - \dot{x}^T(t) P_6^T y(t) - x^T(t-h(t)) P_7^T y(t) \\ & - x^T(t-\eta_2) P_9^T y(t) - x^T(t-\eta(t)) P_{10}^T y(t) - (\int_{t-k(t)}^{t} h(x(s))\, ds)^T P_{11}^T y(t) \\ & - \Delta^T(t) P_1^T Ax(t) - y^T(t) P_2^T Ax(t) - x^T(t) P_3^T Ax(t)) - x^T(t-h_1) P_4^T Ax(t) \\ & - x^T(t-h_2) P_5^T Ax(t) - \dot{x}^T(t) P_6^T Ax(t) - x^T(t-h(t)) P_7^T Ax(t)) \\ & - x^T(t-\eta_1) P_8^T Ax(t) - x^T(t-\eta_2) P_9^T Ax(t) - x^T(t-\eta(t)) P_{10}^T Ax(t) \\ & - (\int_{t-k(t)}^{t} h(x(s))\, ds)^T P_{11}^T Ax(t) + \Delta^T(t) P_1^T W_0 F x(t) + y^T(t) P_2^T W_0 F x(t) \end{aligned}$$

$$+ x^T(t)P_3^TW_0Fx(t) + x^T(t-h_1)P_4^TW_0Fx(t) + x^T(t-h_2)P_5^TW_0Fx(t)$$
$$+ \dot{x}^T(t)P_6^TW_0Fx(t) + x^T(t-h(t))P_7^TW_0Fx(t) + x^T(t) + \eta_1)P_8^TW_0Fx(t)$$
$$+ x^T(t-\eta_2)P_9^TW_0Fx(t) + x^T(t-\eta(t))P_{10}^TW_0Fx(t) - x^T(t-\eta_1)P_8^Ty(t)$$
$$+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^TP_{11}^TW_0Fx(t) + \Delta^T(t)P_1^TW_1Gx(t-h(t))$$
$$+ y^T(t)P_2^TW_1Gx(t-h(t)) + x^T(t)P_3^TW_1Gx(t-h(t)) - x^T(t-h_1)P_4^Ty(t)$$
$$+ x^T(t-h_1)P_4^TW_1Gx(t-h(t)) + x^T(t-h_2)P_5^TW_1Gx(t-h(t))$$
$$+ \dot{x}^T(t)P_6^TW_1Gx(t-h(t)) + x^T(t-h(t))P_7^TW_1Gx(t-h(t))$$
$$+ x^T(t-\eta_1)P_8^TW_1Gx(t-h(t)) + x^T(t-\eta_2)P_9^TW_1Gx(t-h(t))$$
$$+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^TP_{11}^TW_1Gx(t-h(t)) + \Delta^T(t)P_1^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds$$
$$+ y^T(t)P_2^TW_2\int_{t-k(t)}^{t} h(x(s)) + x^T(t)P_3^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds\,ds$$
$$+ x^T(t-h_1)P_4^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h_2)P_5^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds$$
$$+ \dot{x}^T(t)P_6^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h(t))P_7^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds$$
$$+ x^T(t-\eta_1)P_8^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta_2)P_9^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds$$
$$+ x^T(t-\eta(t))P_{10}^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta(t))P_{10}^TW_1Gx(t-h(t))$$
$$+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^TP_{11}^TW_2\int_{t-k(t)}^{t} h(x(s))\,ds - \Delta^T(t)P_{12}^T\Delta(t)$$
$$- y^T(t)P_{13}^T\Delta(t) - x^T(t)P_{14}^T\Delta(t) - x^T(t-h_1)P_{15}^T\Delta(t)$$
$$- x^T(t-h_2)P_{16}^T\Delta(t) - \dot{x}^T(t)P_{17}^T\Delta(t) - x^T(t-h(t))P_{18}^T\Delta(t)$$
$$- x^T(t-\eta_1)P_{19}^T\Delta(t)) - x^T(t-\eta_2)P_{20}^T\Delta(t) - x^T(t-\eta(t))P_{21}^T\Delta(t)$$
$$- (\int_{t-k(t)}^{t} h(x(s))\,ds)^TP_{22}^T\Delta(t) + \Delta^T(t)P_{12}^Tx(t) + y^T(t)P_{13}^Tx(t)$$
$$+ x^T(t)P_{14}^Tx(t) + x^T(t-h_1)P_{15}^Tx(t) + x^T(t-h_2)P_{16}^Tx(t)$$
$$+ \dot{x}^T(t)P_{17}^Tx(t) + x^T(t-h(t))P_{18}^Tx(t) + x^T(t) - \eta_1)P_{19}^Tx(t)$$
$$+ x^T(t-\eta_2)P_{20}^Tx(t) + (\int_{t-k(t)}^{t} h(x(s))\,ds)^TP_{22}^Tx(t) - \Delta^T(t)P_{12}^TA_2x(t-\eta(t))$$
$$+ x^T(t-\eta(t))P_{21}^Tx(t) - y^T(t)P_{13}^TA_2x(t-\eta(t)) - x^T(t)P_{14}^TA_2x(t-\eta(t))$$
$$- x^T(t-h_1)P_{15}^TA_2x(t-\eta(t)) - x^T(t-h_2)P_{16}^TA_2x(t-\eta(t))$$
$$- \dot{x}^T(t)P_{17}^TA_2x(t-\eta(t)) - x^T(t-h(t))P_{18}^TA_2x(t-\eta(t))$$

$$
\begin{aligned}
& - x^T(t-\eta_1)P_{19}^T A_2 x(t-\eta(t)) - x^T(t-\eta_2)P_{20}^T A_2 x(t-\eta(t)) \\
& - x^T(t-\eta(t))P_{21}^T A_2 x(t-\eta(t)) - (\int_{t-k(t)}^t h(x(s))\,ds)^T P_{22}^T A_2 x(t-\eta(t)) \\
& + x^T(t)P_{23}^T \dot{x}(t) - p^T(t)P_{1a}^T y(t) - p^T(t)P_{1a}^T A x(t) + p^T(t)P_{1a}^T W_0 F x(t) \\
& + p^T(t)P_{1a}^T W_1 G x(t-h(t)) + p^T(t)P_{1a}^T W_2 \int_{t-k(t)}^t h(x(s))ds \\
& + p^T(t)P_{1a}^T B p(t) - p^T(t)P_{2a}^T \Delta(t) + p^T(t)P_{2a}^T x(t) - p^T(t)P_{2a}^T A_2 x(t-\eta(t)) \\
& + \Delta^T(t)P_1^T B p(t) + y^T(t)P_2^T B p(t) + x^T(t)P_3^T B p(t) + x^T(t-h_1)P_4^T B p(t) \\
& + x^T(t-h_2)P_5^T B p(t) + \dot{x}^T(t)P_6^T B p(t) + x^T(t-h(t))P_7^T B p(t) \\
& + x^T(t-\eta_1)P_8^T B p(t) + x^T(t-\eta_2)P_9^T B p(t) + x^T(t-\eta(t))P_{10}^T B p(t) \\
& + (\int_{t-k(t)}^t h(x(s))\,ds)^T P_{11}^T B p(t), \qquad (3.12)
\end{aligned}
$$

$$\dot{V}_2 = e^{2\alpha t}x^T(t)Qx(t) - e^{2\alpha(t-h_1)}x^T(t-h_1)Qx(t-h_1), \qquad (3.13)$$

$$\dot{V}_3 = e^{2\alpha t}x^T(t)Qx(t) - e^{2\alpha(t-h_2)}x^T(t-h_2)Qx(t-h_2), \qquad (3.14)$$

$$\dot{V}_4 \le e^{2\alpha t}[h_1^2\dot{x}^T(t)R\dot{x}(t) - h_1\int_{t-h_1}^t e^{2\alpha s}\dot{x}^T(s)R\dot{x}(s)ds], \qquad (3.15)$$

$$\dot{V}_5 \le e^{2\alpha t}[h_2^2\dot{x}^T(t)R\dot{x}(t) - h_2\int_{t-h_2}^t e^{2\alpha s}\dot{x}^T(s)R\dot{x}(s)ds], \qquad (3.16)$$

$$\dot{V}_6 \le e^{2\alpha t}[(h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t) - (h_2-h_1)\int_{t-h_2}^{t-h_1} e^{2\alpha s}\dot{x}^T(s)U\dot{x}(s)ds]. \qquad (3.17)$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_1\int_{t-h_1}^t \dot{x}^T(s)R\dot{x}(s)ds \le & -\Big[\int_{t-h_1}^t \dot{x}(s)ds\Big]^T R\Big[\int_{t-h_1}^t \dot{x}(s)ds\Big] \\
\le & -[x(t)-x(t-h_1)]^T R[x(t)-x(t-h_1)] \\
= & -x^T(t)Rx(t) + 2x^T(t)Rx(t-h_1) \\
& - x^T(t-h_1)Rx(t-h_1). \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
-h_2\int_{t-h_2}^t \dot{x}^T(s)R\dot{x}(s)ds \le & -\Big[\int_{t-h_2}^t \dot{x}(s)ds\Big]^T R\Big[\int_{t-h_2}^t \dot{x}(s)ds\Big] \\
\le & -[x(t)-x(t-h_2)]^T R[x(t)-x(t-h_2)] \\
= & -x^T(t)Rx(t) + 2x^T(t)Rx(t-h_2) \\
& - x^T(t-h_2)Rx(t-h_2). \qquad (3.19)
\end{aligned}
$$

Note that

$$-(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds = -(h_2-h_1)\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds$$

$$
\begin{aligned}
&- (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds \\
=& - (h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) U \dot{x}(s)\, ds \\
&- (h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) U \dot{x}(s)\, ds \\
&- (h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds \\
&- (h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds. \qquad (3.20)
\end{aligned}
$$

Using Lemma 2.3.3 gives

$$
\begin{aligned}
-(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) U \dot{x}(s)\, ds \leq& - \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s) ds \Big]^T U \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s) ds \Big] \\
\leq& - [x(t-h(t)) - x(t-h_2)]^T U [x(t-h(t)) \\
&- x(t-h_2)], \qquad (3.21)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds \leq& - \Big[\int_{t-h(t)}^{t-h_1} \dot{x}(s) ds \Big]^T U \Big[\int_{t-h(t)}^{t-h_1} \dot{x}(s) ds \Big] \\
\leq& - [x(t-h_1) - x(t-h(t))]^T U [x(t-h_1) \\
&- x(t-h(t))]. \qquad (3.22)
\end{aligned}
$$

Since $0 \leq h_1 \leq h(t) \leq h_2$ we get $\beta = \frac{h_2 - h(t)}{h_2 - h_1} \leq 1$. Then

$$
\begin{aligned}
-(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds =& - \beta \int_{t-h(t)}^{t-h_1} (h_2 - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
\leq& - \beta \int_{t-h(t)}^{t-h_1} (h(t) - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
\leq& - \beta [x(t-h_1) - x(t-h(t))]^T U [x(t-h_1) \\
&- x(t-h(t))], \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) U \dot{x}(s)\, ds =& - (1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
\leq& - (1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h(t)) \dot{x}^T(s) U \dot{x}(s)\, ds \\
\leq& - (1-\beta) [x(t-h(t)) - x(t-h_2)]^T \\
&\times U [x(t-h(t)) - x(t-h_2)]. \qquad (3.24)
\end{aligned}
$$

From (3.20)-(3.24), we obtain

$$\begin{aligned}-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds \le &-[x(t-h(t))-x(t-h_2)]^T\\ &\times U[x(t-h(t))-x(t-h_2)]\\ &-[x(t-h_1)-x(t-h(t))]^T\\ &\times U[x(t-h_1)-x(t-h(t))]\\ &-\beta[x(t-h_1)-x(t-h(t))]^T\\ &\times U[x(t-h_1)-x(t-h(t))]\\ &-(1-\beta)[x(t-h(t))-x(t-h_2)]^T\\ &\times U[x(t-h(t))-x(t-h_2)]. \end{aligned} \tag{3.25}$$

$$\dot{V}_7 = e^{2\alpha t}x^T(t)Tx(t) - e^{2\alpha(t-\eta_1)}x^T(t-\eta_1)Tx(t-\eta_1), \tag{3.26}$$

$$\dot{V}_8 = e^{2\alpha t}x^T(t)Tx(t) - e^{2\alpha(t-\eta_2)}x^T(t-\eta_2)Tx(t-\eta_2), \tag{3.27}$$

$$\dot{V}_9 \le e^{2\alpha t}[\eta_1^2\dot{x}^T(t)M\dot{x}(t) - \eta_1\int_{t-\eta_1}^{t} e^{2\alpha s}\dot{x}^T(s)M\dot{x}(s)ds], \tag{3.28}$$

$$\dot{V}_{10} \le e^{2\alpha t}[\eta_2^2\dot{x}^T(t)M\dot{x}(t) - \eta_2\int_{t-\eta_2}^{t} e^{2\alpha s}\dot{x}^T(s)M\dot{x}(s)ds], \tag{3.29}$$

$$\dot{V}_{11} \le e^{2\alpha t}[(\eta_2-\eta_1)^2\dot{x}^T(t)N\dot{x}(t) - (\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1} e^{2\alpha s}\dot{x}^T(s)N\dot{x}(s)ds]. \tag{3.30}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned}-\eta_1\int_{t-\eta_1}^{t}\dot{x}^T(s)M\dot{x}(s)ds \le &-\Big[\int_{t-\eta_1}^{t}\dot{x}(s)\Big]^T M\Big[\int_{t-\eta_1}^{t}\dot{x}(s)\Big]\\ \le &-[x(t)-x(t-\eta_1)]^T M[x(t)-x(t-\eta_1)]\\ = &-x^T(t)Mx(t)+2x^T(t)Mx(t-\eta_1)\\ &-x^T(t-\eta_1)Mx(t-\eta_1). \end{aligned} \tag{3.31}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned}-\eta_2\int_{t-\eta_2}^{t}\dot{x}^T(s)M\dot{x}(s)ds \le &-\Big[\int_{t-\eta_2}^{t}\dot{x}(s)\Big]^T M\Big[\int_{t-\eta_2}^{t}\dot{x}(s)\Big]\\ \le &-[x(t)-x(t-\eta_2)]^T M[x(t)-x(t-\eta_2)]\\ = &-x^T(t)Mx(t)+2x^T(t)Mx(t-\eta_2)\\ &-x^T(t-\eta_2)Mx(t-\eta_2). \end{aligned} \tag{3.32}$$

Note that

$$-(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1}\dot{x}^T(s)N\dot{x}(s)\,ds = -(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta(t)}\dot{x}^T(s)N\dot{x}(s)\,ds$$

$$
\begin{aligned}
&- (\eta_2 - \eta_1) \int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s) N \dot{x}(s)\, ds \\
&= - (\eta_2 - \eta(t)) \int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\quad - (\eta(t) - \eta_1) \int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\quad - (\eta(t) - \eta_1) \int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\quad - (\eta_2 - \eta(t)) \int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s) N \dot{x}(s)\, ds. \qquad (3.33)
\end{aligned}
$$

Using Lemma 2.3.3 gives

$$
\begin{aligned}
-(\eta_2 - \eta(t)) \int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s) N \dot{x}(s)\, ds &\leq - \Big[\int_{t-\eta_2}^{t-\eta(t)} \dot{x}(s) ds \Big]^T N \Big[\int_{t-\eta_2}^{t-\eta(t)} \dot{x}(s) ds \Big] \\
&\leq - [x(t-\eta(t)) - x(t-\eta_2)]^T N [x(t-\eta(t)) \\
&\quad - x(t-\eta_2)], \qquad (3.34)
\end{aligned}
$$

and

$$
\begin{aligned}
-(\eta(t) - \eta_1) \int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s) N \dot{x}(s)\, ds &\leq - \Big[\int_{t-\eta(t)}^{t-\eta_1} \dot{x}(s) ds \Big]^T N \Big[\int_{t-\eta(t)}^{t-\eta_1} \dot{x}(s) ds \Big] \\
&\leq - [x(t-\eta_1) - x(t-\eta(t))]^T N [x(t-\eta_1) \\
&\quad - x(t-\eta(t))]. \qquad (3.35)
\end{aligned}
$$

Since $0 \leq \eta_1 \leq \eta(t) \leq \eta_2$ we get $\alpha = \frac{\eta_2 - \eta(t)}{\eta_2 - \eta_1} \leq 1$. Then

$$
\begin{aligned}
-(\eta_2 - \eta(t)) \int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s) N \dot{x}(s)\, ds &= - \alpha \int_{t-\eta(t)}^{t-\eta_1} (\eta_2 - \eta_1) \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\leq - \alpha \int_{t-\eta(t)}^{t-\eta_1} (\eta(t) - \eta_1) \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\leq - \alpha [x(t-\eta_1) - x(t-\eta(t))]^T N [x(t-\eta_1) \\
&\quad - x(t-\eta(t))], \qquad (3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
-(\eta(t) - \eta_1) \int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s) N \dot{x}(s)\, ds &= - (1-\alpha) \int_{t-\eta_2}^{t-\eta(t)} (\eta_2 - \eta_1) \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\leq - (1-\alpha) \int_{t-\eta_2}^{t-\eta(t)} (\eta_2 - \eta(t)) \dot{x}^T(s) N \dot{x}(s)\, ds \\
&\leq - (1-\alpha) [x(t-\eta(t)) - x(t-\eta_2)]^T \\
&\quad \times N [x(t-\eta(t)) - x(t-\eta_2)]. \qquad (3.37)
\end{aligned}
$$

