CHAPTER V

MULTI-SOURCES SIMULTANEOUS COMMUNICATION is *NP*-complete

In this chapter, we will turn our attention into the mobility model in a three-dimensional grid for more real. The new model is extended from the mobility model of Greenlaw and Kantabutra in a two-dimensional grid. Because there are many users want to communicate with their partners at the same time on a share mobile wireless network, the wireless provider should be able to support these. Particularly, we study a problem called MULTI-SOURCES SIMULTANEOUS COMMUNICATION PROBLEM (MSSCP) in this model. This problem is stated as follows: given a mobility model $M = (S, D, U, L, R, V, C, O)$, k pairs of distinct sources $\{s_1, s_1'\}, \{s_2, s_2'\}, \ldots \{s_k, s_k'\}$, and a time $t \in \mathrm{N}$, can all k pairs of sources simultaneously communicate throughout the duration t of the model without sharing a source? We show that the complexity of this problem is at least as hard as the ONE-IN-THREE 3-SATISFIABILITY. This chapter is organized as follows. Section 5.1 extends the wireless mobility network model in [14] to a three-dimensional grid and introduces a communication protocol in the network. In Section 5.2 MSSCP is defined and the intuitive reduction is discussed. In Section 5.3 all necessary gadgets in the construction are introduced and the detailed construction of the instance from these gadgets is described in Section 5.4. Section 5.5 then shows the *NP*-completeness proof of the MSSCP and its related lemmas.

5.1 Three-Dimensional Mobility Model [15, 16]

Let N be $\{1,2,3,\ldots\}$. A *mobility model* $M = (S, D, U, L, R, V, C, O)$ in a three-dimensional grid is described as follows.

1. The set $S = \{s_1, s_2, \ldots, s_\eta\}$ is a finite collection of *sources*, where $\eta \in \mathrm{N}$. The value η is the *number of sources*. Corresponding to each source s_i, for $1 \le i \le \eta$, an *initial location* (x_i, y_i, z_i) is specified, where $x_i, y_i, z_i \in \mathrm{N}$

2. The set $D = \{0000, 0001, 1000, 0100, 0010, 0110, 1001\}$ is called the *directions*, and these values correspond to no movement, east, west, south, north, up, and down, respectively.

3. The set $U = \{u_1, u_2, \ldots, u_p\}$ is a finite collection of *mobile devices*, where $p \in \mathrm{N}$. The set U is called the set of *users*. The value p is called the *number of users*. Corresponding to each user u_i, for $1 \le i \le p$, an initial location (x_i, y_i, z_i) is specified, where $x_i, y_i, z_i \in \mathrm{N}$.

4. The vector $L = \{l_1, l_2, \ldots, l_p\}$ is a finite collection of "bit strings," where $\tau \in \mathrm{N}$ and $l_i \in D^{\tau}$ for $1 \le i \le p$. Each group of four bits in l_i beginning with the first four defines a step in a given direction for the user u_i's movement or no movement at all if the string is 0000. The value τ is called the *duration of the model*.

5. Let $t(i) \in \mathrm{N}$ for $1 \le i \le \eta$. The vector $R = \{r_1, r_2, \ldots, r_\eta\}$ is a finite collection of "bit strings," where $r_i \in D^{t(i)}$. Each group of four bits in r_i beginning with the first four defines a step in a given direction for the movement of the source s_i or no movement at all if the string is 0000. The vector R is called the *walks* of the mobility model.

6. The vector $V = \{v_1, v_2, \ldots, v_\eta\}$ is a finite collection of numbers, where $v_i \in \mathrm{N}$. The value v_i is the corresponding number of steps from r_i per unit time that s_i will take (i.e., $|r_i|$ modulo $4v_i = 0$). This vector is called the *velocities*.

7. The vector $C = \{c_1, c_2, \ldots, c_\eta\}$ is a finite collection of lengths, where $c_i \in \mathrm{N}$. The value c_i is the corresponding radius of the spherical coverage of source s_i. This vector is called the *coverages*. Each $c_i \le c_{\max}$, where $c_{\max}$ is a constant in N.

8. The set $O = \{o_1, o_2, \ldots, o_d\}$, where $o_i = (x_1, y_1, x_2, y_2, z_1, z_2) \in \mathrm{N}^6$ and $x_2 > x_1$, $y_2 > y_1$, $z_2 > z_1$ is a finite collection of rectangular solids. This set is called the *obstacles*. All $x_i, y_i, z_i \le n_{\max}$, where $n_{\max}$ is a constant in N.

Next, definitions of coverage representation, obstacle representation, and overlapping coverage area in a three-dimensional grid are given.

Definition 5.1.1 (Coverage Representation) *A coverage of radius c in a three-dimensional grid is represented by the set of three-dimensional lattice points within the spherical coverage and on the boundary.*

Definition 5.1.2 (Obstacle Representation) *An obstacle in a three-dimensional grid is represented by the set of three-dimensional lattice points within the obstacle and on the boundary.*

Definition 5.1.3 (Overlapping Coverage Area) *Let s, s' be a coverage or an obstacle in a three-dimensional grid and $s \cap s' = z$. s overlaps s' if and only if $|z| \ge 2$. z is called an overlapping coverage area.*

It is important to note that if the three-dimensional grid is fine enough, this model could represent a real world model. However, the reader should keep in mind that this model is theoretical and simplified, so inherently it cannot be completely realistic. We now turn our attention to the communication protocol which will allow us to illustrate how the model is used. The following communication protocol is needed so that the model works as intended. The sources are always on; they are always broadcasting and listening. Users with mobile devices are moving in and out of the range of each other and various sources. Mobile devices would like to communicate (send and receive messages) with one another. We specify the manner in which they may communicate in what follows. Let $k > 2$ and $k \in \mathrm{N}$.

- At a given instance in time any two sources with overlapping coverages may communicate with each other in full-duplex fashion as long as the intersection of their overlapping coverages is not completely contained inside obstacles. We say that these two sources are *currently in range*. A series $s_1, s_2, \ldots, s_k$ of sources are said to be *currently in range* if s_i and s_{i+1} are currently in range for $1 \le i \le k-1$.

