CHAPTER IV

Square Grid Points Covered by Connected Sources with coverage radius one on a Two-dimensional Grid

So far, we know the limit number of grid points that the communication sources with coverage radius one can cover in both infinite grids and square grids. We also know the lower bound number of sources covering a square grid from the previous section. Here, however, in this chapter, we want to know how we position the communication source to cover an area. Therefore, we take some parts of a theoretical mobility model proposed by Greenlaw and Kantabutra for being our covering model. We exclude the initial location of each source and define the covering problem in general. GRID POINTS COVERAGE PROBLEM (GPC) is to find the minimum number of sources with various coverage to cover some terrain with/without obstacles. However, for our initial work we focus on a particular case of GPC, SQUARE GRID POINTS COVERAGE PROBLEM (SGPC). We focus the area on a square shape because of its symmetry and limited. SGPC want to minimize the number of sources with the coverage radius of one to cover a square grid size of p with the restriction that all sources must be communicable. We also restrict the area to have no obstacles. The status of SGPC is unknown because there are some parts of the work in [11] that is needed to be fixed, so we exclude the parts in this work. However, we keep an APPROX-SQUARE-GRID-POINTS-COVERAGE (ASGC) algorithm. ASGC compute the approximate solution of SGPC and will be useful if SGPC is intractable. ASGC uses the rule that any number can be obtained from the addition of 3, 4 and 5 and then combines 3-gadgets, 4-gadgets and 5-gadgets to specify the position of sources to cover a square grid point size of p. It achieves the approximation ratio of $1+\frac{p-2}{p^2+2}$ and $1+\frac{2p-10}{p^2+2}$ when $5 < p \le 8$ and $p > 8$, respectively.

4.1 Square Grid Points Coverage

Because we are interested in only how to lay the minimum number of sources in order to cover some particular grid points, we reduce M to be 3-tuples. So, our mobility model $M' = (S, C, O)$ is three tuples.

1. The set $S = \{s_1, s_2, ..., s_m\}$ is a finite collection of **sources**, where $m \in \mathrm{N}$. The value m is the **number of sources**. Corresponding to each source s_i, for $1 \le i \le m$.
2. The vector $C = \{c_1, c_2, ..., c_m\}$ is a finite collection of lengths. The value c_i is the corresponding radius of the circular coverage of source s_i. This vector is called the **coverages**.
3. The set $O = \{(x_1, y_1, x_2, y_2) \mid x_1, y_1, x_2, y_2 \in \mathrm{N}, x_2 > x_1 \text{ and } y_2 > y_1\}$ is a finite collection of rectangles in the plane. This set is called the **obstacles**. All coordinates $x_i, y_i \le o_{\max}$, where $o_{\max}$ is a constant in N.

Because our sources also need to have communication with others, we use the same communication protocol which is employed in the model M as stated in Chapter II. Note that we exclude the initial location of the sources in M' because we want to find the places. The GRID POINTS COVERAGE PROBLEM is defined as follows:

<u>GRID POINTS COVERAGE PROBLEM (GPC)</u>

INSTANCE: A mobility model M', a terrain P on a two-dimensional grid, and a variable $k \in \mathrm{N}$.

QUESTION: Can we lay at most k sources to cover the terrain P with the condition that all the sources must be communicable?

GPC is the problem to find the minimum number of sources with various coverage to cover some terrain with/without obstacles. A terrain P on a two-dimensional grid can be varieties. We also interested in laying the minimum number of sources with the coverage radius of one to cover a square grid with the restriction that all sources must be currently in range. The SQUARE GRID POINTS COVERAGE problem is defined as follows:

<u>SQUARE GRID POINTS COVERAGE PROBLEM (SGPC)</u>

INSTANCE: A mobility model M', a square grid size of p, and a variable $k \in \mathrm{N}$.

QUESTION: Can we lay at most *k* sources to cover a square grid size of *p* with the condition that all the sources must be communicable?

In SGPC, we are given a square grid size of *p* with or without obstacles on some positions. We are also given *k* sources with variety coverage and try to find whether we can lay these *k* sources to cover all grid points with the condition that all sources must be communicable. Here, however, we focus on the sources that have the same coverage radius of one to cover a square grid space without obstacle with the restriction that all sources must be communicable, and left the other general version for the future work. Therefore, our mobility model would be $M' = (S = \{s_1, s_2, ..., s_m\}, C = \{1,1,...,1\}, \phi)$.