From (3.33)-(3.37), we obtain

$$
\begin{aligned}
-(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1}\dot{x}^T(s)N\dot{x}(s)\,ds \leq & -[x(t-\eta(t))-x(t-\eta_2)]^T \\
& \times N[x(t-\eta(t))-x(t-\eta_2)] \\
& -[x(t-\eta_1)-x(t-\eta(t))]^T \\
& \times N[x(t-\eta_1)-x(t-\eta(t))] \\
& -\alpha[x(t-\eta_1)-x(t-\eta(t))]^T \\
& \times N[x(t-\eta_1)-x(t-\eta(t))] \\
& -(1-\alpha)[x(t-\eta(t))-x(t-\eta_2)]^T \\
& \times N[x(t-\eta(t))-x(t-\eta_2)]. \qquad (3.38)
\end{aligned}
$$

$$
\dot{V}_{12} \leq e^{2\alpha t}[k^2x^T(t)HD_2Hx(t)-e^{-2\alpha k}(\int_{t-k(t)}^{t}h(x(s))\,ds)^TD_2\int_{t-k(t)}^{t}h(x(s))\,ds]. \qquad (3.39)
$$

By using the following identity relation

$$
\begin{aligned}
& -\dot{x}(t)-Ax(t)+W_0f(x(t))+W_1g(x(t-h(t))+W_2\int_{t-k(t)}^{t}h(x(s))ds \\
& +A_2\dot{x}(t-\eta(t))+Bp(t)=0,
\end{aligned}
$$

we have

$$
\begin{aligned}
& -2\dot{x}^T(t)D\dot{x}(t)-2\dot{x}^T(t)DAx(t)+2\dot{x}^T(t)DW_0f(x(t))+2\dot{x}^T(t)DW_1g(x(t-h(t)) \\
& +2\dot{x}^T(t)DW_2\int_{t-k(t)}^{t}h(x(s))ds+2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t))+2\dot{x}^T(t)DBp(t)=0.
\end{aligned}
$$

Using the condition (3.3), we have

$$
\begin{aligned}
& -2\dot{x}^T(t)D\dot{x}(t)-2\dot{x}^T(t)DAx(t)+2\dot{x}^T(t)DW_0Fx(t)+2\dot{x}^T(t)DW_1Gx(t-h(t)) \\
& +2\dot{x}^T(t)DW_2\int_{t-k(t)}^{t}h(x(s))ds+2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t))+2\dot{x}^T(t)DBp(t)\geq 0. \qquad (3.40)
\end{aligned}
$$

From, Lemma 2.3.4, we have

$$
\begin{aligned}
2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t)) \leq & \ \dot{x}^T(t)D^2\dot{x}(t)+[A_2\dot{x}(t-\eta(t))]^T[A_2\dot{x}(t-\eta(t))] \\
\leq & \ \dot{x}^T(t)D^2\dot{x}(t) \\
& +[\dot{x}(t)+Ax(t)-W_0f(x(t))-W_1g(x(t-h(t)) \\
& -W_2\int_{t-k(t)}^{t}h(x(s))ds-Bp(t)]^T[\dot{x}(t)+Ax(t)-W_0f(x(t)
\end{aligned}
$$

$$
\begin{aligned}
& - W_1 g(x(t-h(t)) - W_2 \int_{t-k(t)}^{t} h(x(s))ds - Bp(t)] \\
\leq\ & \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + \dot{x}^T(t)Ax(t) \\
& - \dot{x}^T(t)W_0 f(x(t) - \dot{x}^T(t)W_1 g(x(t-h(t)) \\
& - \dot{x}^T(t)W_2 \int_{t-k(t)}^{t} h(x(s))ds - \dot{x}^T(t)Bp(t) + x^T(t)A^T\dot{x}(t) \\
& + x^T(t)A^TAx(t) - x^T(t)A^TW_0 f(x(t)) \\
& - x^T(t)A^TW_1 g(x(t-h(t)) - x^T(t)A^TW_2 \int_{t-k(t)}^{t} h(x(s))ds \\
& - x^T(t)A^TBp(t) - f^T(x(t))W_0^T\dot{x}(t) - f^T(x(t))W_0^TAx(t) \\
& + f^T(x(t))W_0^TW_0 f(x(t)) + f^T(x(t))W_0^TW_1 g(x(t-h(t)) \\
& + f^T(x(t))W_0^TW_2 \int_{t-k(t)}^{t} h(x(s))ds + f^T(x(t))W_0^TBp(t) \\
& - g^T(x(t-h(t)))W_1^T\dot{x}(t) - g^T(x(t-h(t)))W_1^TAx(t) \\
& + g^T(x(t-h(t)))W_1^TW_0 f(x(t) \\
& + g^T(x(t-h(t)))W_1^TW_1 g(x(t-h(t)) \\
& + g^T(x(t-h(t)))W_1^TW_2 \int_{t-k(t)}^{t} h(x(s))ds \\
& + g^T(x(t-h(t)))W_1^TBp(t) - (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^T\dot{x}(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TAx(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_0 f(x(t)) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_1 g(x(t-h(t)) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2 \int_{t-k(t)}^{t} h(x(s))ds \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TBp(t) \\
& - p^T(t)B^T\dot{x}(t) - p^T(t)B^TAx(t) \\
& + p^T(t)B^TW_0 f(x(t)) + p^T(t)B^TW_1 g(x(t-h(t)) \\
& + p^T(t)B^TW_2 \int_{t-k(t)}^{t} h(x(s))ds + p^T(t)B^TBp(t) \\
=\ & \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
& - 2\dot{x}^T(t)W_0 f(x(t) - 2\dot{x}^T(t)W_1 g(x(t-h(t))
\end{aligned}
$$

$$
\begin{aligned}
&- 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
&- 2x^T(t)A^TW_0f(x(t)) - 2x^T(t)A^TW_1g(x(t-h(t)) \\
&- 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + f^T(x(t))W_0^TW_0f(x(t)) \\
&+ f^T(x(t))W_0^TW_1g(x(t-h(t)) + 2f^T(x(t))W_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ g^T(x(t-h(t)))W_1^TW_0f(x(t)) + g^T(x(t-h(t)))W_1^TW_1g(x(t-h(t)) \\
&+ 2g^T(x(t-h(t)))W_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ (\int_{t-k_t}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds. \\
&- \dot{x}^T(t)Bp(t) - x^T(t)A^TBp(t) + f^T(x(t))W_0^TBp(t) \\
&+ g^T(x(t-h(t)))W_1^TBp(t) + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TBp(t) \\
&- p^T(t)B^T\dot{x}(t) - p^T(t)B^TAx(t) \\
&+ p^T(t)B^TW_0f(x(t)) + p^T(t)B^TW_1g(x(t-h(t)) \\
&+ p^T(t)B^TW_2\int_{t-k(t)}^{t} h(x(s))ds + p^T(t)B^TBp(t)
\end{aligned}
$$

Using the condition (3.3), we have

$$
\begin{aligned}
2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t)) \leq\ & \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
&- 2\dot{x}^T(t)W_0fx(t) - 2\dot{x}^T(t)W_1gx(t-h(t)) \\
&- 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
&- 2x^T(t)A^TW_0fx(t) - 2x^T(t)A^TW_1gx(t-h(t)) \\
&- 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)FW_0^TW_0Fx(t) \\
&+ x^T(t)FW_0^TW_1Gx(t-h(t) \\
&+ 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ x^T(t-h(t))GW_1^TW_0Fx(t) \\
&+ x^T(t-h(t))GW_1^TW_1Gx(t-h(t)) \\
&+ 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&- \dot{x}^T(t)Bp(t) - x^T(t)A^TBp(t) + f^T(x(t))W_0^TBp(t) \\
&+ g^T(x(t-h(t)))W_1^TBp(t) + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TBp(t) \\
&- p^T(t)B^T\dot{x}(t) - p^T(t)B^TAx(t) \\
&+ p^T(t)B^TW_0f(x(t)) + p^T(t)B^TW_1g(x(t-h(t)) \\
&+ p^T(t)B^TW_2\int_{t-k(t)}^{t} h(x(s))ds + p^T(t)B^TBp(t). \qquad (3.41)
\end{aligned}
$$

From (3.40), (3.41) and Lemma 2.3.4 , we have

$$
\begin{aligned}
&- 2\dot{x}^T(t)D\dot{x}(t) - 2\dot{x}^T(t)DAx(t) + 2\dot{x}^T(t)DW_0Fx(t) + 2\dot{x}^T(t)DW_1Gx(t-h(t) \\
&+ 2\dot{x}^T(t)DW_2\int_{t-k(t)}^{t} h(x(s))ds + \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
&+ \dot{x}^T(t)W_0W_0^T\dot{x}^T(t) + x^T(t)F^2x(t) + \dot{x}^T(t)W_1W_1^T\dot{x}^T(t) \\
&+ x^T(t-h(t))G^2x(t-h(t)) + x^T(t)A^TW_0W_0^TAx(t) + x^T(t)F^2x(t) \\
&+ x^T(t)A^TW_1W_1^TAx(t) + x^T(t-h(t))G^2x(t-h(t)) \\
&- 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
&- 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)FW_0^TW_0Fx(t) \\
&+ x^T(t)FW_0^TW_1Gx(t-h(t) + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ x^T(t-h(t))GW_1^TW_0Fx(t) + x^T(t-h(t))GW_1^TW_1Gx(t-h(t) \\
&+ 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&- \dot{x}^T(t)Bp(t) - x^T(t)A^TBp(t) + f^T(x(t))W_0^TBp(t) \\
&+ g^T(x(t-h(t)))W_1^TBp(t) + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TBp(t) \\
&- p^T(t)B^T\dot{x}(t) - p^T(t)B^TAx(t) + p^T(t)B^TW_0f(x(t)) \\
&+ p^T(t)B^TW_1g(x(t-h(t)) + p^T(t)B^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
&+ p^T(t)B^TBp(t) + 2\dot{x}^T(t)DBp(t) \geq 0. \qquad (3.42)
\end{aligned}
$$

For $\epsilon > 0$, we have

$$- \epsilon p^T(t)p(t) + \epsilon(E_a x(t) + E_0 f(x(t)) + E_1 g(x(t-h(t))) + E_2 \int_{t-k(t)}^{t} h(x(s))ds)^T$$
$$\times (E_a x(t) + E_0 f(x(t)) + E_1 g(x(t-h(t))) + E_2 \int_{t-k(t)}^{t} h(x(s))ds) \geq 0. \qquad (3.43)$$

From (3.12)-(3.17), (3.26)-(3.30), (3.39)and (3.42)-(3.43), we have

$$\begin{aligned}
\dot{V}(t,x_t) \leq\ & e^{2\alpha t}\{2\{\alpha x^T(t)P_{23}x(t) - \Delta^T(t)P_1^T y(t) - y^T(t)P_2^T y(t) - x^T(t)P_3^T y(t) \\
& - x^T(t-h_1)P_4^T y(t) - x^T(t-h_2)P_5^T y(t) - \dot{x}^T(t)P_6^T y(t) \\
& - x^T(t-h(t))P_7^T y(t) - x^T(t-\eta_1)P_8^T y(t) \\
& - x^T(t-\eta_2)P_9^T y(t) - x^T(t-\eta(t))P_{10}^T y(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T y(t) - \Delta^T(t)P_1^T Ax(t) \\
& - y^T(t)P_2^T Ax(t) - x^T(t)P_3^T Ax(t)) - x^T(t-h_1)P_4^T Ax(t) \\
& - x^T(t-h_2)P_5^T Ax(t) - \dot{x}^T(t)P_6^T Ax(t) - x^T(t-h(t))P_7^T Ax(t)) \\
& - x^T(t-\eta_1)P_8^T Ax(t) + x^T(t-\eta_1)P_8^T W_0 Fx(t) \\
& - x^T(t-\eta_2)P_9^T Ax(t) - x^T(t-\eta(t))P_{10}^T Ax(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T Ax(t) + \Delta^T(t)P_1^T W_0 Fx(t) \\
& + y^T(t)P_2^T W_0 Fx(t) + x^T(t-h_2)P_5^T W_0 Fx(t) \\
& + x^T(t)P_3^T W_0 Fx(t) + x^T(t-h_1)P_4^T W_0 Fx(t) \\
& + \dot{x}^T(t)P_6^T W_0 Fx(t) + x^T(t-h(t))P_7^T W_0 Fx(t) \\
& + x^T(t-\eta_2)P_9^T W_0 Fx(t) + x^T(t-\eta(t))P_{10}^T W_0 Fx(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_0 Fx(t) + \Delta^T(t)P_1^T W_1 Gx(t-h(t)) \\
& + y^T(t)P_2^T W_1 Gx(t-h(t)) + x^T(t)P_3^T W_1 Gx(t-h(t)) \\
& + x^T(t-h_1)P_4^T W_1 Gx(t-h(t)) + x^T(t-h_2)P_5^T W_1 Gx(t-h(t)) \\
& + \dot{x}^T(t)P_6^T W_1 Gx(t-h(t)) + x^T(t-h(t))P_7^T W_1 Gx(t-h(t)) \\
& + x^T(t-\eta_1)P_8^T W_1 Gx(t-h(t)) + x^T(t-\eta_2)P_9^T W_1 Gx(t-h(t)) \\
& + x^T(t-\eta(t))P_{10}^T W_1 Gx(t-h(t)) \\
& + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_1 Gx(t-h(t)) \\
& + \Delta^T(t)P_1^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + y^T(t)P_2^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ x^T(t)P_3^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h_1)P_4^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-h_2)P_5^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + \dot{x}^T(t)P_6^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-h(t))P_7^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta_1)P_8^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-\eta_2)P_9^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta(t))P_{10}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds - \Delta^T(t)P_{12}^T\Delta(t) \\
&- y^T(t)P_{13}^T\Delta(t) - x^T(t)P_{14}^T\Delta(t) - x^T(t-h_1)P_{15}^T\Delta(t) - x^T(t-h_2)P_{16}^T\Delta(t) \\
&- \dot{x}^T(t)P_{17}^T\Delta(t) - x^T(t-h(t))P_{18}^T\Delta(t) - x^T(t-\eta_1)P_{19}^T\Delta(t)) \\
&- x^T(t-\eta_2)P_{20}^T\Delta(t) - x^T(t-\eta(t))P_{21}^T\Delta(t) - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T\Delta(t) \\
&+ \Delta^T(t)P_{12}^T x(t) + y^T(t)P_{13}^T x(t) + x^T(t)P_{14}^T x(t) + x^T(t-h_1)P_{15}^T x(t) \\
&+ x^T(t-h_2)P_{16}^T x(t) + \dot{x}^T(t)P_{17}^T x(t) + x^T(t-h(t))P_{18}^T x(t) + x^T(t-\eta_1)P_{19}^T x(t) \\
&+ x^T(t-\eta_2)P_{20}^T x(t) + x^T(t-\eta(t))P_{21}^T x(t) + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T x(t) \\
&- \Delta^T(t)P_{12}^T A_2 x(t-\eta(t)) - y^T(t)P_{13}^T A_2 x(t-\eta(t)) \\
&- x^T(t)P_{14}^T A_2 x(t-\eta(t)) - x^T(t-h_1)P_{15}^T A_2 x(t-\eta(t)) \\
&- x^T(t-h_2)P_{16}^T A_2 x(t-\eta(t)) - \dot{x}^T(t)P_{17}^T A_2 x(t-\eta(t)) \\
&- x^T(t-h(t))P_{18}^T A_2 x(t-\eta(t)) - x^T(t-\eta_1)P_{19}^T A_2 x(t-\eta(t)) \\
&- x^T(t-\eta_2)P_{20}^T A_2 x(t-\eta(t)) - x^T(t-\eta(t))P_{21}^T A_2 x(t-\eta(t)) \\
&- (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T A_2 x(t-\eta(t)) + x^T(t)P_{23}^T\dot{x}(t) \\
&- p^T(t)P_{1a}^T y(t) - p^T(t)P_{1a}^T A x(t) + p^T(t)P_{1a}^T W_0 F x(t) \\
&+ p^T(t)P_{1a}^T W_1 G x(t-h(t)) + p^T(t)P_{1a}^T W_2 \int_{t-k(t)}^{t} h(x(s))ds + p^T(t)P_{1a}^T B p(t) \\
&- p^T(t)P_{2a}^T\Delta(t) + p^T(t)P_{2a}^T x(t) - p^T(t)P_{2a}^T A_2 x(t-\eta(t)) \\
&+ \Delta^T(t)P_1^T B p(t) + y^T(t)P_2^T B p(t) + x^T(t)P_3^T B p(t) + x^T(t-h_1)P_4^T B p(t) \\
&+ x^T(t-h_2)P_5^T B p(t) + \dot{x}^T(t)P_6^T B p(t) + x^T(t-h(t))P_7^T B p(t) \\
&+ x^T(t-\eta_1)P_8^T B p(t) + x^T(t-\eta_2)P_9^T B p(t) + x^T(t-\eta(t))P_{10}^T B p(t) \\
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T B p(t)\} \\
&+ x^T(t)Qx(t) - e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1) + x^T(t)Qx(t)
\end{aligned}
$$