- At any given time, one source can only communicate with one other source.
- Two mobile devices cannot communicate directly with one another.
- A mobile device D_1 always communicates with another mobile device D_2 through a source or series of sources as defined next. The mobile devices D_1 at location (x_1, y_1, z_1) and D_2 at location (x_2, y_2, z_2) *communicate through a single source* s located at (x_3, y_3, z_3) if at a given instance in time the lines between points (x_1, y_1, z_1) and (x_3, y_3, z_3) and points (x_2, y_2, z_2) and (x_3, y_3, z_3) are within the coverage of s, and do not intersect with any obstacle from O. The mobile devices D_1 at location (x_1, y_1, z_1) and D_2 at location (x_2, y_2, z_2) *communicate through a series of sources* s_1 at location (e_1, f_1, g_1), s_2 at location $(e_2, f_2, g_2), \ldots,$ and s_k at location (e_k, f_k, g_k) that are currently in range if the line between points (x_1, y_1, z_1) and (e_1, f_1, g_1) is inside s_1's coverage and does not intersect any obstacle s_k from O and the line between points (x_2, y_2, z_2) and (e_k, f_k, g_k) is inside s_k's coverage and does not intersect any obstacle from O.

In the next section we look at the definition of the MSSCP in the wireless mobility model.

5.2 Multi-Sources Simultaneous Communication Problem [15]

In a military operation, where an attack is strategically planned in advance, military personnel are arranged in a certain movement pattern and several pairs of military personnel may need to communicate with each other throughout the operation. Given this plan of the attack, the military strategist may want to know prior to the operation whether it is possible that all pairs of military personnel will be able to stay in contact throughout the operation. Observe that while the attack is in progress, the area where the operation is taking place may have some mountains or high-rise buildings that may obstruct the communication. In addition, because each battery has limited power, we want to avoid sharing a source. One can certainly imagine a similar scenario in a rescue operation. In this section we introduce the problem called Multi-Sources Simultaneous Communication Problem that exactly models this situation. This problem is defined as follows.

Multi-Sources Simultaneous Communication Problem (MSSCP)

Instance: A mobility model $M = (S, D, U, L, R, V, C, O)$, k pairs of distinct sources $\{s_1, s_1'\}, \{s_2, s_2'\}, \ldots \{s_k, s_k'\}$, and a time $t \in \mathrm{N}$.

Question: Can all *k* pairs of sources simultaneously communicate throughout the duration *t* of the model without sharing a source?

Our goal is to show that the MSSCP is *NP*-complete. We will make a reduction from the One-In-Three 3-Satisfiability (3SAT) stated below. 3SAT is known to be *NP*-complete [20].

One-In-Three 3-Satisfiability (3SAT)

Instance: Collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses on a finite set $U = \{u_1, u_2, \ldots, u_n\}$ of variables such that $|c_i| = 3$ for $1 \le i \le m$ and each variable in c_i is unique.

Question: Is there a truth assignment for U so that each clause has exactly one true literal?

Observe that this version of One-In-Three 3-Satisfiability where each variable in c_i is unique remains *NP*-complete. A simple reduction from the original One-In-Three 3-Satisfiability to this version of One-In-Three 3-Satisfiability can be easily done by appropriately introducing new variables into original clauses. Let us now turn our attention to the reduction of the MSSCP. Because the reduction is rather complex, let us look at a simple example that intuitively explains the communication scheme of the MSSCP before we actually get into the details of the construction in Section 5.5. Figure 5.1 illustrates a schematic reduction of an instance of 3SAT to an instance of MSSCP. The 3SAT instance used in the reduction is $(u_1 \vee \bar{u}_2 \vee u_4)(u_1 \vee u_2 \vee \bar{u}_3)(\bar{u}_2 \vee u_3 \vee u_4)$.

Several remarks are in order. In Figure 5.1 a wire denotes a series of overlapping coverages that enables communication between sources in the model. If two sources are at the ends of a wire, they can communicate through the wire. If two wires intersect at a certain coordinate, the two wires are connected at that particular coordinate. We use a half-circle mark to denote non-connecting wires. In the context of our communication model two wires connected at a coordinate imply a sharing of a single source of communication at that coordinate. On the other hand, non-connecting wires imply no sharing of communication sources. Additionally, a black dot denotes two sources at an identical coordinate. All sources that are used in the proofs are named.

At this point we momentarily ignore the formality of constructing the mobility model M because our purpose is to explain the communication scheme used in the reduction. In this particular instance of MSSCP 34 pairs of sources in consideration are to be the union of $\{S_{i,j}, T_{i,j}\}$ for $1 \le i \le 4$ and some $j \in \{1,2,3\}$, $\{S_{C_i}, T_{C_i}\}$ for $1 \le i \le 3$, $\{S_{u_i}, T_{u_i}\}$ for $1 \le i \le 4$, $\{A_{i,j}, B_{i,j}\}$ for $1 \le i \le 4$ and some $j \in \{1,2,3\}$, and $\{C_{i,w}, D_{i,w}\}$ for $1 \le i \le 4$ and $1 \le w \le 3$. In addition, we only consider a single snapshot of the model. In other words, let $t = 1$.