Note: we position a grid point on (x, y) coordinate from the left to the right and from the top to the bottom. Therefore, the position $(1,1)$ is the top-left point in a square grid.

Because some parts of [11] have to be fixed, the status of SGPC is unknown. The proof to show that SGPC is an *NP*-complete problem is left open. If SGPC is intractable, the approximate algorithm that will be proposed in the next section shall be useful.

4.2 APPROXIMATION ALGORITHM FOR SGPC [11]

Because if SGPC is *NP*-complete, it tend to have no polynomial time algorithm to solve the problem. Therefore, we propose an algorithm called APPROX-SQUARE-GRID-POINTS-COVERAGE (ASGC) to lay sources on a square grid size of p when $p > 5$. The algorithm will give us the source positions and the number of the sources used to cover a square grid size of p. Because any number can be obtained from the addition of 3, 4 and 5, and we want to find the relation of covering a bigger square from some smaller squares, we introduce 3-gadgets, 4-gadgets and 5-gadgets. The three gadgets specify the position of sources to cover a square grid point size of 3, 4 and 5 respectively, as shown in Figure 4.1.

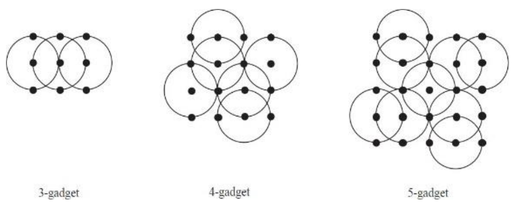


Figure 4.1 3-gadgets, 4-gadgets, and 5-gadgets

From Proposition 3.4.6, the number of sources which coverage radius of one on the 3-gadget is optimal, i.e. three sources. From Theorem 3.4.8, to cover a square grid size of 4 and 5 we must use at least $\frac{4^2+2}{3}$ = 6 and $\frac{5^2+2}{3}$ = 9 sources, respectively. Therefore, the six sources in a 4-gadget and the nine sources in a 5-gadget are optimal. The functions in ASGC is classified into three cases according to three possible fractions of dividing *p* by 3.

Table 4.1 APPROX-SQUARE-GRID-POINTS-COVERAGE pseudocode

```
APPROX-SQUARE-GRID-POINTS-COVERAGE(p)
if p mod 3 equal to 0 then
        FRACTIONZERO(p)
else if p mod 3 equal to 1 then
        FRACTIONONE(p)
else if p mod 3 equal to 2 then
        FRACTIONTWO(p)
end if
```

We shall call the series of grid points in the same *y* (*x*) axis "*a row (column) of grid points*," and call the connected 3-gadgets where all sources are centered on the same *y* (*x*) axis "*a row (column) of connected sources*." In case $p \bmod 3 = 0$, FRACTIONZERO shall be used. In this case, *p* can be written as the sum of 3's, so we use only 3-gadgets to cover our square grid points. We lay 3-gadgets along each three rows of grid points. Let the variables *r* and *c* represent the row number and the column number of grid points respectively. FRACTIONZERO will lay the top-left-corner-point of the 3-gadgets on every other three rows of grid points which have *r* = 1, 4, 7, ... , *p* − 2 and lay the top-left-corner-point on every other three columns which have *c* = 1, 4, 7, ... , *p* − 2, respectively. Because there are $\frac{p}{3}$ rows and $\frac{p}{3}$ columns to lay the 3-gadgets, we use $3*\frac{p}{3}*\frac{p}{3}=\frac{p^2}{3}$ sources to cover the square grid points size of *p*. However, the sources in the different rows of connected sources are uncommunicable, so we add two more sources to connect any two rows of connected sources. We center the new two sources on (2, *r*) and (2, *r* + 1), respectively, and remove one source in each row of connected sources that is in between the addition sources, all the sources are still communicable. We remove the same kind of sources in every row of connected sources except the first and the last rows.