$$- e^{-2\alpha h_2}x^T(t-h_2)Qx(t-h_2) + h_1^2\dot{x}^T(t)R\dot{x}(t) - e^{-2\alpha h_1}x^T(t)Rx(t)$$
$$+ 2e^{-2\alpha h_1}x^T(t)Rx(t-h_1) - e^{-2\alpha h_1}x^T(t-h_1)Rx(t-h_1)$$
$$+ h_2^2\dot{x}^T(t)R\dot{x}(t) - e^{-2\alpha h_2}x^T(t)Rx(t) + 2e^{-2\alpha h_2}x^T(t)Rx(t-h_2)$$
$$- e^{-2\alpha h_2}x^T(t-h_2)Rx(t-h_2) + (h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t)$$
$$- e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h(t)) + e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h_2)$$
$$+ e^{-2\alpha h_2}x^T(t-h_2))Ux(t-h(t)) - e^{-2\alpha h_2}x^T(t-h_2)Ux(t-h_2)$$
$$- e^{-2\alpha h_2}x^T(t-h_1)Ux(t-h_1) + e^{-2\alpha h_2}x^T(t-h_1)Ux(t-h(t))$$
$$+ e^{-2\alpha h_2}x^T(t-h(t)))Ux(t-h_1) - e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h(t))$$
$$- \beta[x(t-h_1) - x(t-h(t))]^T e^{-2\alpha h_2}U[x(t-h_1) - x(t-h(t))]$$
$$- (1-\beta)[x(t-h(t)) - x(t-h_2)]^T \times e^{-2\alpha h_2}U[x(t-h(t)) - x(t-h_2)]$$
$$+ x^T(t)Tx(t) - e^{-2\alpha\eta_1}x^T(t-\eta_1)Tx(t-\eta_1) + x^T(t)Tx(t)$$
$$- e^{-2\alpha\eta_2}x^T(t-\eta_2)Tx(t-\eta_2) + \eta_1^2\dot{x}^T(t)M\dot{x}(t) - e^{-2\alpha\eta_1}x^T(t)Mx(t)$$
$$+ 2e^{-2\alpha\eta_1}x^T(t)Mx(t-\eta_1) - e^{-2\alpha\eta_1}x^T(t-\eta_1)Mx(t-\eta_1)$$
$$+ \eta_2^2\dot{x}^T(t)M\dot{x}(t) - e^{-2\alpha\eta_2}x^T(t)Mx(t) + 2e^{-2\alpha\eta_2}x^T(t)Mx(t-\eta_2)$$
$$- e^{-2\alpha\eta_2}x^T(t-\eta_2)Mx(t-\eta_2) + (\eta_2-\eta_1)^2\dot{x}^T(t)N\dot{x}(t)$$
$$- e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta(t)) + e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta_2)$$
$$+ e^{-2\alpha\eta_2}x^T(t-\eta_2))Nx(t-\eta(t)) - e^{-2\alpha\eta_2}x^T(t-\eta_2)Nx(t-\eta_2)$$
$$- e^{-2\alpha\eta_2}x^T(t-\eta_1)Nx(t-\eta_1) + e^{-2\alpha\eta_2}x^T(t-\eta_1)Nx(t-\eta(t))$$
$$+ e^{-2\alpha\eta_2}x^T(t-\eta(t)))Nx(t-\eta_1) - e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta(t))$$
$$- \alpha[x(t-\eta_1) - x(t-\eta(t))]^T e^{-2\alpha\eta_2}N[x(t-\eta_1) - x(t-\eta(t))]$$
$$- (1-\alpha)[x(t-\eta(t)) - x(t-\eta_2)]^T \times e^{-2\alpha\eta_2}N[x(t-\eta(t)) - x(t-\eta_2)]$$
$$+ 2k^2x^T(t)HD_2Hx(t) - 2e^{-2\alpha k}(\int_{t-k(t)}^{t} h(x(s))\,ds)^T D_2 \int_{t-k(t)}^{t} h(x(s))\,ds$$
$$+ 2\alpha x^T(t)P_{23}x(t) - 2\dot{x}^T(t)D\dot{x}(t) - 2\dot{x}^T(t)DAx(t) + 2\dot{x}^T(t)DW_0Fx(t)$$
$$+ 2\dot{x}^T(t)DW_1Gx(t-h(t) + 2\dot{x}^T(t)DW_2\int_{t-k(t)}^{t} h(x(s))ds$$
$$+ \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)I\dot{x}(t) + 2\dot{x}^T(t)Ax(t)$$
$$+ \dot{x}^T(t)W_0W_0^T\dot{x}^T(t) + x^T(t)F^2x(t) + \dot{x}^T(t)W_1W_1^T\dot{x}^T(t)$$
$$+ x^T(t-h(t))G^2x(t-h(t)) + x^T(t)A^TW_0W_0^TAx(t) + x^T(t)F^2x(t)$$
$$+ x^T(t)A^TW_1W_1^TAx(t) + x^T(t-h(t))G^2x(t-h(t))$$

$$
\begin{aligned}
& - 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
& - 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)FW_0^TW_0Fx(t) \\
& + x^T(t)FW_0^TW_1Gx(t-h(t)) + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + x^T(t-h(t))GW_1^TW_0Fx(t) + x^T(t-h(t))GW_1^TW_1Gx(t-h(t)) \\
& + 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds + 2\dot{x}^T(t)DBp(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& - \dot{x}^T(t)Bp(t) - x^T(t)A^TBp(t) + f^T(x(t))W_0^TBp(t) \\
& + g^T(x(t-h(t)))W_1^TBp(t) + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TBp(t) \\
& - p^T(t)B^T\dot{x}(t) - p^T(t)B^TAx(t) + p^T(t)B^TW_0f(x(t)) \\
& + p^T(t)B^TW_1g(x(t-h(t)) + p^T(t)B^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + p^T(t)B^TBp(t) + 2\dot{x}^T(t)DBp(t) \\
& - \epsilon p^T(t)p(t) + \epsilon(E_ax(t) + E_0f(x(t)) + E_1g(x(t-h(t))) + E_2\int_{t-k(t)}^{t} h(x(s))ds)^T \\
& (E_ax(t) + E_0f(x(t)) + E_1g(x(t-h(t))) + E_2\int_{t-k(t)}^{t} h(x(s))ds)\}. \qquad (3.44)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\dot{V}(t,x_t) \leq\ & e^{2\alpha t}\{\zeta^T(t)\mathcal{M}\zeta(t) - \beta[x(t-h_1) - x(t-h(t))]^Te^{-2\alpha h_2} \\
& \times U[x(t-h_1) - x(t-h(t))] \\
& - (1-\beta)[x(t-h(t)) - x(t-h_2)]^T \\
& \times e^{-2\alpha h_2}U[x(t-h(t)) - x(t-h_2)] + \zeta^T(t)\mathcal{M}\zeta(t) \\
& - \alpha[x(t-\eta_1) - x(t-\eta(t))]^Te^{-2\alpha\eta_2}N[x(t-\eta_1) \\
& - x(t-\eta(t))] - (1-\alpha)[x(t-\eta(t)) - x(t-\eta_2)]^T \\
& \times e^{-2\alpha\eta_2}N[x(t-\eta(t)) - x(t-\eta_2)]\} \\
=\ & e^{2\alpha t}\{\zeta^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 + (1-\alpha)\mathcal{M}_3 \\
& + \alpha\mathcal{M}_4]\zeta(t)\}. \qquad (3.45)
\end{aligned}
$$

Since $0 \leq \beta \leq 1$, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2$ is a convex combination of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $0 \leq \alpha \leq 1$, $(1-\alpha)\mathcal{M}_3 + \alpha\mathcal{M}_4$ is a convex combination of $\mathcal{M}_3$ and $\mathcal{M}_4$. Therefore, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ is equivalent to $\mathcal{M}_1 < 0$ and $\mathcal{M}_2 < 0$, and $(1-\alpha)\mathcal{M}_3 + \alpha\mathcal{M}_4 < 0$

is equivalent to $\mathcal{M}_3 < 0$ and $\mathcal{M}_4 < 0$. From $\dot{V}(x(t)) \leq 0$, we get $V(x(t)) \leq V(x(0))$. Note that,

$$
\begin{aligned}
V_1(x(0)) &= \zeta^T(0)E^T P\zeta(0) \leq \lambda_{\max}(P_{23})\|\phi\|^2, \\
V_2(x(0)) &= \int_{-h_1}^{0} e^{2\alpha s} x^T(s)Qx(s)\,ds \leq \lambda_{\max}(Q)\|M_1\|^2 \int_{-h_1}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(Q)\|M_1\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_3(x(0)) &= \int_{-h_2}^{0} e^{2\alpha s} x^T(s)Qx(s)\,ds \leq \lambda_{\max}(Q)\|M_1\|^2 \int_{-h_2}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(Q)\|M_1\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_4(x(0)) &= h_1 \int_{-h_1}^{0}\int_{s}^{0} e^{2\alpha\tau} \dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(R)\|\phi\|^2 \int_{-h_1}^{0} e^{2\alpha\tau} d\tau \\
&= h_2^2\lambda_{\max}(R)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_5(x(0)) &= h_2 \int_{-h_2}^{0}\int_{s}^{0} e^{2\alpha\tau} \dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(R)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_6(x(0)) &= h_2 \int_{-h_2}^{-h_1}\int_{s}^{0} e^{2\alpha\tau} \dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(U)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_7(x(0)) &= \int_{-\eta_1}^{0} e^{2\alpha s} x^T(s)Tx(s)\,ds \leq \lambda_{\max}(T)\|M_1\|^2 \int_{-\eta_1}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(T)\|M_1\|^2 \frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_8(x(0)) &= \int_{-\eta_2}^{0} e^{2\alpha s} x^T(s)Tx(s)\,ds \leq \lambda_{\max}(T)\|M_1\|^2 \int_{-\eta_2}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(T)\|M_1\|^2 \frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_9(x(0)) &= \eta_1 \int_{-\eta_1}^{0}\int_{s}^{0} e^{2\alpha\tau} \dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds \\
&\leq \eta_2^2\lambda_{\max}(M)\|\phi\|^2 \int_{-\eta_1}^{0} e^{2\alpha\tau} d\tau \\
&= \eta_2^2\lambda_{\max}(M)\|\phi\|^2 \frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_{10}(x(0)) &= \eta_2 \int_{-\eta_2}^{0}\int_{s}^{0} e^{2\alpha\tau} \dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds \\
&\leq \eta_2^2\lambda_{\max}(M)\|\phi\|^2 \frac{1-e^{-2\alpha\eta_2}}{2\alpha},
\end{aligned}
$$

$$
\begin{aligned}
V_{11}(x(0)) &= \eta_2 \int_{-\eta_2}^{-\eta_1} \int_s^0 e^{2\alpha\tau} \dot{x}^T(\tau) N \dot{x}(\tau)\, d\tau\, ds \\
&\leq \eta_2^2 \lambda_{\max}(N) \|\phi\|^2 \frac{1 - e^{-2\alpha\eta_2}}{2\alpha}, \\
V_{12}(x(0)) &= k \int_{-k}^{0} \int_s^0 e^{2\alpha\tau} h^T(x(\tau)) D_2 h(x(\tau)) d\tau\, ds \\
&\leq k^2 \lambda_{\max}(HD_2H) \|\phi\|^2 \int_{-k}^{0} e^{2\alpha\tau} d\tau \\
&= k^2 \lambda_{\max}(HD_2H) \|\phi\|^2 \frac{1 - e^{-2\alpha k}}{2\alpha}.
\end{aligned}
$$

We have $e^{2\alpha t}\lambda_{\min}(P_{23})\|x(t)\|^2 \leq V(t, x_t) \leq V(x(0)),\ \forall t \geq 0$. Then $\|x(t)\| \leq \sqrt{\frac{a\|\phi\|^2 + b\|M_1\|^2}{\lambda_{\min}(P_{23})}}.e^{-\alpha t}$, where $a = \lambda_{\max}(P_{23}) + 2h_2^2\lambda_{\max}(R)\frac{1-e^{-2\alpha h_2}}{2\alpha} + h_2^2\lambda_{\max}(U)\frac{1-e^{-2\alpha h_2}}{2\alpha} + 2\eta_2^2\lambda_{\max}(M)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + \eta_2^2\lambda_{\max}(N)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + k^2\lambda_{\max}(HD_2H)\frac{1-e^{-2\alpha k}}{2\alpha}$, $b = 2\lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha} + 2\lambda_{\max}(T)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}$ and $\|M_1\| = \sup_{-d \leq s \leq 0}\|x(s)\|$. From Definition 2.3.8, we conclude that the equilibrium point is $\alpha-$exponentially stable. This completes the proof.

Remark 3.1.1 *For the case when $h_1 = \eta_1 = 0$, we have the following corollary for delay-dependent stability of system (3.5).*

Corollary 3.1.1 *Given $\alpha > 0$. The system (3.5) is $\alpha-$exponentially stable if there exist $\epsilon > 0$, symmetric positive definite matrices $P_{23}, S, D, Q, U, T, N, D_2$ and matrices $P_{1a}, P_{2a}, P_i, i = 1, 2, ..., 22$, $i \neq 4, 8, 15, 19$ of appropriate dimension such that the following LMIs hold :*

$$
\begin{aligned}
\mathcal{M}_1 = &\mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
\mathcal{M}_2 = &\mathcal{M} - \begin{bmatrix} 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
\mathcal{M}_3 = &\mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
\mathcal{M}_4 = &\mathcal{M} - \begin{bmatrix} 0 & 0 & -I & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & -I & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.49}
$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{15} & \phi_{16} & \phi_{17} & \phi_{19} & \phi_{1a} & \phi_{1b} & \phi_{1c} \\ * & \phi_{22} & \phi_{23} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{29} & \phi_{2a} & \phi_{2b} & \phi_{2c} \\ * & * & \omega_{33} & \omega_{35} & \phi_{36} & \omega_{37} & \omega_{39} & \omega_{3a} & \phi_{3b} & \phi_{3c} \\ * & * & * & \omega_{55} & 0 & \phi_{57} & 0 & \phi_{5a} & \phi_{5b} & 0 \\ * & * & * & * & \omega_{66} & \phi_{67} & 0 & \phi_{6a} & \phi_{6b} & \phi_{6c} \\ * & * & * & * & * & \phi_{77} & \phi_{79} & \phi_{7a} & \phi_{7b} & \phi_{7c} \\ * & * & * & * & * & * & \omega_{99} & \phi_{9a} & \phi_{9b} & 0 \\ * & * & * & * & * & * & * & \phi_{10a} & \phi_{10b} & \phi_{10c} \\ * & * & * & * & * & * & * & * & \phi_{11b} & \phi_{11c} \\ * & * & * & * & * & * & * & * & * & \phi_{12c} \end{bmatrix},$$

where

$$\begin{aligned}
\phi_{11} &= -P_{12}^T - P_{12}, \\
\phi_{12} &= -P_1^T - P_{13}, \\
\phi_{13} &= -P_1^T A + P_1^T W_0 F + P_{12}^T - P_{14}, \\
\phi_{14} &= -P_{15}, \\
\phi_{15} &= -P_{16}, \\
\phi_{16} &= -P_{17}, \\
\phi_{17} &= P_1^T W_1 G - P_{18}, \\
\phi_{18} &= -P_{19}, \\
\phi_{19} &= -P_{20}, \\
\phi_{1a} &= -P_{12}^T A_2 - P_{21}, \\
\phi_{1b} &= P_1^T W_2 - P_{22}, \\
\phi_{1c} &= -P_{2a}^T + P_1^T B, \\
\phi_{22} &= -P_2^T - P_2, \\
\phi_{23} &= -P_2^T A + P_2^T W_0 F + P_{13}^T - P_3, \\
\phi_{24} &= -P_4, \\
\phi_{25} &= -P_5, \\
\phi_{26} &= -P_6, \\
\phi_{27} &= P_2^T W_1 G - P_7, \\
\phi_{28} &= -P_8, \\
\phi_{29} &= -P_9,
\end{aligned}$$