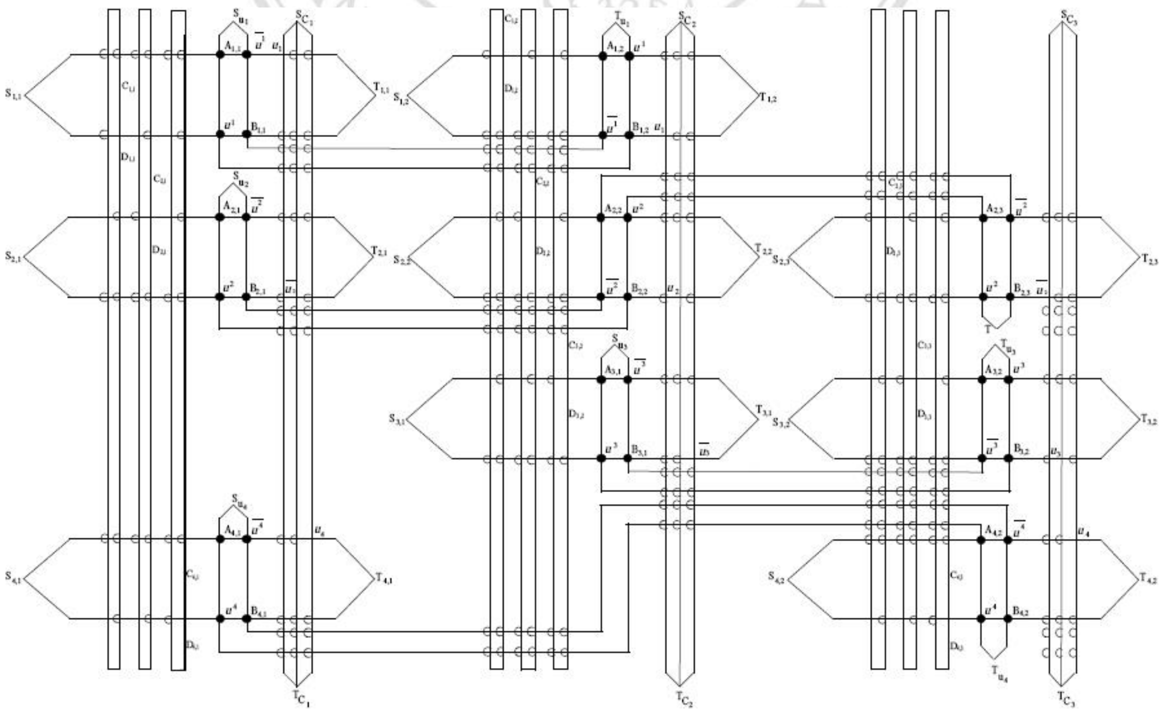

Figure 5.1 Communication scheme in the reduction

In Figure 5.1 observe that all 34 pairs of sources simultaneously communicate throughout the duration 1 of the model without sharing sources if and only if there is a truth assignment for U that satisfies all the clauses in C in exactly 1 in 3 manner.

5.3 Introducing Gadgets

Before we get into details of the reduction construction, a few remarks are in order. Since we are working with three-dimensional space, we need to specify coordinate axes in order to view our construction. Throughout this article, we define x, y and z axes as in Figure 5.2. These axes together define top, left, and front views. In what follows we will occasionally refer to these views when we discuss the construction.

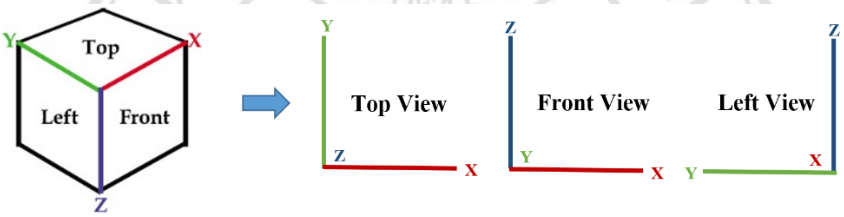

Figure 5.2 Axes x, y and z that define top, left, and front view

In this section we introduce 3 gadgets in the construction: *variable*, *clause*, and *consistency* gadgets. A variable gadget corresponds to a literal in a clause. A clause gadget groups 3 variable gadgets together and corresponds to a clause in the 3SAT. A consistency gadget ascertains that each variable gadget is set in a consistent manner. These three types of gadgets permit us to simulate the 3SAT with the mobility model. These gadgets are composed of a number of sources of different radii. Each source is presented by a spherical coverage and is shown in three dimensions. The smaller source has $r = 1$ and the bigger sources has $r = 2$. All sources in variable, clause, consistency gadgets are placed at $z = a + 6, z = a + 8, z = a$, respectively, where $a \in \mathrm{N}$. Therefore, the consistency gadgets are positioned under the variable gadgets, and both the consistency and the variable gadgets are positioned under the clause gadgets. Notice that the sources in different gadgets are uncommunicable.

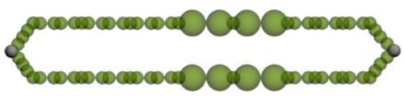

Figure 5.3 Variable gadget

The variable gadget is shown in Figure 5.3. It composed of 68 sources with coverage radius of one and 8 sources with coverage radius of two. There are some positions that the sources do not overlap. The variable gadget has two paths of sources, namely *upper path* and *lower path*, to communicate between the leftmost source and the rightmost source, $S_{i,j}, T_{i,j}$. Both paths have 5.2.3.2.3.2.3.3.3.3.3.5 sources, from the left to the right, "." is the location that the sources do not overlap. Choosing each path corresponds to setting a corresponding variable to true or false, depending upon categories of variable gadgets and how literals are assigned. This setting will be clarified momentarily.

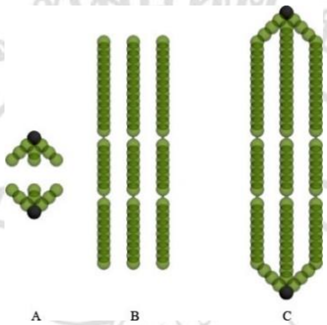

Figure 5.4 Clause gadget

Figure 5.4-C illustrates a clause gadget. A clause gadget compose of the *end part* and the *middle part*, as shown in Figure 5.4-A, B, respectively. There are 9 sources in the end part, and there are 13.7.9 sources in the middle part, from the top to the bottom. Like the variable gadget, this gadget comprises a number of sources and there are some positions that the sources do not overlap. Each source and its spherical coverage are shown in three dimensions. Three paths of overlapping sources are used to communicate between a communication pair S_{C_i}, T_{C_i} positioned at the topmost source and the bottommost source. Choosing each path corresponds to setting a corresponding variable in the clause to an appropriate truth value. This gadget extends to cover three variable gadgets in the clause by connecting the middle part vertically.

A consistency gadget is shown in Figure 5.5-D. A consistency gadget composed of the *end part*, *middle part*, and the *horizontal part* as show in Figure 5.5-A, B, C, respectively. There are two communication path on the left and right of the dark green source of the end part. The shorter path has 6.3.3 sources and the longer path has 6.3.6 sources. The middle part composed of 6.3.3 and 3.3.6 sources, and the horizontal part has 50 sources and 66 sources on the shorter and longer path, respectively.