Therefore, we shall add $2*\left(\frac{p}{3}-1\right)=2*\frac{p}{3}-2$ sources and remove $\frac{p}{3}-2$ sources. As a result, we use $\left(\frac{p^2}{3}\right)+2*\left(\frac{p}{3}\right)-2-\left(\frac{p}{3}-2\right)=\frac{p(p+1)}{3}$ sources to cover a square grid point size of p. FRACTIONZERO takes $O(p^2)$ which is in time polynomial, and the pseudocode of FRACTIONZERO is shown in Table 4.2.

Table 4.2 Pseudocode of FRACTIONZERO

```
FRACTIONZERO(p)
//lay 3-gadget along each three rows of grid points:
r, c ← 1
for r ← 1 to p − 2 do
        for c ← 1 to p − 2 do
                lay 3-gadget on (c, r)
                c = c + 3,
        end for
        r = r + 3
end for
//add two sources to connect each two rows of connected sources and delete some redundant
    sources
for r ← 2 to p − 7 do
        lay sources on (2, r) and (2, r + 1)
        delete source on (2, r + 2)
        r = r + 3
end for
lay sources on (2, r) and (2, r + 1)
```

The position of 3-gadgets and the example of laying the gadgets on some square grid are shown in Figure 4.2.

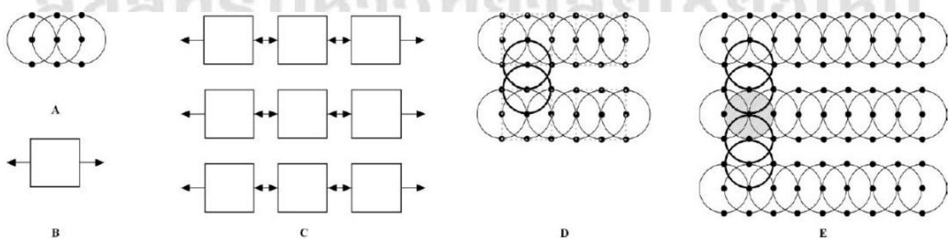


Figure 4.2 The example of laying 3-gadgets to cover the square grids size of 6 and 9

Figure 4.2-A represents a 3-gadget. Figure 4.2-B shows a box represents a 3-gadget. The end of the two arrows pointing out of the box represents the position of the points that the 3-gadget cover, but are outside the boundary of the square grid size of 3 that the 3-gadget

cover. Figure 4.2-C shows the position of laying 3-gadgets cover a square grid size of 9. Figure 4.2-D, E show the sources position obtained from FRACTIONZERO to cover square grid size of 6 and 9 respectively. The two additional sources which connect any two row of connected sources are shown by the thick circles, and the removed source is shown by the shaded circle.

In case $p \bmod 3 = 1$, FRACTIONONE shall be used. In this case p can be written as an addition of some 3s and a 4, so we use 3-gadgets and a 4-gadget. The pseudocode of FRACTIONONE is shown in Table 4.3.

Table 4.3 Pseudocode of FRACTIONONE

```
FRACTIONONE(p)
// lay 4-gadget along diagonal line:
c ← 1
for c ← 1 to p − 3 do
        lay 4-gadget on (c, c)
        FLIP(4-gadget)
        c = c + 3
end for
// lay 3-gadget along each three rows of grid points on the left of the 4-gadgets:
c_max ← 1
while c_max ≤ p − 6 do
        for c ← 1 to c_max do
                lay 3-gadget on (c, c_max + 4)
                c = c + 3
        end for
        c_max = c_max + 3
end while
// lay 3-gadget along each three rows of grid points on the right of the 4-gadgets
c_min ← 5
while c_min ≥ p − 2 do
        for c ← c_min to p − 2 do
                lay 3-gadget on (c, c_min − 4)
                c = c + 3
        end for
        c_min = c_min + 3
end while
```