$$
\begin{aligned}
\phi_{2a} &= -P_{13}^T A_2 - P_{10}, \\
\phi_{2b} &= P_2^T W_2 - P_{11}, \\
\phi_{2c} &= -P_{1a} + P_2^T B, \\
\phi_{33} &= -A^T P_3 - P_3^T A + F^T W_0^T P_3 + P_3^T W_0 F + P_{14}^T + P_{14} + \alpha P_{23}^T + \alpha P_{23} \\
&\quad + Q + Q^T - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + T + T^T - e^{-2\alpha \eta_1} M - e^{-2\alpha \eta_2} M \\
&\quad + K^2 H D_2 H + K^2 H^T D_2^T H^T + A^T A + F W_0^T W_0 F + 2F^2 + A^T W_0 W_0^T A \\
&\quad + A^T W_1 W_1^T A + \epsilon E_a^T E_a + \epsilon E_a^T E_0 F + \epsilon F^T E_0^T E_a + \epsilon F^T E_0^T E_0 F, \\
\phi_{34} &= -A^T P_4 + F^T W_0^T P_4 + P_{15} + e^{-2\alpha h_1} R, \\
\phi_{35} &= -A^T P_5 + F^T W_0^T P_5 + P_{16} + e^{-2\alpha h_2} R, \\
\phi_{36} &= P_{23}^T - A^T P_6 + F^T W_0^T P_6 + P_{17} + A^T - A^T D^T + F^T W_0^T D^T, \\
\phi_{37} &= P_3^T W_1 G - A^T P_7 + F^T W_0^T P_7 + P_{18} + F^T W_0^T W_1 G + \epsilon E_a^T E_1 G, \\
&\quad + \epsilon F^T E_0^T E_1 G, \\
\phi_{38} &= -A^T P_8 + F^T W_0^T P_8 + P_{19} + e^{-2\alpha \eta_1} M, \\
\phi_{39} &= -A^T P_9 + F^T W_0^T P_9 + P_{20} + e^{-2\alpha \eta_2} M, \\
\phi_{3a} &= -P_{14}^T A_2 - A^T P_{10} + F^T W_0^T P_{10} + P_{21}, \\
\phi_{3b} &= P_3^T W_2 - A^T P_{11} + F^T W_0^T P_{11} + P_{22} - A^T W_2 + F W_0^T W_2 + \epsilon E_a^T E_2 \\
&\quad + \epsilon F^T E_0^T E_2, \\
\phi_{3c} &= -A^T P_{1a} + F^T W_0^T P_{1a} + P_{2a} + P_3^T B, \\
\phi_{55} &= -e^{-2\alpha h_2} Q - e^{-2\alpha h_2} R - e^{-2\alpha h_2} U, \\
\phi_{57} &= P_5^T W_1 G + e^{-2\alpha h_2} U, \\
\phi_{5a} &= -P_{16}^T A_2, \\
\phi_{5b} &= P_5^T W_2, \\
\phi_{5c} &= P_5^T B, \\
\phi_{66} &= h_1^2 R + h_2^2 R + (h_1 - h_2)^2 U + \eta_1^2 M + \eta_2^2 M + (\eta_1 - \eta_2)^2 N \\
&\quad - 2D + I + D^2 + W_0 W_0^T + W_1 W_1^T, \\
\phi_{67} &= P_6^T W_1 G + D W_1 G, \\
\phi_{6a} &= -P_{17}^T A_2, \\
\phi_{6b} &= P_6^T W_2 - W_2 + D W_2, \\
\phi_{6c} &= DB + P_6^T B, \\
\phi_{77} &= P_7^T W_1 G + G^T W_1^T P_7 - e^{-2\alpha h_2} U - e^{-2\alpha h_2} U^T + G W_1^T W_1 G + \epsilon G^T E_1^T E_1 G
\end{aligned}
$$

$$
\begin{aligned}
& +2G^2, \\
\phi_{78} &= G^T W_1^T P_8, \\
\phi_{79} &= G^T W_1^T P_9, \\
\phi_{7a} &= -P_{18}^T A_2 + G^T W_1^T P_{10}, \\
\phi_{7b} &= P_7^T W_2 + G^T W_1^T P_{11} + G W_1^T W_2 + \epsilon G^T E_1^T E_2, \\
\phi_{7c} &= G^T W_1^T P_{1a} + P_7^T B, \\
\phi_{99} &= -e^{-2\alpha\eta_2} T - e^{-2\alpha\eta_2} M - e^{-2\alpha\eta_2} N, \\
\phi_{9a} &= -P_{20}^T A_2 + e^{-2\alpha\eta_2} N, \\
\phi_{9b} &= P_9^T W_2, \\
\phi_{9c} &= P_9^T B, \\
\phi_{10a} &= -P_{21}^T A_2 - A_2^T P_{21} - e^{-2\alpha\eta_2} N - e^{-2\alpha\eta_2} N^T, \\
\phi_{10b} &= P_{10}^T W_2 - A_2^T P_{22}, \\
\phi_{10c} &= -A_2^T P_{2a} + P_{10}^T B, \\
\phi_{11b} &= P_{11}^T W_2 + W_2^T P_{11} - e^{-2\alpha k} D_2 - e^{-2\alpha k} D_2^T + W_2^T W_2 + \epsilon E_2^T E_2, \\
\phi_{11c} &= W_2^T P_{1a} + P_{11}^T B, \\
\phi_{12c} &= B^T P_{1a} - \epsilon I.
\end{aligned}
$$

Proof. Choose a Lyapunov-Krasovskii functional candidate as

$$
V(t, x_t) = \sum_{i=1}^{6} V_i, \tag{3.50}
$$

where

$$
\begin{aligned}
V_1 &= e^{2\alpha t} \zeta^T(t) E^T P \zeta(t), \\
V_2 &= \int_{t-h_2}^{t} e^{2\alpha s} x^T(s) Q x(s)\, ds, \\
V_3 &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha\tau} \dot{x}^T(\tau) U \dot{x}(\tau)\, d\tau\, ds, \\
V_4 &= \int_{t-\eta_2}^{t} e^{2\alpha s} x^T(s) T x(s)\, ds, \\
V_5 &= \eta_2 \int_{-\eta_2}^{0} \int_{t+s}^{t} e^{2\alpha\tau} \dot{x}^T(\tau) N \dot{x}(\tau)\, d\tau\, ds, \\
V_6 &= k \int_{-k}^{t} \int_{t+s}^{t} e^{2\alpha\tau} h^T(x(\tau)) D_2 h(x(\tau))\, d\tau\, ds,
\end{aligned}
$$

with $\zeta(t) = [\Delta^T(t) \;\; y^T(t) \;\; x^T(t) \;\; x^T(t-h_2) \;\; \dot{x}^T(t) \;\; x^T(t-h(t)) \;\; x^T(t-\eta_2)$

$x^T(t-\eta(t))\ \ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T\ \ p^T(t)]^T$, and

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 & P_5 & P_6 & P_7 & P_9 & P_{10} & P_{11} & P_{1a} \\ P_{12} & P_{13} & P_{14} & P_{16} & P_{17} & P_{18} & P_{20} & P_{21} & P_{22} & P_{2a} \\ 0 & 0 & P_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the similar method as in Theorem 3.1.1, we have

$$\begin{aligned} \dot{V}(t,x_t) \ &\leq\ e^{2\alpha t}\{\zeta^T(t)\mathcal{M}\zeta(t) - \beta[x(t)-x(t-h(t))]^T e^{-2\alpha h_2} \\ &\quad \times U[x(t)-x(t-h(t))] \\ &\quad -(1-\beta)[x(t-h(t))-x(t-h_2)]^T \\ &\quad \times e^{-2\alpha h_2}U[x(t-h(t))-x(t-h_2)] + \zeta^T(t)\mathcal{M}\zeta(t) \\ &\quad -\alpha[x(t)-x(t-\eta(t))]^T e^{-2\alpha\eta_2}N[x(t)-x(t-\eta(t))] \\ &\quad -(1-\alpha)[x(t-\eta(t))-x(t-\eta_2)]^T \\ &\quad \times e^{-2\alpha\eta_2}N[x(t-\eta(t))-x(t-\eta_2)]\}. \end{aligned}$$

remaining part of the proof immediately follows from Theorem 3.1.1. This completes the proof. □

3.2 Exponential stability of neutral-type neural networks with interval non-differentiable and distributed time-varying delays

Throughout this section, we assume that all eigenvalues of matrix A_2 are inside the unit circle. Let $B=0$, from system (3.5) we have the following system:

$$\begin{aligned} y(t) &= \dot{x}(t) - A_2\dot{x}(t-\eta(t)) \\ y(t) &= -Ax(t) + W_0 f(x(t) + W_1 g(x(t-h(t)) + W_2\int_{t-k(t)}^{t} h(x(s))ds. \end{aligned} \tag{3.51}$$

Theorem 3.2.1 *Given $\alpha > 0$. The system (3.51) is $\alpha-$exponentially stable if there exist symmetric positive definite matrices $P_{23}, S, D, Q, R, U, T, M, N, D_2$ and matrices $P_i, i = 1,2,...,22$ of appropriate dimension such that the following LMIs hold :*

$$
\begin{aligned}
\mathcal{M}_1 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
\mathcal{M}_2 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.53}
$$

$$
\begin{aligned}
\mathcal{M}_3 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha \eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.54}
$$

$$
\begin{aligned}
\mathcal{M}_4 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & I & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha \eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & I & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.55}
$$

$$
\mathcal{M} = \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} & \phi_{17} & \phi_{18} & \phi_{19} & \phi_{1a} & \phi_{1b} \\
* & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{28} & \phi_{29} & \phi_{2a} & \phi_{2b} \\
* & * & \phi_{33} & \phi_{34} & \phi_{35} & \phi_{36} & \phi_{37} & \phi_{38} & \phi_{39} & \phi_{3a} & \phi_{3b} \\
* & * & * & \phi_{44} & 0 & 0 & \phi_{47} & 0 & 0 & \phi_{4a} & \phi_{4b} \\
* & * & * & * & \phi_{55} & 0 & \phi_{57} & 0 & 0 & \phi_{5a} & \phi_{5b} \\
* & * & * & * & * & \phi_{66} & \phi_{67} & 0 & 0 & \phi_{6a} & \phi_{6b} \\
* & * & * & * & * & * & \phi_{77} & \phi_{78} & \phi_{79} & \phi_{7a} & \phi_{7b} \\
* & * & * & * & * & * & * & \phi_{88} & 0 & \phi_{8a} & \phi_{8b} \\
* & * & * & * & * & * & * & * & \phi_{99} & \phi_{9a} & \phi_{9b} \\
* & * & * & * & * & * & * & * & * & \phi_{10a} & \phi_{10b} \\
* & * & * & * & * & * & * & * & * & * & \phi_{11b}
\end{bmatrix},
$$

where

$$
\begin{aligned}
\phi_{11} &= -P_{12}^T - P_{12}, \\
\phi_{12} &= -P_1^T - P_{13}, \\
\phi_{13} &= -P_1^T A + P_1^T W_0 F + P_{12}^T - P_{14}, \\
\phi_{14} &= -P_{15}, \\
\phi_{15} &= -P_{16}, \\
\phi_{16} &= -P_{17},
\end{aligned}
$$

$$
\begin{aligned}
\phi_{17} &= P_1^T W_1 G - P_{18}, \\
\phi_{18} &= -P_{19}, \\
\phi_{19} &= -P_{20}, \\
\phi_{1a} &= -P_{12}^T A_2 - P_{21}, \\
\phi_{1b} &= P_1^T W_2 - P_{22}, \\
\phi_{22} &= -P_2^T - P_2, \\
\phi_{23} &= -P_2^T A + P_2^T W_0 F + P_{13}^T - P_3, \\
\phi_{24} &= -P_4, \\
\phi_{25} &= -P_5, \\
\phi_{26} &= -P_6, \\
\phi_{27} &= P_2^T W_1 G - P_7, \\
\phi_{28} &= -P_8, \\
\phi_{29} &= -P_9, \\
\phi_{2a} &= -P_{13}^T A_2 - P_{10}, \\
\phi_{2b} &= P_2^T W_2 - P_{11}, \\
\phi_{33} &= -A^T P_3 - P_3^T A + F^T W_0^T P_3 + P_3^T W_0 F + P_{14}^T + P_{14} + \alpha P_{23}^T + \alpha P_{23} \\
&\quad + Q + Q^T - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + T + T^T - e^{-2\alpha \eta_1} M - e^{-2\alpha \eta_2} M \\
&\quad + K^2 H D_2 H + K^2 H^T D_2^T H^T + A^T A + F W_0^T W_0 F \\
&\quad + 2F^2 + A^T W_0 W_0^T A + A^T W_1 W_1^T A, \\
\phi_{34} &= -A^T P_4 + F^T W_0^T P_4 + P_{15} + e^{-2\alpha h_1} R, \\
\phi_{35} &= -A^T P_5 + F^T W_0^T P_5 + P_{16} + e^{-2\alpha h_2} R, \\
\phi_{36} &= P_{23}^T - A^T P_6 + F^T W_0^T P_6 + P_{17} + A^T - A^T D^T + F^T W_0^T D^T, \\
\phi_{37} &= P_3^T W_1 G - A^T P_7 + F^T W_0^T P_7 + P_{18} + F^T W_0^T W_1 G, \\
\phi_{38} &= -A^T P_8 + F^T W_0^T P_8 + P_{19} + e^{-2\alpha \eta_1} M, \\
\phi_{39} &= -A^T P_9 + F^T W_0^T P_9 + P_{20} + e^{-2\alpha \eta_2} M, \\
\phi_{3a} &= -P_{14}^T A_2 - A^T P_{10} + F^T W_0^T P_{10} + P_{21}, \\
\phi_{3b} &= P_3^T W_2 - A^T P_{11} + F^T W_0^T P_{11} + P_{22} - A^T W_2 + F W_0^T W_2, \\
\phi_{44} &= -e^{-2\alpha h_1} Q - e^{-2\alpha h_1} R - e^{-2\alpha h_2} U, \\
\phi_{47} &= P_4^T W_1 G + e^{-2\alpha h_2} U, \\
\phi_{4a} &= -P_{15}^T A_2,
\end{aligned}
$$

$$\begin{aligned}
\phi_{4b} &= P_4^T W_2, \\
\phi_{55} &= -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \\
\phi_{57} &= P_5^T W_1 G + e^{-2\alpha h_2}U, \\
\phi_{5a} &= -P_{16}^T A_2, \\
\phi_{5b} &= P_5^T W_2, \\
\phi_{66} &= h_1^2 R + h_2^2 R + (h_1 - h_2)^2 U + \eta_1^2 M + \eta_2^2 M + (\eta_1 - \eta_2)^2 N - 2D + I + D^2 \\
&\quad + W_0 W_0^T + W_1 W_1^T, \\
\phi_{67} &= P_6^T W_1 G + D W_1 G, \\
\phi_{6a} &= -P_{17}^T A_2, \\
\phi_{6b} &= P_6^T W_2 - W_2 + D W_2, \\
\phi_{77} &= P_7^T W_1 G + G^T W_1^T P_7 - e^{-2\alpha h_2}U - e^{-2\alpha h_2}U^T + G W_1^T W_1 G + 2G^2, \\
\phi_{78} &= G^T W_1^T P_8, \\
\phi_{79} &= G^T W_1^T P_9, \\
\phi_{7a} &= -P_{18}^T A_2 + G^T W_1^T P_{10}, \\
\phi_{7b} &= P_7^T W_2 + G^T W_1^T P_{11} + G W_1^T W_2, \\
\phi_{88} &= -e^{-2\alpha\eta_1}T - e^{-2\alpha\eta_1}M - e^{-2\alpha\eta_2}N, \\
\phi_{8a} &= -P_{19}^T A_2 + e^{-2\alpha\eta_2}N, \\
\phi_{8b} &= P_8^T W_2, \\
\phi_{99} &= -e^{-2\alpha\eta_2}T - e^{-2\alpha\eta_2}M - e^{-2\alpha\eta_2}N, \\
\phi_{9a} &= -P_{20}^T A_2 + e^{-2\alpha\eta_2}N, \\
\phi_{9b} &= P_9^T W_2, \\
\phi_{10a} &= -P_{21}^T A_2 - A_2^T P_{21} - e^{-2\alpha\eta_2}N - e^{-2\alpha\eta_2}N^T, \\
\phi_{10b} &= P_{10}^T W_2 - A_2^T P_{22}, \\
\phi_{11b} &= P_{11}^T W_2 + W_2^T P_{11} - e^{-2\alpha k}D_2 - e^{-2\alpha k}D_2^T + W_2^T W_2.
\end{aligned}$$

The solution $x(t)$ of the system satisfies,

$$\|x(t)\| \leq \sqrt{\frac{a\|\phi\|^2 + b\|M_1\|^2}{\lambda_{\min}(P_{23})}}.e^{-\alpha t}, \tag{3.56}$$

where $a = \lambda_{\max}(P_{23}) + 2h_2^2\lambda_{\max}(R)\frac{1-e^{-2\alpha h_2}}{2\alpha} + h_2^2\lambda_{\max}(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}$
$+ 2\eta_2^2\lambda_{\max}(M)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + \eta_2^2\lambda_{\max}(N)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + k^2\lambda_{\max}(HD_2H)\frac{1-e^{-2\alpha k}}{2\alpha}$,
$b = 2\lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha} + 2\lambda_{\max}(T)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}$ and $\|M_1\| = \sup_{-d\leq s\leq 0}\|x(s)\|$.