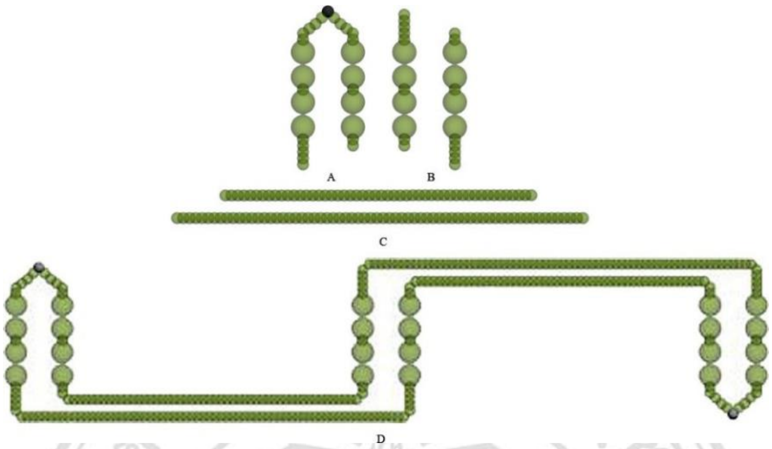

Figure 5.5 Consistency gadget

We combined these three parts together to cover all relevant variable gadgets. This gadget is made up of a number of sources with different radii. The small one has $r = 1$ and the big one has $r = 2$. The communication pair is S_{u_i}, T_{u_i} positioned at the end of the consistency gadget represented by the dark green sources. Observe that two paths of communication between the pair of dark green sources exist. Choosing each path corresponds to setting a truth value in a consistent manner. When consistency gadget is connected to a variable gadget, this truth value corresponds to the value chosen in the variable gadget.

Before we describe the composition of these gadgets in greater detail, we would like to note the shape similarity of these gadgets to the corresponding communication substructures in Figure 5.1. In the next section we will build an instance of the MSSCP from an instance of 3SAT by applying this scheme of communication.

5.4 Instance Construction

This section elaborates, given an instance of the 3SAT, how a corresponding instance of the MSSCP can be constructed in polynomial time using the 3 gadgets introduced in the previous section. The algorithm to construct a MSSCP instance can be described as follows. We note that the construction is viewed in a frame of n rows and m columns, where $|U| = n$ and $|C| = m$. In the construction, the row numbers are in the increasing order from top to bottom and the column numbers are in the increasing order from left to right. The first phase is to place $3m$ variable gadgets in the $n \times m$ frame. Notice that each path of the variable gadget has 5.2.3.2.3.2.3.3.3.3.3.5 sources, there are three sources with $r = 2$ and some positions that the sources do not overlap, if $S_{i,j}$ is positioned on $(x, y, a + 6)$, $T_{i,j}$ will be position on $(x + 57, y, a + 6)$. Therefore, along each row, if we lay $S_{i,j}$ on $(x, y, a + 6)$, we will lay $S_{i,j+1}$ on $(x + 61, y, a + 6)$. Along each column, if we lay $S_{i,j}$ on $(x, y, a + 6)$, we will lay $S_{i+1,j}$ on $(x, y + 31, a + 6)$. See Figure 5.6 for an example that corresponds to the reduction in Figure 5.1.

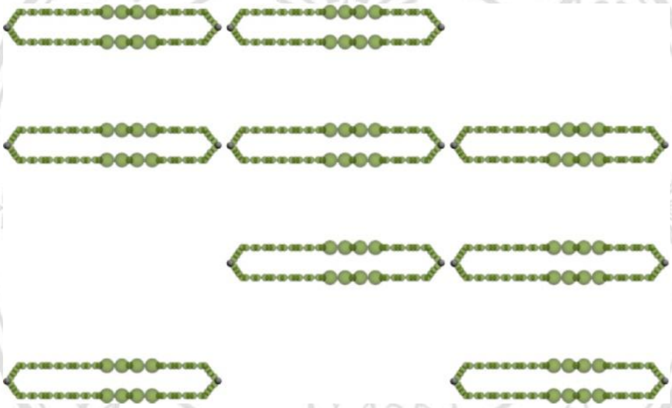

Figure 5.6 First phase in the construction

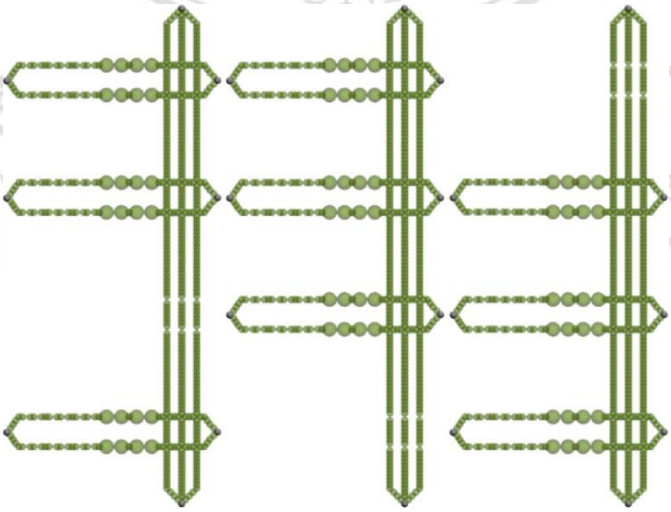

Figure 5.7 Second phase in the construction

The second phase is to place m clause gadgets in the $n \times m$ frame. We lay the six un-overlap position of the middle part of the clause gadget over the six rightmost un-overlap position of the variable gadget. That is, if the two un-overlap sources of the variable gadget are centered on $(x, y, a+6)$ and $(x+2, y, a+6)$, the corresponding two un-overlap sources of the consistency gadget will be on $(x+1, y-1, a+8)$ and $(x+1, y+1, a+8)$. We extend each clause, by connecting the middle part, to covers three relevant variables (See Figure 5.7).