First, we lay 4-gadgets along a diagonal line in a square grid. FRACTIONONE will lay the top-left-corner-point of 4-gadgets along a diagonal line in a square grid in the position (r, c) where $r = c = (1, 4, 7, \ldots, p - 3)$. We also vertically mirror every other 4-gadgets to make all the 4-gadgets communicable. Therefore, we use $\frac{p-1}{3}$ of 4-gadgets. Because each 4-gadget has 6 sources, we use $\left(\frac{p-1}{3}\right)*6 = 2(p-1)$ sources. The positions of laying 4-gadgets will separate uncovered grid points into the left side and the right side of the communicable 4-gadgets. On the left side of the 4-gadgets, we start laying the top-left-corner-point of one 3-gadget on the row number equal to 5. We increase laying one gadget in every row until the row number is equal to $p - 6$ where we have $\frac{p-4}{3}$ of 3-gadgets laid in this row. Therefore, we use $\sum_{n=1}^{\frac{p-4}{3}} n$ of 3-gadgets or use $3*\sum_{n=1}^{\frac{p-4}{3}} n$ sources. On the right side of the 4-gadgets, we start laying $\frac{p-4}{3}$ of 3-gadgets on the column number equal to 5, we increase column number by three and decrease one gadget in every row until there is only one 3-gadget in the last row. Therefore, we use $\sum_{n=1}^{\frac{p-4}{3}} n$ of 3-gadgets or use $3*\sum_{n=1}^{\frac{p-4}{3}} n$ sources. The number of sources in the 3-gadgets laying on both side of the 4-gadgets are equal. As a result, we use $2(p-1)+2*3*\sum_{n=1}^{\frac{p-4}{3}} n = \frac{(p-1)(p+2)}{3}$ sources to cover a square grid size of p and FRACTIONONE takes $O(p^2)$ which is in time polynomial. The position of 3-gadgets, 4-gadgets and the examples of laying the gadgets on the square grids size of 10 are shown in Figure 4.3.

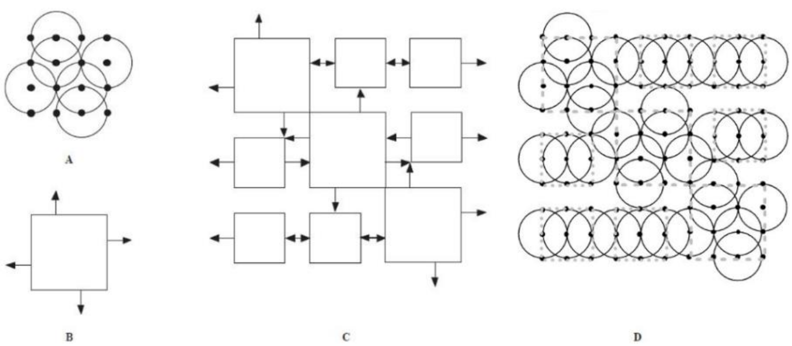


Figure 4.3 Examples of laying 3-gadgets and 4-gadgets
to cover square grids size of 10

Figure 4.3-A represents a 4-gadget. Figure 4.3-B shows a box represents a 4-gadget. The end of the arrows pointing out of the box represent the position of the points that the 4-gadget covers, but outside the square grid size of 4 under the 4-gadget. Figure 4.3-C, D

show the position of laying 3-gadgets and 4-gadgets, and the sources positions obtained from FractionOne to cover square grid size of 10, respectively.

In case $p \bmod 3 = 2$, FractionTwo shall be used. In this case p can be written as an addition of some 3s and a 5, so we use 3-gadgets and a 5-gadget. The pseudocode of FractionOne is shown in Table 4.4.

Table 4.4 The pseudocode of FractionTwo

```
FractionTwo(p)
// lay 5-gadget along diagonal line:
for c ← 1 to p − 4 do
        lay 5-gadget on (c, c)
        c = c + 3
end for
// lay 3-gadget along each three rows on the left of the 5-gadgets
c_max ← 1
while c_max ≤ p − 7 do
        for c ← 1 to c_max do
                lay 3-gadget on (c, c_max + 5)
                c = c + 3
        end for
        c_max = c_max + 3
end while
// lay 3-gadget along each three rows on the right of the 5-gadgets
c_min ← 6
while c_min ≥ p − 2 do
        for c ← c_min to p − 2 do
                lay 3-gadget on (c, c_min − 5)
                c = c + 3
        end for
        c_min = c_min + 3
end while
```