Proof. We consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{12} V_i, \tag{3.57}$$

where

$$\begin{aligned}
V_1 &= e^{2\alpha t}\zeta^T(t)E^T P\zeta(t),\\
V_2 &= \int_{t-h_1}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_3 &= \int_{t-h_2}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_4 &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_5 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_6 &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds,\\
V_7 &= \int_{t-\eta_1}^{t} e^{2\alpha s}x^T(s)Tx(s)\,ds,\\
V_8 &= \int_{t-\eta_2}^{t} e^{2\alpha s}x^T(s)Tx(s)\,ds,\\
V_9 &= \eta_1\int_{-\eta_1}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds,\\
V_{10} &= \eta_2\int_{-\eta_2}^{0}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds,\\
V_{11} &= (\eta_2-\eta_1)\int_{-\eta_2}^{-\eta_1}\int_{t+s}^{t} e^{2\alpha\tau}\dot{x}^T(\tau)N\dot{x}(\tau)\,d\tau\,ds,\\
V_{12} &= k\int_{-k}^{t}\int_{t+s}^{t} e^{2\alpha\tau}h^T(x(\tau))D_2h(x(\tau))\,d\tau\,ds,
\end{aligned}$$

where $\zeta(t) = [\Delta^T(t)\ \ y^T(t)\ \ x^T(t)\ \ x^T(t-h_1)\ \ x^T(t-h_2)\ \ \dot{x}^T(t)\ \ x^T(t-h(t))$ $x^T(t-\eta_1)\ \ x^T(t-\eta_2)\ \ x^T(t-\eta(t))\ \ \ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T]^T$,
$\Delta(t) = x(t) - A_2x(t-\eta(t))$ and

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11}\\ P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} & P_{20} & P_{21} & P_{22}\\ 0 & 0 & P_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking the derivative of $V(x_t)$ along any trajectory of solution of system (3.51), we have

$$\begin{aligned}\dot{V}_1 &= 2\alpha e^{2\alpha t}x^T(t)P_{23}x(t) + 2e^{2\alpha t}x^T(t)P_{23}\dot{x}(t)\\ &= 2\alpha e^{2\alpha t}x^T(t)P_{23}x(t) + 2e^{2\alpha t}\zeta^T(t)P^T\begin{bmatrix}0\\0\\\dot{x}(t)\end{bmatrix}.\end{aligned}$$

From (3.51), we have

$$\begin{aligned}\dot{V}_1 = \quad & 2\alpha e^{2\alpha t}x^T(t)P_{23}x(t) + 2e^{2\alpha t}\zeta^T(t)P^T\\ & \times\begin{bmatrix}-y(t) - Ax(t) + W_0f(x(t) + W_1g(x(t-h(t)))\\ +W_2\int_{t-k_t}^{t}h(x(s))ds,\\ -\Delta(t) + x(t) - A_2x(t-\eta(t)),\\ \dot{x}(t)\end{bmatrix}.\end{aligned}$$

Using condition (3.3) we have,

$$\begin{aligned}\dot{V}_1 \leq \quad & 2e^{2\alpha t}\{\alpha x^T(t)P_{23}x(t) - \Delta^T(t)P_1^Ty(t) - y^T(t)P_2^Ty(t) - x^T(t)P_3^Ty(t)\\
& - x^T(t-h_2)P_5^Ty(t) - \dot{x}^T(t)P_6^Ty(t) - x^T(t-h(t))P_7^Ty(t)\\
& - x^T(t-\eta_1)P_8^Ty(t) - x^T(t-\eta_2)P_9^Ty(t) - x^T(t-\eta(t))P_{10}^Ty(t)\\
& - (\int_{t-k(t)}^{t}h(x(s))\,ds)^TP_{11}^Ty(t) - \Delta^T(t)P_1^TAx(t) - y^T(t)P_2^TAx(t)\\
& - x^T(t)P_3^TAx(t)) - x^T(t-h_1)P_4^TAx(t) - x^T(t-h_2)P_5^TAx(t)\\
& - \dot{x}^T(t)P_6^TAx(t) - x^T(t-h(t))P_7^TAx(t)) - x^T(t-h_1)P_4^Ty(t)\\
& - x^T(t-\eta_1)P_8^TAx(t) - x^T(t-\eta_2)P_9^TAx(t) - x^T(t-\eta(t))P_{10}^TAx(t)\\
& - (\int_{t-k(t)}^{t}h(x(s))\,ds)^TP_{11}^TAx(t) + \Delta^T(t)P_1^TW_0Fx(t) + y^T(t)P_2^TW_0Fx(t)\\
& + x^T(t)P_3^TW_0Fx(t) + x^T(t-h_1)P_4^TW_0Fx(t) + x^T(t-h_2)P_5^TW_0Fx(t)\\
& + \dot{x}^T(t)P_6^TW_0Fx(t) + x^T(t-h(t))P_7^TW_0Fx(t) + x^T(t) + \eta_1)P_8^TW_0Fx(t)\\
& + x^T(t-\eta_2)P_9^TW_0Fx(t) + x^T(t-\eta(t))P_{10}^TW_0Fx(t)\\
& + (\int_{t-k(t)}^{t}h(x(s))\,ds)^TP_{11}^TW_0Fx(t) + \Delta^T(t)P_1^TW_1Gx(t-h(t))\\
& + y^T(t)P_2^TW_1Gx(t-h(t)) + x^T(t)P_3^TW_1Gx(t-h(t))\\
& + x^T(t-h_1)P_4^TW_1Gx(t-h(t)) + x^T(t-h_2)P_5^TW_1Gx(t-h(t))\\
& + \dot{x}^T(t)P_6^TW_1Gx(t-h(t)) + x^T(t-h(t))P_7^TW_1Gx(t-h(t))\\
& + x^T(t-\eta_1)P_8^TW_1Gx(t-h(t)) + x^T(t-\eta_2)P_9^TW_1Gx(t-h(t))\\
& + x^T(t-\eta_2)P_9^TW_1Gx(t-h(t)) + x^T(t-\eta(t))P_{10}^TW_1Gx(t-h(t))\end{aligned}$$

$$
\begin{aligned}
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_1 G x(t-h(t)) + \Delta^T(t) P_1^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ y^T(t) P_2^T W_2 \int_{t-k(t)}^{t} h(x(s)) + x^T(t) P_3^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds\,ds \\
&+ x^T(t-h_1) P_4^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h_2) P_5^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ \dot{x}^T(t) P_6^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h(t)) P_7^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-\eta_1) P_8^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta_2) P_9^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-\eta(t)) P_{10}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds - \Delta^T(t) P_{12}^T \Delta(t) \\
&- y^T(t) P_{13}^T \Delta(t) - x^T(t) P_{14}^T \Delta(t) - x^T(t-h_1) P_{15}^T \Delta(t) \\
&- x^T(t-h_2) P_{16}^T \Delta(t) - \dot{x}^T(t) P_{17}^T \Delta(t) - x^T(t-h(t)) P_{18}^T \Delta(t) \\
&- x^T(t-\eta_1) P_{19}^T \Delta(t)) - x^T(t-\eta_2) P_{20}^T \Delta(t) - x^T(t-\eta(t)) P_{21}^T \Delta(t) \\
&- (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T \Delta(t) + \Delta^T(t) P_{12}^T x(t) + y^T(t) P_{13}^T x(t) \\
&+ x^T(t) P_{14}^T x(t) + x^T(t-h_1) P_{15}^T x(t) + x^T(t-h_2) P_{16}^T x(t) \\
&+ \dot{x}^T(t) P_{17}^T x(t) + x^T(t-h(t)) P_{18}^T x(t) + x^T(t) - \eta_1) P_{19}^T x(t) \\
&+ x^T(t-\eta_2) P_{20}^T x(t) + x^T(t-\eta(t)) P_{21}^T x(t) \\
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T x(t) - \Delta^T(t) P_{12}^T A_2 x(t-\eta(t)) \\
&- y^T(t) P_{13}^T A_2 x(t-\eta(t)) - x^T(t) P_{14}^T A_2 x(t-\eta(t)) \\
&- x^T(t-h_1) P_{15}^T A_2 x(t-\eta(t)) - x^T(t-h_2) P_{16}^T A_2 x(t-\eta(t)) \\
&- \dot{x}^T(t) P_{17}^T A_2 x(t-\eta(t)) - x^T(t-h(t)) P_{18}^T A_2 x(t-\eta(t)) - x^T(t) \\
&- \eta_1) P_{19}^T A_2 x(t-\eta(t)) - x^T(t-\eta_2) P_{20}^T A_2 x(t-\eta(t)) - x^T(t \\
&- \eta(t)) P_{21}^T A_2 x(t-\eta(t)) - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T A_2 x(t-\eta(t)) \\
&+ x^T(t) P_{23} \dot{x}(t),
\end{aligned} \tag{3.58}
$$

$$
\dot{V}_2 = e^{2\alpha t} x^T(t) Q x(t) - e^{-2\alpha(t-h_1)} x^T(t-h_1) Q x(t-h_1), \tag{3.59}
$$

$$
\dot{V}_3 = e^{2\alpha t} x^T(t) Q x(t) - e^{-2\alpha(t-h_2)} x^T(t-h_2) Q x(t-h_2), \tag{3.60}
$$

$$
\dot{V}_4 \le e^{2\alpha t}[h_1^2 \dot{x}^T(t) R \dot{x}(t) - h_1 \int_{t-h_1}^{t} e^{2\alpha s} \dot{x}^T(s) R \dot{x}(s) ds], \tag{3.61}
$$

$$\dot{V}_5 \leq e^{2\alpha t}[h_2^2 \dot{x}^T(t)R\dot{x}(t) - h_2 \int_{t-h_2}^{t} e^{2\alpha s}\dot{x}^T(s)R\dot{x}(s)ds], \tag{3.62}$$

$$\dot{V}_6 \leq e^{2\alpha t}[(h_2 - h_1)^2 \dot{x}^T(t)U\dot{x}(t) - (h_2 - h_1) \int_{t-h_2}^{t-h_1} e^{2\alpha s}\dot{x}^T(s)U\dot{x}(s)ds]. \tag{3.63}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned} -h_1 \int_{t-h_1}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq & - \Big[\int_{t-h_1}^{t} \dot{x}(s)\Big]^T R \Big[\int_{t-h_1}^{t} \dot{x}(s)\Big] \\ \leq & - [x(t) - x(t-h_1)]^T R[x(t) - x(t-h_1)] \\ = & - x^T(t)Rx(t) + 2x^T(t)Rx(t-h_1) \\ & - x^T(t-h_1)Rx(t-h_1). \end{aligned} \tag{3.64}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned} -h_2 \int_{t-h_2}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq & - \Big[\int_{t-h_2}^{t} \dot{x}(s)\Big]^T R \Big[\int_{t-h_2}^{t} \dot{x}(s)\Big] \\ \leq & - [x(t) - x(t-h_2)]^T R[x(t) - x(t-h_2)] \\ = & - x^T(t)Rx(t) + 2x^T(t)Rx(t-h_2) \\ & - x^T(t-h_2)Rx(t-h_2). \end{aligned} \tag{3.65}$$

Note that

$$\begin{aligned} -(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds = & - (h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds \\ & - (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \\ = & - (h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds \\ & - (h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds \\ & - (h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \\ & - (h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds. \end{aligned} \tag{3.66}$$

Using Lemma 2.3.3 gives

$$\begin{aligned} -(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds \leq & - \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big]^T U \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big] \\ \leq & - [x(t-h(t)) - x(t-h_2)]^T U[x(t-h(t)) \\ & - x(t-h_2)], \end{aligned} \tag{3.67}$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds \leq &-\Big[\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds\Big]^T U\Big[\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds\Big] \\
\leq &-[x(t-h_1)-x(t-h(t))]^T U[x(t-h_1) \\
&-x(t-h(t))]. \qquad (3.68)
\end{aligned}
$$

Since $0 \leq h_1 \leq h(t) \leq h_2$ we get $\beta = \frac{h_2-h(t)}{h_2-h_1} \leq 1$. Then

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds = &-\beta\int_{t-h(t)}^{t-h_1}(h_2-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
\leq &-\beta\int_{t-h(t)}^{t-h_1}(h(t)-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
\leq &-\beta[x(t-h_1)-x(t-h(t))]^T U[x(t-h_1) \\
&-x(t-h(t))], \qquad (3.69)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds = &-(1-\beta)\int_{t-h_2}^{t-h(t)}(h_2-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
\leq &-(1-\beta)\int_{t-h_2}^{t-h(t)}(h_2-h(t))\dot{x}^T(s)U\dot{x}(s)\,ds \\
\leq &-(1-\beta)[x(t-h(t))-x(t-h_2)]^T \\
&\times U[x(t-h(t))-x(t-h_2)]. \qquad (3.70)
\end{aligned}
$$

From (3.66)-(3.70), we obtain

$$
\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds \leq &-[x(t-h(t))-x(t-h_2)]^T \\
&\times U[x(t-h(t))-x(t-h_2)] \\
&-[x(t-h_1)-x(t-h(t))]^T \\
&\times U[x(t-h_1)-x(t-h(t))] \\
&-\beta[x(t-h_1)-x(t-h(t))]^T \\
&\times U[x(t-h_1)-x(t-h(t))] \\
&-(1-\beta)[x(t-h(t))-x(t-h_2)]^T \\
&\times U[x(t-h(t))-x(t-h_2)]. \qquad (3.71)
\end{aligned}
$$

$$
\dot{V}_7 = e^{2\alpha t}x^T(t)Tx(t) - e^{2\alpha(t-\eta_1)}x^T(t-\eta_1)Tx(t-\eta_1), \qquad (3.72)
$$

$$
\dot{V}_8 = e^{2\alpha t}x^T(t)Tx(t) - e^{2\alpha(t-\eta_2)}x^T(t-\eta_2)Tx(t-\eta_2), \qquad (3.73)
$$

$$\dot{V}_9 \leq e^{2\alpha t}[\eta_1^2 \dot{x}^T(t)M\dot{x}(t) - \eta_1 \int_{t-\eta_1}^{t} e^{2\alpha s}\dot{x}^T(s)M\dot{x}(s)ds], \tag{3.74}$$

$$\dot{V}_{10} \leq e^{2\alpha t}[\eta_2^2 \dot{x}^T(t)M\dot{x}(t) - \eta_2 \int_{t-\eta_2}^{t} e^{2\alpha s}\dot{x}^T(s)M\dot{x}(s)ds], \tag{3.75}$$

$$\dot{V}_{11} \leq e^{2\alpha t}[(\eta_2-\eta_1)^2 \dot{x}^T(t)N\dot{x}(t) - (\eta_2-\eta_1) \int_{t-\eta_2}^{t-\eta_1} e^{2\alpha s}\dot{x}^T(s)N\dot{x}(s)ds]. \tag{3.76}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned}
-\eta_1 \int_{t-\eta_1}^{t} \dot{x}^T(s)M\dot{x}(s)ds \leq & -\Big[\int_{t-\eta_1}^{t} \dot{x}(s)\Big]^T M \Big[\int_{t-\eta_1}^{t} \dot{x}(s)\Big] \\
\leq & -[x(t) - x(t-\eta_1)]^T M [x(t) - x(t-\eta_1)] \\
= & -x^T(t)Mx(t) + 2x^T(t)Mx(t-\eta_1) \\
& - x^T(t-\eta_1)Mx(t-\eta_1).
\end{aligned} \tag{3.77}$$

Applying Lemma 2.3.3 and the Leibniz-Newton formula, we have

$$\begin{aligned}
-\eta_2 \int_{t-\eta_2}^{t} \dot{x}^T(s)M\dot{x}(s)ds \leq & -\Big[\int_{t-\eta_2}^{t} \dot{x}(s)\Big]^T M \Big[\int_{t-\eta_2}^{t} \dot{x}(s)\Big] \\
\leq & -[x(t) - x(t-\eta_2)]^T M [x(t) - x(t-\eta_2)] \\
= & -x^T(t)Mx(t) + 2x^T(t)Mx(t-\eta_2) \\
& - x^T(t-\eta_2)Mx(t-\eta_2).
\end{aligned} \tag{3.78}$$

Note that

$$\begin{aligned}
-(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1} \dot{x}^T(s)N\dot{x}(s)\,ds = & -(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s)N\dot{x}(s)\,ds \\
& -(\eta_2-\eta_1)\int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s)N\dot{x}(s)\,ds \\
= & -(\eta_2-\eta(t))\int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s)N\dot{x}(s)\,ds \\
& -(\eta(t)-\eta_1)\int_{t-\eta_2}^{t-\eta(t)} \dot{x}^T(s)N\dot{x}(s)\,ds \\
& -(\eta(t)-\eta_1)\int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s)N\dot{x}(s)\,ds \\
& -(\eta_2-\eta(t))\int_{t-\eta(t)}^{t-\eta_1} \dot{x}^T(s)N\dot{x}(s)\,ds.
\end{aligned} \tag{3.79}$$

Using Lemma 2.3.3 gives

$$\begin{aligned}
-(\eta_2-\eta(t))\int_{t-\eta_2}^{t-\eta(t)}\dot{x}^T(s)N\dot{x}(s)\,ds \leq & -\Big[\int_{t-\eta_2}^{t-\eta(t)}\dot{x}(s)ds\Big]^T N\Big[\int_{t-\eta_2}^{t-\eta(t)}\dot{x}(s)ds\Big]\\
\leq & -[x(t-\eta(t))-x(t-\eta_2)]^T N[x(t-\eta(t))\\
& -x(t-\eta_2)],
\end{aligned} \tag{3.80}$$

and

$$\begin{aligned}
-(\eta(t)-\eta_1)\int_{t-\eta(t)}^{t-\eta_1}\dot{x}^T(s)N\dot{x}(s)\,ds \leq & -\Big[\int_{t-\eta(t)}^{t-\eta_1}\dot{x}(s)ds\Big]^T N\Big[\int_{t-\eta(t)}^{t-\eta_1}\dot{x}(s)ds\Big]\\
\leq & -[x(t-\eta_1)-x(t-\eta(t))]^T N[x(t-\eta_1)\\
& -x(t-\eta(t))].
\end{aligned} \tag{3.81}$$