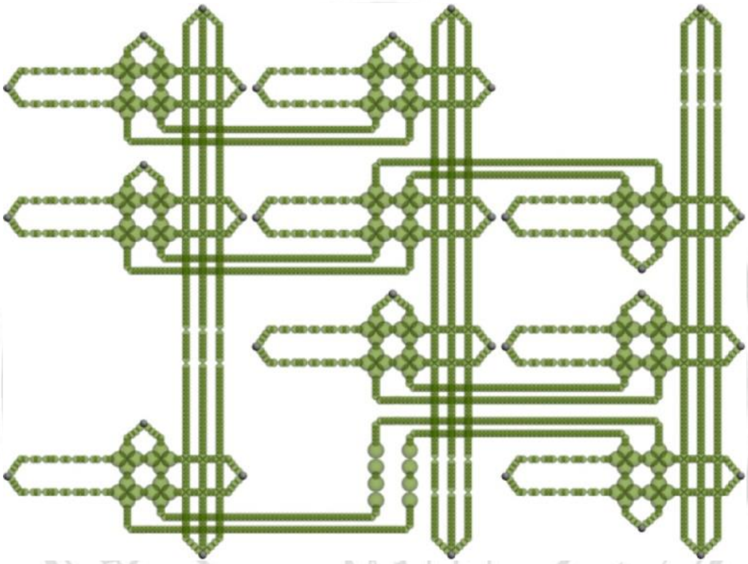

Figure 5.8 Third phase in the construction

In the next phase of the construction, we place n consistency gadgets in the $n \times m$ frame (See Figure 5.8). We let the eight un-overlap position of the bigger sources of the consistency gadget positioned under the eight un-overlap position of the bigger sources of the variable gadget. Each consistency gadget is alternately extended downward and upward respectively along the m columns. It always does downward on the first variable gadget.

Once three types of gadgets are put in the $n \times m$ frame as in Figure 5.8, three additional adjustments are needed to make the construction effectively simulate the 3SAT instance. The adjustments will make the sources on the different types of gadgets communicable in some ways. These adjustments are shown in Figure 5.9.

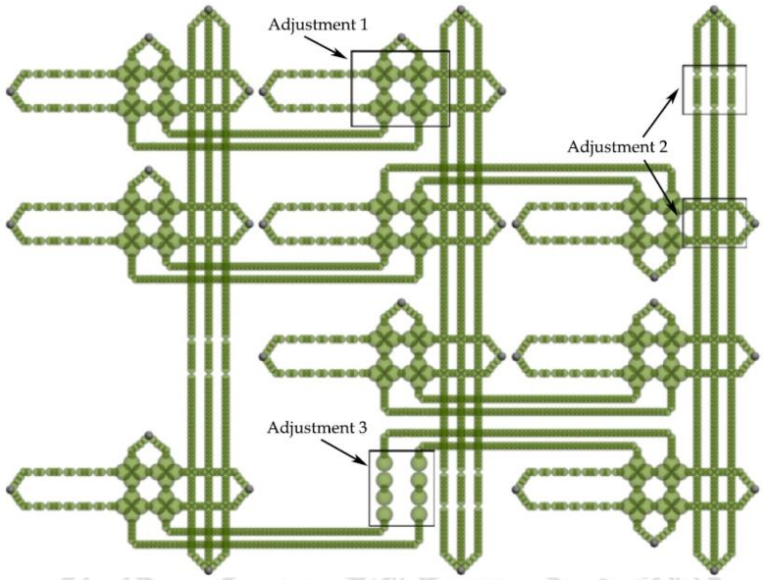

Figure 5.9 Adjustment phase in the construction

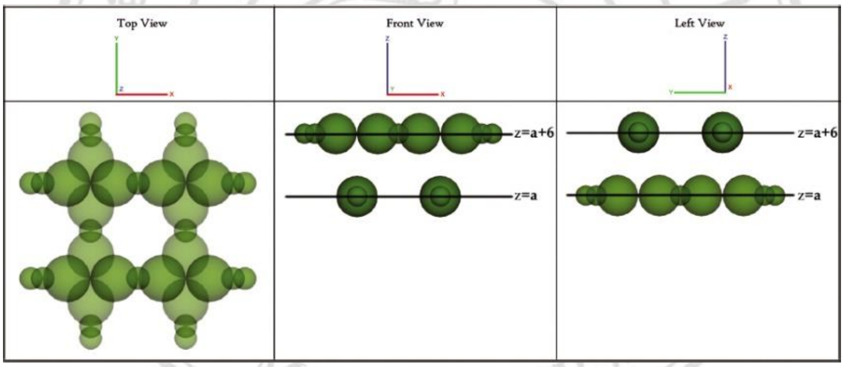

Figure 5.10 The part connecting variable and consistency gadgets

Figure 5.10 illustrates upclose the part of the construction where Adjustment 1 will be applied. This part shows where the connection between variable and consistency gadgets will be made. Observe that in this figure sources in consistency and variable gadgets cannot communicate to each other. In Figure 5.1 this part is where the 4 black dots are located. We will modify this part to have the effect of the 4 black dots. Observe that these 4 black dots allow simultaneous communication at their coordinates.The modification at each variable and consistency gadget pair is classified into two groups, depending upon source locations. The first is the group called $A_{i,j}$ and $B_{i,j}$. The second is the group called u^i and $\bar{u}^i$. In our modification, we use the sources with $r = 4$ centered on $z = a + 3$, and denote each source in the first group with a black coverage and each source in the second group with a green coverage.

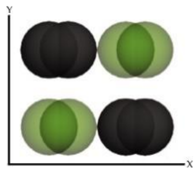

Figure 5.11 Sources in the Adjustment 1 from the top view

Figure 5.11 illustrates the top view of the sources that will be placed in the construction in order for the simulation to have the same effect as the four black dots in Figure 5.1. Each pair of overlapping sources is placed to the corresponding location (i.e., the black dot) in the construction. After the placement, the views at each location are shown in Figure 5.12. Figure 5.12 only shows the placement of sources at a location in the group $A_{i,j}$ and $B_{i,j}$. If the bigger sources of the variable gadget centered on $(x, y, a+6)$ and $(x+4, y, a+6)$, the bigger sources of the consistency gadget would be centered on $(x+2, y-2, a)$ and $(x+2, y+2, a)$. We will lay the two sources of Adjustment 1 on $(x+1, y, a+3)$ and $(x+3, y, a+3)$. The placement of sources at a location in the group u^i and $\bar{u}^i$ is precisely the same except black coverages become green coverages and is shown in Figure 5.13.