First, we lay 5-gadgets along a diagonal line in a square grid. FractionTwo lay the top-left-corner-point of 5-gadgets along a diagonal line in a square grid on the position (r, c) where $r = c = (1, 4, 7, \ldots, p - 4)$. In this part, we use $\frac{p-2}{3}$ of 5- gadgets. Since each 5-gadget has 9 sources, we use $\frac{p-2}{3} * 9 = 3(p-2)$ sources. The positions of laying 5-gadgets will separate uncovered grid points into the left side and the right side of the communicable 5-gadgets. On the left side of the 5-gadgets, we start laying the top-left-corner-point of one 3-gadget on the row number equal to 6. We increase laying one gadget in every row until the row number is equal to $p-7$ where we have $\frac{p-5}{3}$ of 3-gadgets laid in this row. Therefore, we use $\sum_{n=1}^{\frac{p-5}{3}} n$ of 3-gadgets or use $3*\sum_{n=1}^{\frac{p-5}{3}} n$ sources. On the right side of the 5-gadgets, we start laying $\frac{p-5}{3}$ of 3-gadgets on the column number equal to 6, we increase column number by three and decrease one gadget in every row until there is only one 3-gadgets in the last row. Therefore, we use $\sum_{n=1}^{\frac{p-5}{3}} n$ of 3-gadgets or use $3*\sum_{n=1}^{\frac{p-5}{3}} n$ sources. The number of sources in 3-gadgets laying on both sides of the 5-gadgets are equal. As a result, we use $2(p-1)+2*3*\sum_{n=1}^{\frac{p-5}{3}} n = \frac{(p-2)(p+4)}{3}$ sources to cover a square grid size of p and FRACTIONTWO will take $O(p^2)$ which is in time polynomial. The position of 3-gadgets, 5-gadgets and some examples of laying the gadgets on the square grids size of 11 are shown in Figure 4.4.

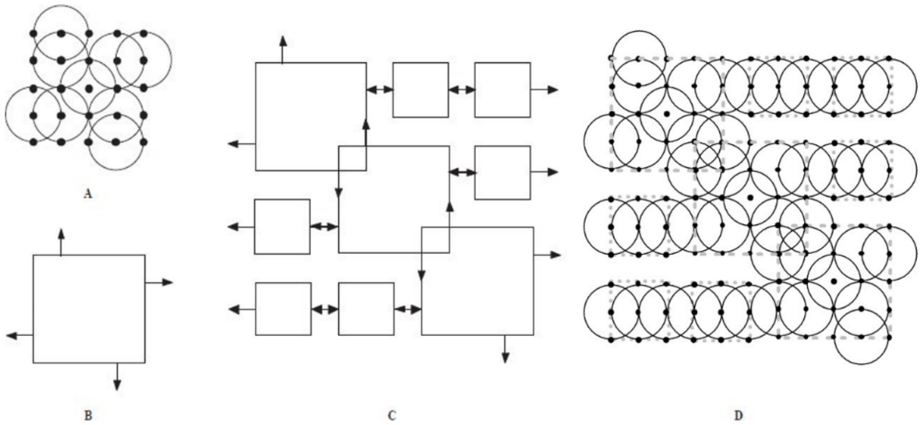


Figure 4.4 Examples of laying 3-gadgets and 5-gadgets to cover square grids size of 11

Figure 4.4-A represents a 5-gadget. Figure 4.4-B shows a box represents a 5-gadget. The end of the arrows pointing out of the box represent the position of the points that the 5-gadget covers, but outside the square grid size of 5 under the 5-gadgets. Figure 4.4-C, D show the position of laying 3-gadgets and 5-gadgets, and the source positions obtained from FRACTIONTWO to cover a square grid size of 11, respectively.

Next, we will show the approximation ratio of ASGC.

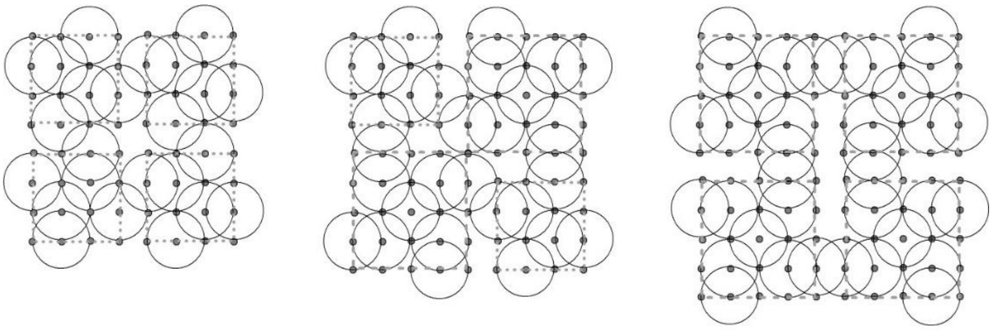
Theorem 4.3.1 IF SQUARE GRID POINTS COVERAGE is *NP*-complete, APPROX-SQUARE-GRID-POINTS-COVERAGE is $\left(1+\frac{p-2}{p^2+2}\right)$-*approximation algorithm when* $5<p\le 8$ and $\left(1+\frac{2p-10}{p^2+2}\right)$-*approximation algorithm when* $p>8$.