Since $0\leq\eta_1\leq\eta(t)\leq\eta_2$ we get $\alpha=\frac{\eta_2-\eta(t)}{\eta_2-\eta_1}\leq 1$. Then

$$\begin{aligned}
-(\eta_2-\eta(t))\int_{t-\eta(t)}^{t-\eta_1}\dot{x}^T(s)N\dot{x}(s)\,ds = & -\alpha\int_{t-\eta(t)}^{t-\eta_1}(\eta_2-\eta_1)\dot{x}^T(s)N\dot{x}(s)\,ds\\
\leq & -\alpha\int_{t-\eta(t)}^{t-\eta_1}(\eta(t)-\eta_1)\dot{x}^T(s)N\dot{x}(s)\,ds\\
\leq & -\alpha[x(t-\eta_1)-x(t-\eta(t))]^T N[x(t-\eta_1)\\
& -x(t-\eta(t))],
\end{aligned} \tag{3.82}$$

and

$$\begin{aligned}
-(\eta(t)-\eta_1)\int_{t-\eta_2}^{t-\eta(t)}\dot{x}^T(s)N\dot{x}(s)\,ds = & -(1-\alpha)\int_{t-\eta_2}^{t-\eta(t)}(\eta_2-\eta_1)\dot{x}^T(s)N\dot{x}(s)\,ds\\
\leq & -(1-\alpha)\int_{t-\eta_2}^{t-\eta(t)}(\eta_2-\eta(t))\dot{x}^T(s)N\dot{x}(s)\,ds\\
\leq & -(1-\alpha)[x(t-\eta(t))-x(t-\eta_2)]^T\\
& \times N[x(t-\eta(t))-x(t-\eta_2)].
\end{aligned} \tag{3.83}$$

From (3.79)-(3.83), we obtain

$$\begin{aligned}
-(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1}\dot{x}^T(s)N\dot{x}(s)\,ds \leq & -[x(t-\eta(t))-x(t-\eta_2)]^T\\
& \times N[x(t-\eta(t))-x(t-\eta_2)]\\
& -[x(t-\eta_1)-x(t-\eta(t))]^T\\
& \times N[x(t-\eta_1)-x(t-\eta(t))]\\
& -\alpha[x(t-\eta_1)-x(t-\eta(t))]^T\\
& \times N[x(t-\eta_1)-x(t-\eta(t))]
\end{aligned}$$

$$- (1-\alpha)[x(t-\eta(t)) - x(t-\eta_2)]^T$$
$$\times N[x(t-\eta(t)) - x(t-\eta_2)]. \qquad (3.84)$$

$$\dot{V}_{12} \leq \; e^{2\alpha t}[k^2 x^T(t) H D_2 H x(t) - e^{-2\alpha k}(\int_{t-k(t)}^{t} h(x(s))\, ds)^T D_2 \int_{t-k(t)}^{t} h(x(s))\, ds]. \qquad (3.85)$$

By using the following identity relation

$$- \dot{x}(t) - Ax(t) + W_0 f(x(t) + W_1 g(x(t-h(t)) + W_2 \int_{t-k(t)}^{t} h(x(s))ds$$
$$+ A_2 \dot{x}(t-\eta(t)) = 0,$$

we have

$$- 2\dot{x}^T(t) D\dot{x}(t) - 2\dot{x}^T(t) DAx(t) + 2\dot{x}^T(t) DW_0 f(x(t) + 2\dot{x}^T(t) DW_1 g(x(t-h(t))$$
$$+ 2\dot{x}^T(t) DW_2 \int_{t-k(t)}^{t} h(x(s))ds + 2\dot{x}^T(t) DA_2 \dot{x}(t-\eta(t)) = 0.$$

Using the condition (3.3), we have

$$- 2\dot{x}^T(t) D\dot{x}(t) - 2\dot{x}^T(t) DAx(t) + 2\dot{x}^T(t) DW_0 Fx(t) + 2\dot{x}^T(t) DW_1 Gx(t-h(t)$$
$$+ 2\dot{x}^T(t) DW_2 \int_{t-k(t)}^{t} h(x(s))ds + 2\dot{x}^T(t) DA_2 \dot{x}(t-\eta(t)) \geq 0. \qquad (3.86)$$

Consider

$$\begin{aligned}
2\dot{x}^T(t) DA_2 \dot{x}(t-\eta(t)) \leq \; & \dot{x}^T(t) DD\dot{x}(t) + (A_2 \dot{x}(t-\eta(t)))^T A_2 \dot{x}(t-\eta(t)) \\
\leq \; & \dot{x}^T(t) DD\dot{x}(t) \\
& + [\dot{x}(t) + Ax(t) - W_0 f(x(t) - W_1 g(x(t-h(t)) \\
& - W_2 \int_{t-k(t)}^{t} h(x(s))ds]^T [\dot{x}(t) + Ax(t) - W_0 f(x(t) \\
& - W_1 g(x(t-h(t)) - W_2 \int_{t-k(t)}^{t} h(x(s))ds]^T \\
\leq \; & \dot{x}^T(t) D^2 \dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + \dot{x}^T(t) Ax(t) \\
& - \dot{x}^T(t) W_0 f(x(t) - \dot{x}^T(t) W_1 g(x(t-h(t)) \\
& - \dot{x}^T(t) W_2 \int_{t-k(t)}^{t} h(x(s))ds + x^T(t) A^T \dot{x}(t) \\
& + x^T(t) A^T Ax(t) - x^T(t) A^T W_0 f(x(t) \\
& - x^T(t) A^T W_1 g(x(t-h(t)) - x^T(t) A^T W_2 \int_{t-k(t)}^{t} h(x(s))ds
\end{aligned}$$

$$
\begin{aligned}
& - f^T(x(t))W_0^T\dot{x}(t) - f^T(x(t))W_0^TAx(t) \\
& + f^T(x(t))W_0^TW_0f(x(t) + f^T(x(t))W_0^TW_1g(x(t-h(t)) \\
& + f^T(x(t))W_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& - g^T(x(t-h(t)))W_1^T\dot{x}(t) - g^T(x(t-h(t)))W_1^TAx(t) \\
& + g^T(x(t-h(t)))W_1^TW_0f(x(t) \\
& + g^T(x(t-h(t)))W_1^TW_1g(x(t-h(t)) \\
& + g^T(x(t-h(t)))W_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& - (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^T\dot{x}(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TAx(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_0f(x(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_1g(x(t-h(t)) \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds.
\end{aligned}
$$

$$
\begin{aligned}
2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t)) \leq\ & \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
& - 2\dot{x}^T(t)W_0f(x(t)) - 2\dot{x}^T(t)W_1g(x(t-h(t)) \\
& - 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
& - 2x^T(t)A^TW_0f(x(t)) - 2x^T(t)A^TW_1g(x(t-h(t)) \\
& - 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + f^T(x(t))W_0^TW_0f(x(t)) \\
& + f^T(x(t))W_0^TW_1g(x(t-h(t)) \\
& + 2f^T(x(t))W_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + g^T(x(t-h(t)))W_1^TW_0f(x(t)) \\
& + g^T(x(t-h(t)))W_1^TW_1g(x(t-h(t)) \\
& + 2g^T(x(t-h(t)))W_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + (\int_{t-k_t}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds.
\end{aligned}
$$

Using the condition (3.3), we have

$$
\begin{aligned}
2\dot{x}^T(t)DA_2\dot{x}(t-\eta(t)) \leq\ & \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
& - 2\dot{x}^T(t)W_0fx(t) - 2\dot{x}^T(t)W_1gx(t-h(t)) \\
& - 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
& - 2x^T(t)A^TW_0Fx(t) - 2x^T(t)A^TW_1Gx(t-h(t)) \\
& - 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)FW_0^TW_0Fx(t) \\
& + x^T(t)FW_0^TW_1Gx(t-h(t) \\
& + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + x^T(t-h(t))GW_1^TW_0Fx(t) \\
& + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + (\int_{t-k(t)}^{t} h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^{t} h(x(s))ds. \qquad (3.87)
\end{aligned}
$$

From (3.86), (3.87) and Lemma 2.3.4, we have

$$
\begin{aligned}
& - 2\dot{x}^T(t)D\dot{x}(t) - 2\dot{x}^T(t)DAx(t) + 2\dot{x}^T(t)DW_0Fx(t) + 2\dot{x}^T(t)DW_1Gx(t-h(t) \\
& + 2\dot{x}^T(t)DW_2\int_{t-k(t)}^{t} h(x(s))ds + \dot{x}^T(t)D^2\dot{x}(t) + \dot{x}^T(t)I\dot{x}(t) + 2\dot{x}^T(t)Ax(t) \\
& + \dot{x}^T(t)W_0W_0^T\dot{x}^T(t) + x^T(t)F^2x(t) + \dot{x}^T(t)W_1W_1^T\dot{x}^T(t) \\
& + x^T(t-h(t))G^2x(t-h(t)) + x^T(t)A^TW_0W_0^TAx(t) + x^T(t)F^2x(t) \\
& + x^T(t)A^TW_1W_1^TAx(t) + x^T(t-h(t))G^2x(t-h(t)) \\
& - 2\dot{x}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)A^TAx(t) \\
& - 2x^T(t)A^TW_0Fx(t) - 2x^T(t)A^TW_1Gx(t-h(t)) \\
& - 2x^T(t)A^TW_2\int_{t-k(t)}^{t} h(x(s))ds + x^T(t)FW_0^TW_0Fx(t) \\
& + x^T(t)FW_0^TW_1Gx(t-h(t) + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^{t} h(x(s))ds \\
& + x^T(t-h(t))GW_1^TW_0Fx(t) + x^T(t-h(t))GW_1^TW_1Gx(t-h(t) \\
& + 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^{t} h(x(s))ds
\end{aligned}
$$

$$+(\int_{t-k(t)}^{t} h(x(s))ds)^T W_2^T W_2 \int_{t-k(t)}^{t} h(x(s))ds \geq 0. \tag{3.88}$$

From (3.58)-(3.65), (3.71-3.78), (3.84-3.85) and (3.88) we have

$$\begin{aligned}
\dot{V}(t,x_t) \leq\ & 2e^{2\alpha t}\{\alpha x^T(t)P_{23}x(t) - \Delta^T(t)P_1^T y(t) - y^T(t)P_2^T y(t) - x^T(t)P_3^T y(t) \\
& - x^T(t-h_1)P_4^T y(t) - x^T(t-h_2)P_5^T y(t) - \dot{x}^T(t)P_6^T y(t) \\
& - x^T(t-h(t))P_7^T y(t) - x^T(t-\eta_1)P_8^T y(t) \\
& - x^T(t-\eta_2)P_9^T y(t) - x^T(t-\eta(t))P_{10}^T y(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T y(t) - \Delta^T(t)P_1^T Ax(t) \\
& - x^T(t)P_2^T Ax(t) - x^T(t)P_3^T Ax(t)) - x^T(t-h_1)P_4^T Ax(t) \\
& - x^T(t-h_2)P_5^T Ax(t) - \dot{x}^T(t)P_6^T Ax(t) \\
& - x^T(t-h(t))P_7^T Ax(t)) - x^T(t-\eta_1)P_8^T Ax(t) \\
& - x^T(t-\eta_2)P_9^T Ax(t) - x^T(t-\eta(t))P_{10}^T Ax(t) \\
& - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T Ax(t) + \Delta^T(t)P_1^T W_0 Fx(t) \\
& + y^T(t)P_2^T W_0 Fx(t) + x^T(t-h_2)P_5^T W_0 Fx(t) + \eta_1)P_8^T W_0 Fx(t) \\
& + x^T(t)P_3^T W_0 Fx(t) + x^T(t-h_1)P_4^T W_0 Fx(t) \\
& + \dot{x}^T(t)P_6^T W_0 Fx(t) + x^T(t-h(t))P_7^T W_0 Fx(t) + x^T(t) \\
& + x^T(t-\eta_2)P_9^T W_0 Fx(t) + x^T(t-\eta(t))P_{10}^T W_0 Fx(t) \\
& + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_0 Fx(t) + \Delta^T(t)P_1^T W_1 Gx(t-h(t))\} \\
& + 2\{y^T(t)P_2^T W_1 Gx(t-h(t)) + x^T(t)P_3^T W_1 Gx(t-h(t)) \\
& + x^T(t-h_1)P_4^T W_1 Gx(t-h(t)) + x^T(t-h_2)P_5^T W_1 Gx(t-h(t)) \\
& + \dot{x}^T(t)P_6^T W_1 Gx(t-h(t)) + x^T(t-h(t))P_7^T W_1 Gx(t-h(t)) \\
& + x^T(t-\eta_1)P_8^T W_1 Gx(t-h(t)) + x^T(t-\eta_2)P_9^T W_1 Gx(t-h(t)) \\
& + x^T(t-\eta(t))P_{10}^T W_1 Gx(t-h(t)) \\
& + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_1 Gx(t-h(t)) \\
& + \Delta^T(t)P_1^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + y^T(t)P_2^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
& + x^T(t)P_3 W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-h_1)P_4^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
& + x^T(t-h_2)P_5^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + \dot{x}^T(t)P_6^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ x^T(t-h(t))P_7^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta_1)P_8^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ x^T(t-\eta_2)P_9^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds + x^T(t-\eta(t))P_{10}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds \\
&+ (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{11}^T W_2 \int_{t-k(t)}^{t} h(x(s))\,ds - \Delta^T(t)P_{12}^T\Delta(t) \\
&- y^T(t)P_{13}\Delta(t) - x^T(t)P_{14}^T\Delta(t) - x^T(t-h_1)P_{15}^T\Delta(t) - x^T(t-h_2)P_{16}^T\Delta(t) \\
&- \dot{x}^T(t)P_{17}^T\Delta(t) - x^T(t-h(t))P_{18}^T\Delta(t) - x^T(t-\eta_1)P_{19}^T\Delta(t)) \\
&- x^T(t-\eta_2)P_{20}^T\Delta(t) - x^T(t-\eta(t))P_{21}^T\Delta(t) - (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T\Delta(t) \\
&+ \Delta^T(t)P_{12}^T x(t) + y^T(t)P_{13}^T x(t) + x^T(t)P_{14}^T x(t) + x^T(t-h_1)P_{15}^T x(t) \\
&+ x^T(t-h_2)P_{16}^T x(t) + \dot{x}^T(t)P_{17}^T x(t) + x^T(t-h(t))P_{18}^T x(t) + x^T(t-\eta_1)P_{19}^T x(t) \\
&+ x^T(t-\eta_2)P_{20}^T x(t) + x^T(t-\eta(t))P_{21}^T x(t) + (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T x(t) \\
&- \Delta^T(t)P_{12}^T A_2 x(t-\eta(t)) - y^T(t)P_{13}^T A_2 x(t-\eta(t)) \\
&- x^T(t)P_{14}^T A_2 x(t-\eta(t)) - x^T(t-h_1)P_{15}^T A_2 x(t-\eta(t)) \\
&- x^T(t-h_2)P_{16}^T A_2 x(t-\eta(t)) - \dot{x}^T(t)P_{17}^T A_2 x(t-\eta(t)) \\
&- x^T(t-h(t))P_{18}^T A_2 x(t-\eta(t)) - x^T(t-\eta_1)P_{19}^T A_2 x(t-\eta(t)) \\
&- x^T(t-\eta_2)P_{20}^T A_2 x(t-\eta(t)) - x^T(t-\eta(t))P_{21}^T A_2 x(t-\eta(t)) \\
&- (\int_{t-k(t)}^{t} h(x(s))\,ds)^T P_{22}^T A_2 x(t-\eta(t)) + x^T(t)P_{23}^T\dot{x}(t)\} \\
&+ x^T(t)Qx(t) - e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1) + x^T(t)Qx(t) \\
&- e^{-2\alpha h_2}x^T(t-h_2)Qx(t-h_2) + h_1^2\dot{x}^T(t)R\dot{x}(t) - e^{-2\alpha h_1}x^T(t)Rx(t) \\
&+ 2e^{-2\alpha h_1}x^T(t)Rx(t-h_1) - e^{-2\alpha h_1}x^T(t-h_1)Rx(t-h_1) \\
&+ h_2^2\dot{x}^T(t)R\dot{x}(t) - e^{-2\alpha h_2}x^T(t)Rx(t) + 2e^{-2\alpha h_2}x^T(t)Rx(t-h_2) \\
&- e^{-2\alpha h_2}x^T(t-h_2)Rx(t-h_2) + (h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t) \\
&- e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h(t)) + e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h_2) \\
&+ e^{-2\alpha h_2}x^T(t-h_2))Ux(t-h(t)) - e^{-2\alpha h_2}x^T(t-h_2)Ux(t-h_2) \\
&- e^{-2\alpha h_2}x^T(t-h_1)Ux(t-h_1) + e^{-2\alpha h_2}x^T(t-h_1)Ux(t-h(t)) \\
&+ e^{-2\alpha h_2}x^T(t-h(t)))Ux(t-h_1) - e^{-2\alpha h_2}x^T(t-h(t))Ux(t-h(t)) \\
&- \beta[x(t-h_1)-x(t-h(t))]^T e^{-2\alpha h_2}U[x(t-h_1)-x(t-h(t))] \\
&- (1-\beta)[x(t-h(t))-x(t-h_2)]^T \times e^{-2\alpha h_2}U[x(t-h(t))-x(t-h_2)] \\
&+ x^T(t)Tx(t) - e^{-2\alpha\eta_1}x^T(t-\eta_1)Tx(t-\eta_1) + x^T(t)Tx(t)
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-2\alpha\eta_2}x^T(t-\eta_2)Tx(t-\eta_2) + \eta_1^2\dot{x}^T(t)M\dot{x}(t) - e^{-2\alpha\eta_1}x^T(t)Mx(t)\\
&+ 2e^{-2\alpha\eta_1}x^T(t)Mx(t-\eta_1) - e^{-2\alpha\eta_1}x^T(t-\eta_1)Mx(t-\eta_1)\\
&+ \eta_2^2\dot{x}^T(t)M\dot{x}(t) - e^{-2\alpha\eta_2}x^T(t)Mx(t) + 2e^{-2\alpha\eta_2}x^T(t)Mx(t-\eta_2)\\
&- e^{-2\alpha\eta_2}x^T(t-\eta_2)Mx(t-\eta_2) + (\eta_2-\eta_1)^2\dot{x}^T(t)N\dot{x}(t)\\
&- e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta(t)) + e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta_2)\\
&+ e^{-2\alpha\eta_2}x^T(t-\eta_2))Nx(t-\eta(t)) - e^{-2\alpha\eta_2}x^T(t-\eta_2)Nx(t-\eta_2)\\
&- e^{-2\alpha\eta_2}x^T(t-\eta_1)Nx(t-\eta_1) + e^{-2\alpha\eta_2}x^T(t-\eta_1)Nx(t-\eta(t))\\
&+ e^{-2\alpha\eta_2}x^T(t-\eta(t)))Nx(t-\eta_1) - e^{-2\alpha\eta_2}x^T(t-\eta(t))Nx(t-\eta(t))\\
&- \alpha[x(t-\eta_1)-x(t-\eta(t))]^Te^{-2\alpha\eta_2}N[x(t-\eta_1)-x(t-\eta(t))]\\
&- (1-\alpha)[x(t-\eta(t))-x(t-\eta_2)]^T \times e^{-2\alpha\eta_2}N[x(t-\eta(t))-x(t-\eta_2)]\\
&+ 2k^2x^T(t)HD_2Hx(t) - 2e^{-2\alpha k}(\int_{t-k(t)}^t h(x(s))\,ds)^TD_2\int_{t-k(t)}^t h(x(s))\,ds\\
&+ 2\alpha x^T(t)P_{23}x(t) - 2\dot{x}^T(t)D\dot{x}(t) - 2\dot{x}^T(t)DAx(t) + 2\dot{x}^T(t)DW_0Fx(t)\\
&+ 2\dot{x}^T(t)DW_1Gx(t-h(t) + 2\dot{x}^T(t)DW_2\int_{t-k_t}^t h(x(s))ds + \dot{x}^T(t)D^2\dot{x}(t)\\
&+ \dot{x}^T(t)I\dot{x}(t) + 2\dot{x}^T(t)Ax(t) - 2\dot{x}^T(t)W_0Fx(t) - 2\dot{x}^T(t)W_1Gx(t-h(t))\\
&- 2\dot{x}^T(t)W_2\int_{t-k(t)}^t h(x(s))ds + x^T(t)A^TAx(t) - 2x^T(t)A^TW_0Fx(t)\\
&- 2x^T(t)A^TW_1Gx(t-h(t)) - 2x^T(t)A^TW_2\int_{t-k(t)}^t h(x(s))ds + x^T(t)FW_0^TW_0Fx(t)\\
&- 2x^T(t)A^TW_2\int_{t-k(t)}^t h(x(s))ds + x^T(t)FW_0^TW_0Fx(t)\\
&+ x^T(t)FW_0^TW_1Gx(t-h(t)) + 2x^T(t)FW_0^TW_2\int_{t-k(t)}^t h(x(s))ds\\
&+ x^T(t-h(t))GW_1^TW_0Fx(t) + x^T(t-h(t))GW_1^TW_1Gx(t-h(t))\\
&+ 2x^T(t-h(t))GW_1^TW_2\int_{t-k(t)}^t h(x(s))ds\\
&+ (\int_{t-k(t)}^t h(x(s))ds)^TW_2^TW_2\int_{t-k(t)}^t h(x(s))ds. \qquad (3.89)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\dot{V}(t,x_t) \leq\ & e^{2\alpha t}\{\zeta^T(t)\mathcal{M}\zeta(t) - \beta[x(t-h_1)-x(t-h(t))]^Te^{-2\alpha h_2}\\
&\times U[x(t-h_1)-x(t-h(t))]\\
&- (1-\beta)[x(t-h(t))-x(t-h_2)]^T\\
&\times e^{-2\alpha h_2}U[x(t-h(t))-x(t-h_2)] + \zeta^T(t)\mathcal{M}\zeta(t)
\end{aligned}
$$