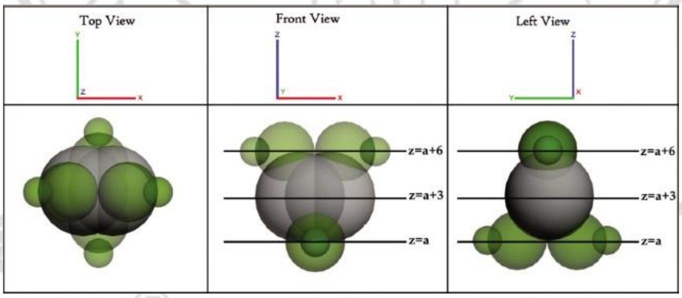

Figure 5.12 Modification at a location in the group $A_{i,j}$ and $B_{i,j}$

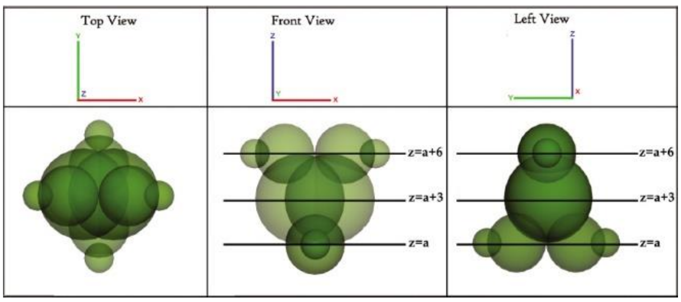

Figure 5.13 Modification at a location in the group u^i and $\bar{u}^i$

Next we discuss Adjustment 2. The part of the construction where Adjustment 2 will be applied is shown upclose in Figure 5.14.

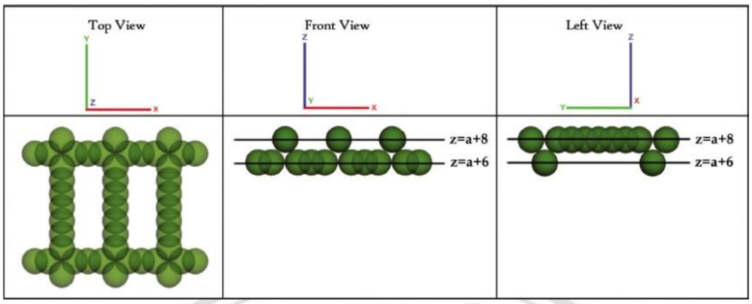

Figure 5.14 The part connecting Variable and Clause Gadget

This part is where clause and variable gadgets are connected. In Figure 5.1 this part comprises six locations indicated by five half-circle marks (non-connecting wires) and one intersected coordinate (connected wires). Observe that there are seven possible configurations of connection in this part, depending upon the location of connected wires plus one without a variable gadget connection. The top views of these corresponding configurations in the construction are illustrated in Figure 5.15. From this view, there are six positions of the sources and the sources are colored by red and green. In the Configuration 1 to 6, the red coverage are the lower-right, lower-left, upper-middle, lower-middle, upper-right, and upper-left sources of the six sources, respectively. The red coverage will make the sources on the clause gadget and the variable gadget communicable. For any $x, y, a \in \mathrm{N}$, the red coverage will be on $z = a + 7$. If the red coverage is centered on $(x, y, a + 7)$, it will overlap two sources of variable gadget centered on $(x + 1, y, a + 6)$ and $(x - 1, y, a + 6)$, and it will overlap two sources of clause gadget centered on $(x, y + 1, a + 8)$ and $(x, y - 1, a + 8)$. In Configuration 1 to 6, a green coverage represent two sources centered on $(x, y, a + 6)$ and $(x, y, a + 8)$ which will overlap two sources of variable gadget centered on $(x + 1, y, a + 6)$ and $(x - 1, y, a + 6)$, and overlap two sources of clause gadget centered on $(x, y + 1, a + 8)$ and $(x, y - 1, a + 8)$. However, because there are some positions that clause gadgets do not intersect with any variable gadget, the green coverage in Configuration 7 is a source centered on $(x, y, a + 8)$ that will overlap the two sources centered on $(x, y + 1, a + 8)$ and $(x, y - 1, a + 8)$ of the clause gadgets.

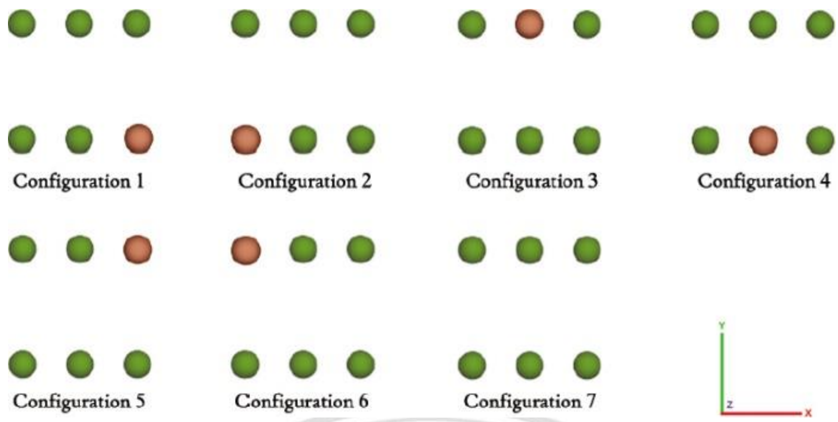

Figure 5.15 Top views of the seven corresponding possible configurations

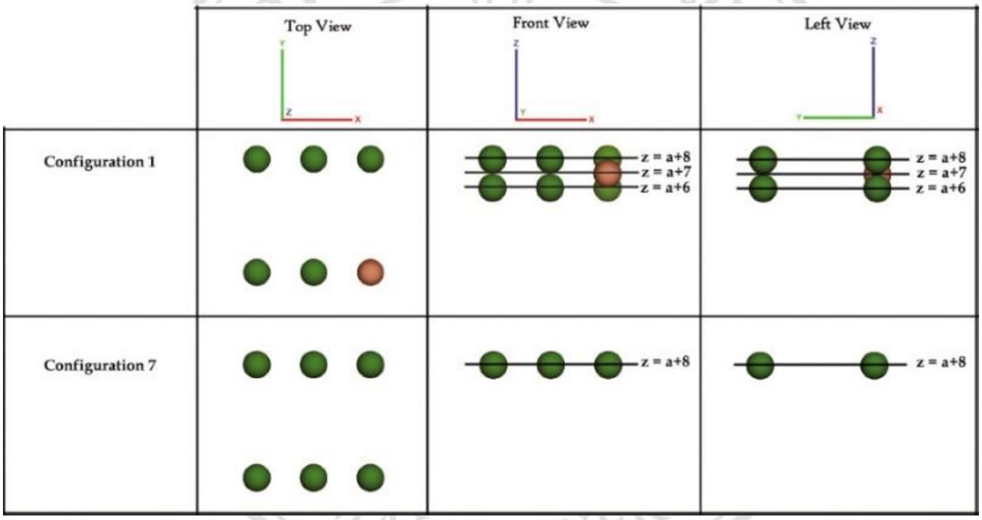

Figure 5.16 Top, front, and left views of red and green coverages

Figure 5.16 shows the top, front, and left views of the additional sources of the red and green coverages that will be placed in the construction in Figure 5.14. The top row in Figure 5.16 shows how additional sources are arranged for Configuration 1. In the other five configurations with a red coverage, the placement of additional sources is done in the similar manner. We note that the red coverage in each configuration enables the communication between the higher and lower layers of sources. For the one without a variable gadget connection (i.e., all green coverages), we place additional sources as shown in the bottom row of Figure 5.16.