Proof. We have already shown that ASGC runs in polynomial time. Let $|A^*|$ denote the minimum number of sources to cover a square grid size of p. Let $|A|$ denote the number of sources used to cover a square grid size of p obtained from ASGC. From considering all three cases in ASGC, the number of sources use for covering a square grid size of p are equal to $\frac{p(p+1)}{3}$, $\frac{(p-1)(p+2)}{3}$, and $\frac{(p-2)(p+4)}{3}$, or equal to $\frac{p^2+p}{3}$, $\frac{p^2+p}{3}-\frac{2}{3}$, $\frac{p^2+p}{3}+\frac{p-8}{3}$ in case $p \bmod 3 = 0$, 1, and 2, respectively. Therefore, $|A|\le\frac{p^2+p}{3}$ when $5<p\le 8$ and $|A|\le\frac{p^2+p}{3}+\frac{p-8}{3}$ when $p>8$. From Theorem 3.4.8, we know that $\frac{p^2+2}{3}\le|A^*|$. Therefore, $\frac{|A|}{|A^*|}\le\frac{p^2+p}{p^2+2}=1+\frac{p-2}{p^2+2}$ when $5<p\le 8$ and $\frac{|A|}{|S^*|}\le\frac{p^2+2p-8}{p^2+2}=1+\frac{2p-10}{p^2+2}$ when $p>8$. Thus, the approximation ratio are equal to $\left(1+\frac{p-2}{p^2+2}\right)$ and $\left(1+\frac{2p-10}{p^2+2}\right)$ when $5<p\le 8$ and $p>8$, respectively □

So far, we know an approximate algorithm for SGPC which give the approximation ratio equal to $\left(1+\frac{p-2}{p^2+2}\right)$ and $\left(1+\frac{2p-10}{p^2+2}\right)$ when $5<p\le 8$ and $p>8$, respectively. This approximation algorithm will be useful if SGPC is intractable.

Note that we use 3, 4, and 5-gadgets to cover a square grid size of p from the intuition that the smaller gadgets would cover an area better than the bigger ones. However, a number can also be obtained from the addition of 3, 4 and 5 which is not from the addition of 3s, the addition of 3s and a 4, or the addition of 3s and a 5. For example, 8 = 4+4, 9 = 4+5, and 10 = 5+5. We can apply our 3, 4 and 5 gadgets to cover the square grid size of

8, 9, and 10 as shown in Figure 4.5. However, we left the approximation algorithm to cover the bigger square and its approximation ratio for the future work.



A B C

Figure 4.5 Examples of laying 4-gadgets and 5-gadgets
to cover square grids size of 8, 9, 10

4.3 Summary

This chapter, we focus on the point coverage problem. We take some part of Greenlaw and Kantabutra's mobility model to be our model and proposed GRID POINTS COVERAGE PROBLEM (GPC). However, we are interested in the case of covering square grid points without obstacles by sources with coverage radius of one with the restriction that all sources must be communicable, SQUARE GRID POINTS COVERAGE PROBLEM (SGPC) with $M' = (S = \{s_1, s_2, ..., s_m\}, C = \{1,1,...,1\}, \phi)$. An approximation algorithm to compute an approximate solution to SGPC is also proposed. APPROX-SQUARE-GRID-POINTS-COVERAGE (ASGC) gives an approximate solution to SGPC and gives the approximation ratio equal to $\left(1+\frac{p-2}{p^2+2}\right)$ and $\left(1+\frac{2p-10}{p^2+2}\right)$ when $5 < p \leq 8$ and $p > 8$, respectively. Note that ASGC will be useful if SGPC is *NP*-complete.