$$
\begin{aligned}
& - \alpha[x(t-\eta_1) - x(t-\eta(t))]^T e^{-2\alpha\eta_2} N[x(t-\eta_1) \\
& - x(t-\eta(t))] - (1-\alpha)[x(t-\eta(t)) - x(t-\eta_2)]^T \\
& \times e^{-2\alpha\eta_2} N[x(t-\eta(t)) - x(t-\eta_2)]\} \\
= \; & e^{2\alpha t}\{\zeta^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 + (1-\alpha)\mathcal{M}_3 \\
& + \alpha\mathcal{M}_4]\zeta(t)\}. \qquad (3.90)
\end{aligned}
$$

Since $0 \leq \beta \leq 1$, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2$ is a convex combination of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $0 \leq \alpha \leq 1$, $(1-\alpha)\mathcal{M}_3 + \alpha\mathcal{M}_4$ is a convex combination of $\mathcal{M}_3$ and $\mathcal{M}_4$. Therefore, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ is equivalent to $\mathcal{M}_1 < 0$ and $\mathcal{M}_2 < 0$, and $(1-\alpha)\mathcal{M}_3 + \alpha\mathcal{M}_4 < 0$ is equivalent to $\mathcal{M}_3 < 0$ and $\mathcal{M}_4 < 0$. From $\dot{V}(x(t)) \leq 0$, we get $V(x(t)) \leq V(x(0))$. Note that,

$$
\begin{aligned}
V_1(x(0)) &= \zeta^T(0)E^T P\zeta(0) \leq \lambda_{\max}(P_{23})\|\phi\|^2, \\
V_2(x(0)) &= \int_{-h_1}^{0} e^{2\alpha s} x^T(s)Qx(s)\,ds \leq \lambda_{\max}(Q)\|M_1\|^2 \int_{-h_1}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(Q)\|M_1\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_3(x(0)) &= \int_{-h_2}^{0} e^{2\alpha s} x^T(s)Qx(s)\,ds \leq \lambda_{\max}(Q)\|M_1\|^2 \int_{-h_2}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(Q)\|M_1\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_4(x(0)) &= h_1 \int_{-h_1}^{0}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(R)\|\phi\|^2 \int_{-h_1}^{0} e^{2\alpha\tau} d\tau \\
&= h_2^2\lambda_{\max}(R)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_5(x(0)) &= h_2 \int_{-h_2}^{0}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(R)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_6(x(0)) &= h_2 \int_{-h_2}^{-h_1}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds \\
&\leq h_2^2\lambda_{\max}(U)\|\phi\|^2 \frac{1-e^{-2\alpha h_2}}{2\alpha}, \\
V_7(x(0)) &= \int_{-\eta_1}^{0} e^{2\alpha s} x^T(s)Tx(s)\,ds \leq \lambda_{\max}(T)\|M_1\|^2 \int_{-\eta_1}^{0} e^{2\alpha s} ds \\
&= \lambda_{\max}(T)\|M_1\|^2 \frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_8(x(0)) &= \int_{-\eta_2}^{0} e^{2\alpha s} x^T(s)Tx(s)\,ds \leq \lambda_{\max}(T)\|M_1\|^2 \int_{-\eta_2}^{0} e^{2\alpha s} ds
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_{\max}(T)\|M_1\|^2\frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_9(x(0)) &= \eta_1\int_{-\eta_1}^{0}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds \\
&\leq \eta_2^2\lambda_{\max}(M)\|\phi\|^2\int_{-\eta_1}^{0} e^{2\alpha\tau}d\tau \\
&= \eta_2^2\lambda_{\max}(M)\|\phi\|^2\frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_{10}(x(0)) &= \eta_2\int_{-\eta_2}^{0}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)M\dot{x}(\tau)\,d\tau\,ds \\
&\leq \eta_2^2\lambda_{\max}(M)\|\phi\|^2\frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_{11}(x(0)) &= \eta_2\int_{-\eta_2}^{-\eta_1}\int_{s}^{0} e^{2\alpha\tau}\dot{x}^T(\tau)N\dot{x}(\tau)\,d\tau\,ds \\
&\leq \eta_2^2\lambda_{\max}(N)\|\phi\|^2\frac{1-e^{-2\alpha\eta_2}}{2\alpha}, \\
V_{12}(x(0)) &= k\int_{-k}^{0}\int_{s}^{0} e^{2\alpha\tau}h^T(x(\tau))D_2h(x(\tau))d\tau\,ds \\
&\leq k^2\lambda_{\max}(HD_2H)\|\phi\|^2\int_{-k}^{0} e^{2\alpha\tau}d\tau \\
&= k^2\lambda_{\max}(HD_2H)\|\phi\|^2\frac{1-e^{-2\alpha k}}{2\alpha}.
\end{aligned}
$$

We have $e^{2\alpha t}\lambda_{\min}(P_{23})\|x(t)\|^2 \leq V(t,x_t) \leq V(x(0)),\ \forall t \geq 0$. Then $\|x(t)\| \leq \sqrt{\frac{a\|\phi\|^2+b\|M_1\|^2}{\lambda_{\min}(P_{23})}}.e^{-\boldsymbol{\alpha t}}$, where $a = \lambda_{\max}(P_{23})+2h_2^2\lambda_{\max}(R)\frac{1-e^{-2\alpha h_2}}{2\alpha}+h_2^2\lambda_{\max}(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}+2\eta_2^2\lambda_{\max}(M)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}+\eta_2^2\lambda_{\max}(N)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}+k^2\lambda_{\max}(HD_2H)\frac{1-e^{-2\alpha k}}{2\alpha}$, $b = 2\lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha}+2\lambda_{\max}(T)\frac{1-e^{-2\alpha\eta_2}}{2\alpha}$ and $\|M_1\| = \sup_{-d\leq s\leq 0}\|x(s)\|$. From Definition 2.3.8, we conclude that the equilibrium point is $\alpha-$exponentially stable. This completes the proof.

Remark 3.2.1 *For the case when $h_1 = \eta_1 = 0$, we have the following corollary for delay-dependent stability of system (3.51).*

Corollary 3.2.1 *Given $\alpha > 0$. The system (3.51) is $\alpha-$exponentially stable if there exist symmetric positive definite matrices $P_{23}, S, D, Q, U, T, N, D_2$,*
matrices $P_i, i = 1,2,...,22$, $i \neq 4,8,15,19$ of appropriate dimension such that the following LMIs hold

$$
\begin{aligned}
\mathcal{M}_1 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha h_2}U\begin{bmatrix} 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.91}
$$

$$
\begin{aligned}
\mathcal{M}_2 =& \mathcal{M} - \begin{bmatrix} 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}^T \\
&\times e^{-2\alpha h_2}U\begin{bmatrix} 0 & 0 & -I & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{3.92}
$$

$$\mathcal{M}_3 = \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 \end{bmatrix}^T \times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 \end{bmatrix} < 0, \quad (3.93)$$

$$\mathcal{M}_4 = \mathcal{M} - \begin{bmatrix} 0 & 0 & -I & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}^T \times e^{-2\alpha\eta_2} N \begin{bmatrix} 0 & 0 & -I & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} < 0, \quad (3.94)$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{15} & \phi_{16} & \phi_{17} & \phi_{19} & \phi_{1a} & \phi_{1b} \\ * & \phi_{22} & \phi_{23} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{29} & \phi_{2a} & \phi_{2b} \\ * & * & \omega_{33} & \omega_{35} & \phi_{36} & \omega_{37} & \omega_{39} & \omega_{3a} & \phi_{3b} \\ * & * & * & \omega_{55} & 0 & \phi_{57} & 0 & \phi_{5a} & \phi_{5b} \\ * & * & * & * & \omega_{66} & \phi_{67} & 0 & \phi_{6a} & \phi_{6b} \\ * & * & * & * & * & \phi_{77} & \phi_{79} & \phi_{7a} & \phi_{7b} \\ * & * & * & * & * & * & \omega_{99} & \phi_{9a} & \phi_{9b} \\ * & * & * & * & * & * & * & \phi_{10a} & \phi_{10b} \\ * & * & * & * & * & * & * & * & \phi_{11b} \end{bmatrix},$$

where

$$\begin{aligned}
\phi_{11} &= -P_{12}^T - P_{12}, \\
\phi_{12} &= -P_1^T - P_{13}, \\
\phi_{13} &= -P_1^T A + P_1^T W_0 F + P_{12}^T - P_{14}, \\
\phi_{14} &= -P_{15}, \\
\phi_{15} &= -P_{16}, \\
\phi_{16} &= -P_{17}, \\
\phi_{17} &= P_1^T W_1 G - P_{18}, \\
\phi_{18} &= -P_{19}, \\
\phi_{19} &= -P_{20}, \\
\phi_{1a} &= -P_{12}^T A_2 - P_{21}, \\
\phi_{1b} &= P_1^T W_2 - P_{22}, \\
\phi_{22} &= -P_2^T - P_2, \\
\phi_{23} &= -P_2^T A + P_2^T W_0 F + P_{13}^T - P_3, \\
\phi_{24} &= -P_4, \\
\phi_{25} &= -P_5,
\end{aligned}$$

$$
\begin{aligned}
\phi_{26} &= -P_6, \\
\phi_{27} &= P_2^T W_1 G - P_7, \\
\phi_{28} &= -P_8, \\
\phi_{29} &= -P_9, \\
\phi_{2a} &= -P_{13}^T A_2 - P_{10}, \\
\phi_{2b} &= P_2^T W_2 - P_{11}, \\
\phi_{33} &= -A^T P_3 - P_3^T A + F^T W_0^T P_3 + P_3^T W_0 F + P_{14}^T + P_{14} + \alpha P_{23}^T + \alpha P_{23} \\
&\quad + Q + Q^T - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + T + T^T - e^{-2\alpha \eta_1} M - e^{-2\alpha \eta_2} M \\
&\quad + K^2 H D_2 H + K^2 H^T D_2^T H^T + A^T A + F W_0^T W_0 F \\
&\quad + 2F^2 + A^T W_0 W_0^T A + A^T W_1 W_1^T A, \\
\phi_{34} &= -A^T P_4 + F^T W_0^T P_4 + P_{15} + e^{-2\alpha h_1} R, \\
\phi_{35} &= -A^T P_5 + F^T W_0^T P_5 + P_{16} + e^{-2\alpha h_2} R, \\
\phi_{36} &= P_{23}^T - A^T P_6 + F^T W_0^T P_6 + P_{17} + A^T - A^T D^T + F^T W_0^T D^T, \\
\phi_{37} &= P_3^T W_1 G - A^T P_7 + F^T W_0^T P_7 + P_{18} + F^T W_0^T W_1 G, \\
\phi_{38} &= -A^T P_8 + F^T W_0^T P_8 + P_{19} + e^{-2\alpha \eta_1} M, \\
\phi_{39} &= -A^T P_9 + F^T W_0^T P_9 + P_{20} + e^{-2\alpha \eta_2} M, \\
\phi_{3a} &= -P_{14}^T A_2 - A^T P_{10} + F^T W_0^T P_{10} + P_{21}, \\
\phi_{3b} &= P_3^T W_2 - A^T P_{11} + F^T W_0^T P_{11} + P_{22} - A^T W_2 + F W_0^T W_2, \\
\phi_{55} &= -e^{-2\alpha h_2} Q - e^{-2\alpha h_2} R - e^{-2\alpha h_2} U, \\
\phi_{57} &= P_5^T W_1 G + e^{-2\alpha h_2} U, \\
\phi_{5a} &= -P_{16}^T A_2, \\
\phi_{5b} &= P_5^T W_2, \\
\phi_{66} &= h_1^2 R + h_2^2 R + (h_1 - h_2)^2 U + \eta_1^2 M + \eta_2^2 M + (\eta_1 - \eta_2)^2 N - 2D + I + D^2 \\
&\quad + W_0 W_0^T + W_1 W_1^T, \\
\phi_{67} &= P_6^T W_1 G + D W_1 G, \\
\phi_{6a} &= -P_{17}^T A_2, \\
\phi_{6b} &= P_6^T W_2 - W_2 + D W_2, \\
\phi_{77} &= P_7^T W_1 G + G^T W_1^T P_7 - e^{-2\alpha h_2} U - e^{-2\alpha h_2} U^T + G W_1^T W_1 G + 2G^2, \\
\phi_{78} &= G^T W_1^T P_8, \\
\phi_{79} &= G^T W_1^T P_9,
\end{aligned}
$$

$$
\begin{aligned}
\phi_{7a} &= -P_{18}^T A_2 + G^T W_1^T P_{10}, \\
\phi_{7b} &= P_7^T W_2 + G^T W_1^T P_{11} + G W_1^T W_2, \\
\phi_{99} &= -e^{-2\alpha\eta_2} T - e^{-2\alpha\eta_2} M - e^{-2\alpha\eta_2} N, \\
\phi_{9a} &= -P_{20}^T A_2 + e^{-2\alpha\eta_2} N, \\
\phi_{9b} &= P_9^T W_2, \\
\phi_{10a} &= -P_{21}^T A_2 - A_2^T P_{21} - e^{-2\alpha\eta_2} N - e^{-2\alpha\eta_2} N^T, \\
\phi_{10b} &= P_{10}^T W_2 - A_2^T P_{22}, \\
\phi_{11b} &= P_{11}^T W_2 + W_2^T P_{11} - e^{-2\alpha k} D_2 - e^{-2\alpha k} D_2^T + W_2^T W_2.
\end{aligned}
$$