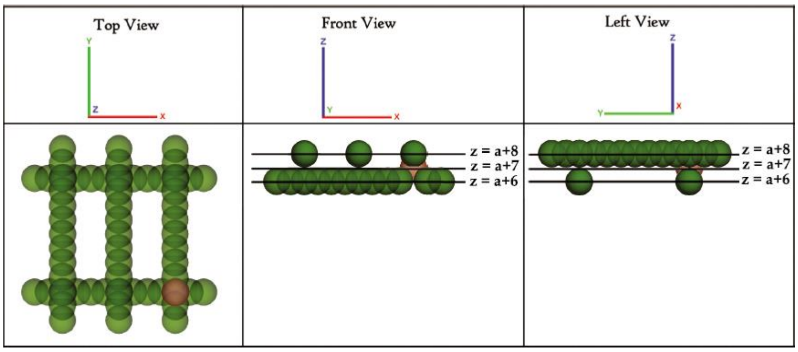

Figure 5.17 The part connecting variable and clause gadgets after Adjustment 2

After Adjustment 2 has been applied to the instance construction, we have Figure 5.17. We now describe a rule in selecting a configuration. We view the construction in a $n \times m$ frame. Each column represents a clause. Each row represents a variable. In a column we have three variable gadgets representing three variables in the clause. Variable gadgets are divided into *negative* and *positive* categories. A negative (positive) variable gadget is a variable gadget in which the upper communication path contains a source denoted by $\bar{u}^i$ (u^i) at the intersection of variable and consistency gadgets. In each row the categories of variable gadgets are selected according to the following rules. First, each row always starts with a negative variable gadget. Second, the category of the next variable gadget in column j depends on the column i of the immediately preceding variable gadget. If $j-i$ is even, the category of the next variable gadget is the same as that of the immediately preceding variable gadget. On the other hand, if $j-i$ is odd, the category of the next variable gadget is the opposite of that of the immediately preceding variable gadget. Without loss of generality, we assume three literals u_i ($\bar{u}_i$) in each clause in a given 3SAT instance are in the increasing order of i. In a clause gadget there are three parallel paths of communication between the top and bottom sources. Each path represents a corresponding literal u_i ($\bar{u}_i$) in the clause. In our construction we denote these paths by the corresponding three literals u_i ($\bar{u}_i$) in the clause in the increasing order of i from left to right. We now describe a rule to choose configurations in Figure 5.15. Observe that the position of a red coverage in each configuration identifies a corresponding intersection between variable and clause gadgets (See Figure 5.1). We have two cases in the construction. The first case is the case when a clause gadget intersects a negative variable gadget. If the literal in a clause is u_i ($\bar{u}_i$), we choose the configuration where the red coverage is located at the intersection between the upper (lower) communication path of the variable gadget and the path representing the corresponding literal u_i ($\bar{u}_i$) in the clause gadget. The second case is the case when a clause gadget intersects a positive variable gadget. If the literal in a clause is u_i ($\bar{u}_i$), we choose the configuration where the red coverage is located at the intersection between the lower (upper) communication path of the variable gadget and the path representing the corresponding literal u_i ($\bar{u}_i$) in the clause gadget.

We now discuss Adjustment 3. Adjustment 3 is the simplest adjustment of the three adjustments in this construction. Its objective is to enable the two parallel communication paths of a consistency gadget to communicate. Figure 5.18 illustrates top, front, and left views of the part of the construction where Adjustment 3 will be applied. Observe that there is no overlapping coverage that allows communication. We fix this part by placing more overlapping coverages as done in Figure 5.19.