Proof. Choose a Lyapunov-Krasovskii functional candidate as

$$
V(t, x_t) = \sum_{i=1}^{6} V_i, \tag{3.95}
$$

where

$$
\begin{aligned}
V_1 &= e^{2\alpha t} \zeta^T(t) E^T P \zeta(t), \\
V_2 &= \int_{t-h_2}^{t} e^{2\alpha s} x^T(s) Q x(s)\, ds, \\
V_3 &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha\tau} \dot{x}^T(\tau) U \dot{x}(\tau)\, d\tau\, ds, \\
V_4 &= \int_{t-\eta_2}^{t} e^{2\alpha s} x^T(s) T x(s)\, ds, \\
V_5 &= \eta_2 \int_{-\eta_2}^{0} \int_{t+s}^{t} e^{2\alpha\tau} \dot{x}^T(\tau) N \dot{x}(\tau)\, d\tau\, ds, \\
V_6 &= k \int_{-k}^{t} \int_{t+s}^{t} e^{2\alpha\tau} h^T(x(\tau)) D_2 h(x(\tau))\, d\tau\, ds,
\end{aligned}
$$

with $\zeta(t) = [\Delta^T(t) \quad y^T(t) \quad x^T(t) \quad x^T(t-h_2) \quad \dot{x}^T(t) \quad x^T(t-h(t)) \quad x^T(t-\eta_2) \quad x^T(t-\eta(t)) \quad (\int_{t-k(t)}^{t} h(x(s))\, ds)^T]^T$, and

$$
E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
P = \begin{bmatrix} P_1 & P_2 & P_3 & P_5 & P_6 & P_7 & P_9 & P_{10} & P_{11} \\ P_{12} & P_{13} & P_{14} & P_{16} & P_{17} & P_{18} & P_{20} & P_{21} & P_{22} \\ 0 & 0 & P_{23} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Using the similar method as in Theorem 3.1.1, we have

$$
\dot{V}(t, x_t) \leq e^{2\alpha t} \{ \zeta^T(t) \mathcal{M} \zeta(t) - \beta [x(t) - x(t-h(t))]^T e^{-2\alpha h_2}
$$

$$
\begin{aligned}
&\times U[x(t) - x(t-h(t))] \\
&-(1-\beta)[x(t-h(t)) - x(t-h_2)]^T \\
&\times e^{-2\alpha h_2} U[x(t-h(t)) - x(t-h_2)] + \zeta^T(t)\mathcal{M}\zeta(t) \\
&-\alpha[x(t) - x(t-\eta(t))]^T e^{-2\alpha\eta_2} N[x(t) - x(t-\eta(t))] \\
&-(1-\alpha)[x(t-\eta(t)) - x(t-\eta_2)]^T \\
&\times e^{-2\alpha\eta_2} N[x(t-\eta(t)) - x(t-\eta_2)]\}.
\end{aligned}
$$

Remaining part of the proof immediately follows from Theorem 3.1.1. This completes the proof. □

3.3 Numerical examples

Example 3.1.1 Consider the neutral-type neural networks with interval non-differentiable and distributed time-varying delays (3.5) which was considered in [13] where

$$
A = \begin{pmatrix} 1.2 & 1 \\ 0 & 1.2 \end{pmatrix}, A_2 = \begin{pmatrix} 0.15 & 0 \\ 0.1 & 0.15 \end{pmatrix}, W_0 = \begin{pmatrix} -2.1 & 0.1 \\ 0.1 & -1.1 \end{pmatrix},
$$
$$
W_1 = \begin{pmatrix} 0.2 & 0.3 \\ 0.05 & 0.2 \end{pmatrix}, W_2 = \begin{pmatrix} 0.2 & -0.1 \\ 0.05 & -0.02 \end{pmatrix}, F = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix},
$$
$$
G = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, H = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_a = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},
$$
$$
E_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},
$$
$$
\alpha = 0.01, k = 0.1, h_1 = 0.02, h_2 = 0.03, \eta_1 = 0.02, \eta_2 = 0.03.
$$

By using the Matlab LMI toolbox, we can solve for symmetric positive definite matrices $P_{23}, S, D, Q, R, U, T, M, N, D_2$ and matrices $P_i, i = 1, 2, ..., 22$, P_{1a}, P_{2a}, ϵ which satisfy the criterion of Theorem 3.1.1 as follows.

$$
P_{23} = \begin{pmatrix} 30.0470 & 0.8150 \\ 0.8150 & 32.6045 \end{pmatrix}, S = \begin{pmatrix} 68.6912 & 0 \\ 0 & 68.6912 \end{pmatrix},
$$
$$
D = \begin{pmatrix} 12.3451 & -1.3047 \\ -1.3047 & 14.3630 \end{pmatrix}, Q = \begin{pmatrix} 68.5725 & 5.0212 \\ 5.0212 & 63.9157 \end{pmatrix},
$$

$$R = \begin{pmatrix} 21.6097 & 1.4802 \\ 1.4802 & 21.2744 \end{pmatrix}, U = \begin{pmatrix} 18.4001 & -0.0335 \\ -0.0335 & 18.2014 \end{pmatrix},$$

$$M = \begin{pmatrix} 30.8753 & 3.0190 \\ 3.0190 & 29.0900 \end{pmatrix}, N = \begin{pmatrix} 23.2963 & 0.7278 \\ 0.7278 & 19.0371 \end{pmatrix},$$

$$T = \begin{pmatrix} 15.5821 & 4.2186 \\ 4.7458 & 21.5037 \end{pmatrix}, D_2 = \begin{pmatrix} 72.0443 & 7.8536 \\ 7.8536 & 41.0385 \end{pmatrix},$$

$$P_1 = 10^4 \begin{pmatrix} 5.8690 & -1.6304 \\ 0.6868 & 2.3363 \end{pmatrix}, P_2 = 10^4 \begin{pmatrix} 0.0027 & -1.1552 \\ 1.1542 & 0.0037 \end{pmatrix},$$

$$P_3 = 10^4 \begin{pmatrix} -5.8927 & -0.2397 \\ 0.8255 & -3.4808 \end{pmatrix}, P_4 = \begin{pmatrix} 5.8532 & -1.7476 \\ 2.3400 & 7.0178 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 5.8457 & -1.7458 \\ 2.3367 & 7.0090 \end{pmatrix}, P_6 = \begin{pmatrix} 13.5713 & -4.5346 \\ 2.8440 & 17.2363 \end{pmatrix},$$

$$P_7 = 10^3 \begin{pmatrix} -0.2898 & 1.1540 \\ -0.6939 & 1.0382 \end{pmatrix}, P_8 = \begin{pmatrix} 11.7629 & 2.2980 \\ 4.5635 & 10.6139 \end{pmatrix},$$

$$P_9 = \begin{pmatrix} 11.7591 & 2.2987 \\ 4.5618 & 10.6100 \end{pmatrix}, P_{10} = 10^4 \begin{pmatrix} 1.5667 & 2.0908 \\ 0.1030 & 0.3503 \end{pmatrix},$$

$$P_{11} = 10^3 \begin{pmatrix} -0.5782 & 2.3082 \\ 0.2310 & -1.1541 \end{pmatrix}, P_{12} = 10^3 \begin{pmatrix} 0.0342 & 3.5345 \\ -3.5354 & 0.0227 \end{pmatrix},$$

$$P_{13} = 10^4 \begin{pmatrix} -5.8671 & -0.6862 \\ 1.6301 & -2.3350 \end{pmatrix}, P_{14} = 10^4 \begin{pmatrix} -9.5369 & -1.4191 \\ -3.3207 & -3.7382 \end{pmatrix},$$

$$P_{15} = \begin{pmatrix} -6.0381 & 0.9958 \\ 1.3263 & -3.2297 \end{pmatrix}, P_{16} = \begin{pmatrix} -6.0313 & 0.9943 \\ 1.3244 & -3.2262 \end{pmatrix},$$

$$P_{17} = \begin{pmatrix} 4.7420 & 4.8654 \\ 0.2997 & 2.9305 \end{pmatrix}, P_{18} = 10^3 \begin{pmatrix} 5.4608 & 1.2703 \\ 4.3032 & 2.0192 \end{pmatrix},$$

$$P_{19} = \begin{pmatrix} -4.7528 & 13.5161 \\ 2.2939 & -2.1053 \end{pmatrix}, P_{20} = \begin{pmatrix} -4.7497 & 13.5151 \\ 2.2930 & -2.1038 \end{pmatrix},$$

$$P_{21} = 10^3 \begin{pmatrix} -3.5359 & 0.5294 \\ -0.5304 & -0.0002 \end{pmatrix}, P_{22} = 10^4 \begin{pmatrix} 1.0923 & 0.2541 \\ -0.5543 & -0.1154 \end{pmatrix},$$

$$P_{1a} = \begin{pmatrix} -20.2034 & -6.9092 \\ 6.7307 & -20.5050 \end{pmatrix}, P_{2a} = \begin{pmatrix} -15.4310 & -8.6513 \\ 6.1200 & -6.9284 \end{pmatrix},$$

$\epsilon = 2.9050.$

Therefore, the zero solution of system (3.5) $\alpha-$exponentially stable.

Example 3.1.2 Consider the neutral-type neural networks with interval non-differentiable and distributed time-varying delays (3.51) which was considered in [13] where

$$A=\begin{pmatrix}3.6 & 0\\ 0 & 3.6\end{pmatrix}, A_2=\begin{pmatrix}0.15 & 0\\ 0 & 0.15\end{pmatrix}, W_0=\begin{pmatrix}-1.198 & 0.1\\ 0.1 & -1.198\end{pmatrix},$$
$$W_1=\begin{pmatrix}0.1 & 0.16\\ 0.05 & 0.1\end{pmatrix}, W_2=\begin{pmatrix}0.3 & -0.15\\ 0.5 & -0.2\end{pmatrix}, F=\begin{pmatrix}0.4 & 0\\ 0 & 0.5\end{pmatrix},$$
$$G=\begin{pmatrix}0.1 & 0\\ 0 & 0.2\end{pmatrix}, H=\begin{pmatrix}0.5 & 0\\ 0 & 0.3\end{pmatrix}, I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \alpha=0.01, k=0.1,$$
$$h_1=0.01, h_2=0.02, \eta_1=0.01, \eta_2=0.02.$$

By using the Matlab LMI toolbox, we can solve for symmetric positive definite matrices $P_{23}, S, D, Q, R, U, T, M, N, D_2$ and matrices $P_i, i = 1, 2, ..., 22$ which satisfy the criterion of Theorem 3.2.1 as follow.

$$P_{23}=10^3\begin{pmatrix}2.1218 & -0.0306\\ -0.0306 & 2.1505\end{pmatrix}, S=10^3\begin{pmatrix}2.2073 & 0\\ 0 & 2.2073\end{pmatrix},$$
$$D=\begin{pmatrix}520.8801 & -3.1925\\ -3.1925 & 515.7363\end{pmatrix}, Q=10^3\begin{pmatrix}2.3305 & -0.0162\\ -0.0162 & 2.3165\end{pmatrix},$$
$$R=\begin{pmatrix}836.1007 & -8.8390\\ -8.8390 & 828.3419\end{pmatrix}, U=\begin{pmatrix}598.2972 & 0.3117\\ 0.3117 & 598.6342\end{pmatrix},$$
$$M=10^3\begin{pmatrix}1.3371 & -0.0206\\ -0.0206 & 1.3269\end{pmatrix}, N=\begin{pmatrix}640.4151 & 0.1757\\ 0.1757 & 640.5371\end{pmatrix},$$
$$D_2=10^3\begin{pmatrix}1.2711 & -0.0189\\ -0.0189 & 1.2358\end{pmatrix}, T=\begin{pmatrix}576.4712 & 0.1817\\ 0.1817 & 576.4712\end{pmatrix},$$
$$P_1=\begin{pmatrix}6.4712 & 0.1817\\ 0.0063 & 6.1073\end{pmatrix}, P_2=10^3\begin{pmatrix}0.2937 & -2.8980\\ 2.8909 & 0.2729\end{pmatrix},$$
$$P_3=10^4\begin{pmatrix}-6.5251 & -1.3612\\ 1.2104 & -6.1325\end{pmatrix}, P_4=\begin{pmatrix}180.1920 & -3.7038\\ -6.3561 & 170.8700\end{pmatrix},$$
$$P_5=\begin{pmatrix}180.1558 & -3.7031\\ -6.3553 & 170.8356\end{pmatrix}, P_6=\begin{pmatrix}-5.9101 & -7.2824\\ -9.0571 & -16.8555\end{pmatrix},$$

$$P_7 = \begin{pmatrix} -15.9028 & 28.3103 \\ -62.4622 & 90.0565 \end{pmatrix}, P_8 = \begin{pmatrix} 313.4930 & -6.9121 \\ -8.0246 & 299.1290 \end{pmatrix},$$

$$P_9 = \begin{pmatrix} 313.4352 & -6.9107 \\ -8.0230 & 299.0739 \end{pmatrix}, P_{10} = 10^3 \begin{pmatrix} 9.6587 & 0.2730 \\ 0.0101 & 9.1147 \end{pmatrix},$$

$$P_{11} = 10^3 \begin{pmatrix} -1.4533 & 0.8576 \\ 0.5823 & -0.4302 \end{pmatrix}, P_{12} = 10^3 \begin{pmatrix} 1.0505 & 0.9696 \\ -0.9627 & 1.0561 \end{pmatrix},$$

$$P_{13} = 10^4 \begin{pmatrix} -6.4325 & -0.0071 \\ -0.1824 & -6.0699 \end{pmatrix}, P_{14} = 10^5 \begin{pmatrix} -2.6294 & 0.0123 \\ -0.0342 & -2.5546 \end{pmatrix},$$

$$P_{15} = \begin{pmatrix} -49.8768 & -7.7773 \\ -11.8317 & -54.6362 \end{pmatrix}, P_{16} = \begin{pmatrix} -49.8671 & -7.7759 \\ -11.8303 & -54.6258 \end{pmatrix},$$

$$P_{17} = \begin{pmatrix} -12.1795 & -19.3401 \\ -19.4301 & -29.2370 \end{pmatrix}, P_{18} = 10^3 \begin{pmatrix} 0.6554 & 0.3056 \\ 2.1045 & 1.2220 \end{pmatrix},$$

$$P_{19} = \begin{pmatrix} 27.6424 & -11.8826 \\ -11.8833 & 21.5242 \end{pmatrix}, P_{20} = \begin{pmatrix} 27.6582 & -11.8805 \\ -11.8811 & 21.5411 \end{pmatrix},$$

$$P_{21} = \begin{pmatrix} -51.8431 & 147.4216 \\ -142.4171 & -49.9921 \end{pmatrix}, P_{22} = 10^4 \begin{pmatrix} 2.0369 & 3.0629 \\ -1.0093 & -1.2254 \end{pmatrix},$$

Therefore, the zero solution of system (3.51) is $\alpha-$exponentially stable.

Example 3.1.3 Consider the neutral-type neural networks with interval time-varying delay $\dot{x}(t) - A_2\dot{x}(t-\eta(t)) = -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))$
which was considered in [13] where

$$A = \begin{pmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{pmatrix},$$

$$W_0 = \begin{pmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0364 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -0.4425 & 1.0562 & 1.5111 & -0.2044 \\ 0.3762 & -0.9163 & 0.5464 & 0.7713 \\ 1.8495 & 1.1717 & -0.7028 & 0.2815 \\ -1.1143 & 0.5008 & 1.1512 & 0.7171 \end{pmatrix},$$

$$F = \begin{pmatrix} 0.4557 & 0 & 0 & 0 \\ 0 & 0.6575 & 0 & 0 \\ 0 & 0 & 0.3734 & 0 \\ 0 & 0 & 0 & 0.8657 \end{pmatrix},$$

$$G = \begin{pmatrix} 0.1863 & 0 & 0 & 0 \\ 0 & 0.5317 & 0 & 0 \\ 0 & 0 & 0.6259 & 0 \\ 0 & 0 & 0 & 0.4096 \end{pmatrix}.$$

Table 3.1: The maximum allowable upper bound h_2 for Example 3.1.3

Method	h_1	μ	$\alpha = 0$	$\alpha = 1.5$
[12]	0	0	1.932	-
[5]	0	0	1.674	0.983
[16]	0	1.08	1.541	0.633
[23]	0	0	1.7674	0.5382
[13]	0.491	1.434	3.201	2.311
corollary 3.2.1	0	unknown	3.521	2.942

Table 3.1. gives comparison of maximum allowable value of h_2. We see that, the maximum allowable bounds for h_2 obtained from corollary 3.2.1 are much better than that obtained in [12], [5], [16], [23], [13].