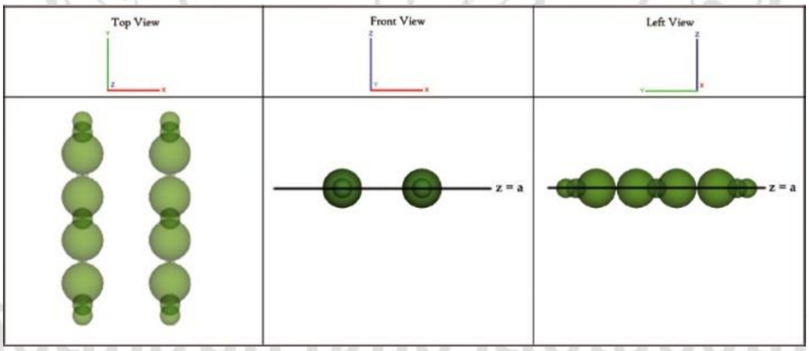

Figure 5.18 Top, front, and left views of the part before Adjustment 3

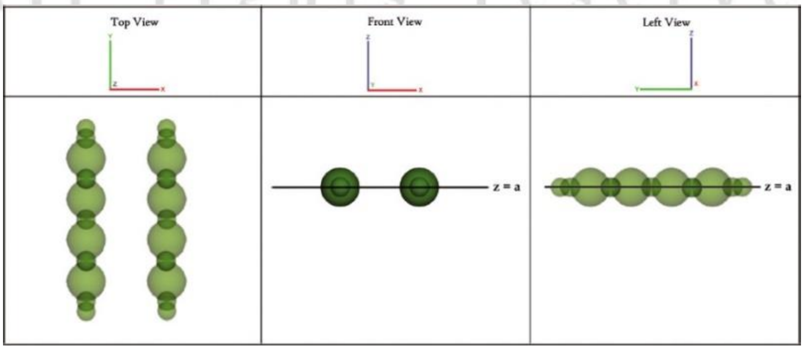

Figure 5.19 Top, front, and left views of the part after Adjustment 3

The final phase of the construction is the addition of three "cycles" of sources to each column. In each column three cycles of sources are placed as in Figure 5.20. Each cycle intersects each of the three variable gadgets exactly once. For each variable gadget in a column, if literal u_i or $\bar{u}_i$ is at the intersection of the upper (lower) path of the variable gadget and the clause gadget, the corresponding cycle intersects the lower (upper) path of the variable gadget. The construction of each intersection is done in a similar manner to that in Adjustment 2 (Figure 5.17). Additionally, we name a pair of sources $C_{y,w}$ and $D_{y,w}$ in each cycle (y as in variable u_y and w as in clause w. Source $C_{y,w}$ and source $D_{y,w}$ are placed in the corresponding cycle such that they are on different sides of the intersection.

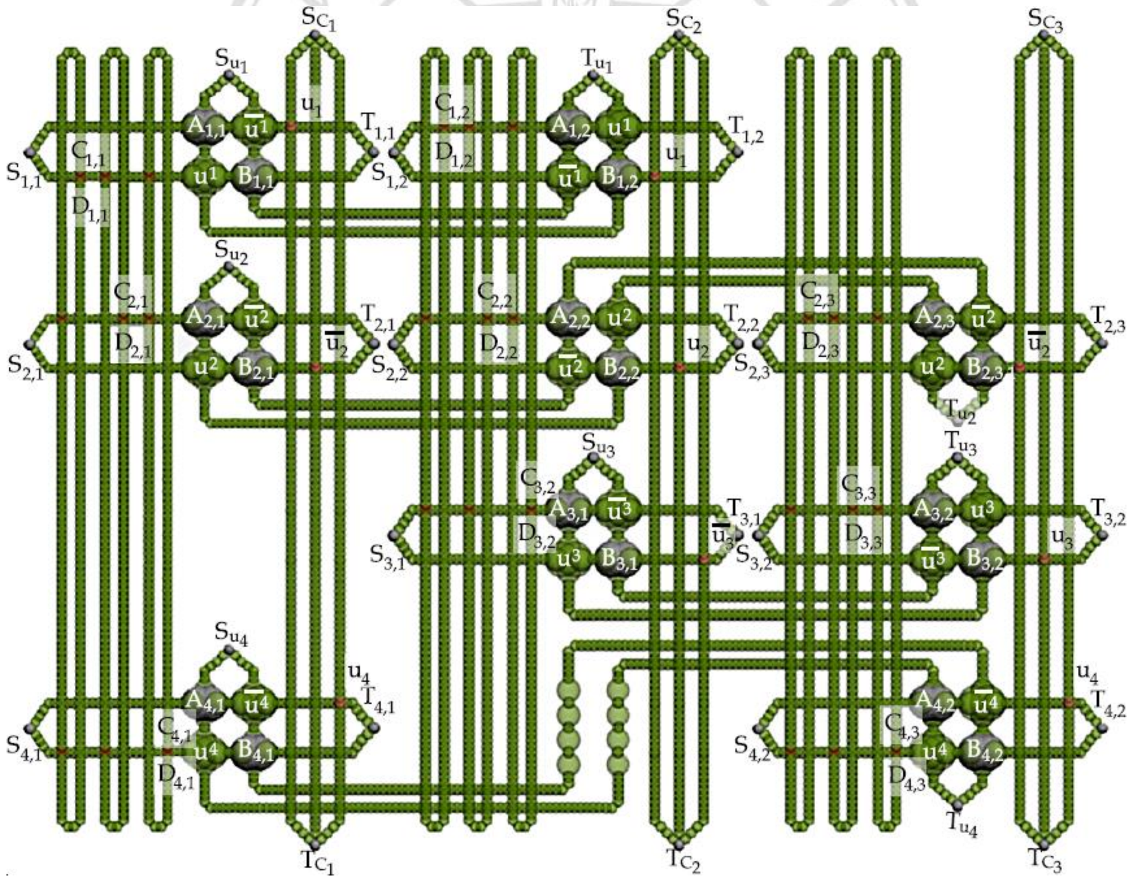

Figure 5.20 The complete construction

The whole construction process is complete (See Figure 5.20). Note that the construction process provides exactly the communication scheme shown in Figure 5.1 for the same

3SAT instance. We will momentarily show that, given a 3SAT instance, this construction process also produces a corresponding MSSCP instance in the general cases.

5.5 *NP*-Completeness of the MSSCP

In this section we will show that the MSSCP is *NP*-complete. Before we show the proof of *NP*-completeness, let us see roughly why the MSSCP instance in Figure 5.1 simulates $(u_1 \vee \bar{u}_2 \vee u_4)(u_1 \vee u_2 \vee \bar{u}_3)(\bar{u}_2 \vee u_3 \vee u_4)$. Let $s_1 \sim s_2$ denote a communicating path between sources s_1 and s_2. Remember that the 34 pairs of communicating sources were originally divided into 5 groups $\{S_{i,j}, T_{i,j}\}$ for $1 \le i \le 4$ and some $j \in \{1,2,3\}$, $\{S_{u_i}, T_{u_i}\}$ for $1 \le i \le 4$, $\{S_{C_i}, T_{C_i}\}$ for $1 \le i \le 3$, $\{A_{i,j}, B_{i,j}\}$ for $1 \le i \le 4$ and some $j \in \{1,2,3\}$, and $\{C_{i,w}, D_{i,w}\}$ for $1 \le i \le 4$ and $1 \le w \le 3$. Suppose u_1, u_2, u_3, u_4 are all assigned *FALSE*. The 5 groups are as follows.

- 3 communicating $\{S_{C_i}, T_{C_i}\}$-pairs are $S_{C_{1i}} \sim \bar{u}_2 \sim T_{C_1}$, $S_{C_{2i}} \sim \bar{u}_3 \sim T_{C_2}$ and $S_{C_{3i}} \sim \bar{u}_3 \sim T_{C_3}$.

- 9 communicating $\{S_{i,j}, T_{i,j}\}$-pairs are $S_{1,1} \sim A_{1,1} \sim \bar{u}^1 \sim u_1 \sim T_{1,1}$, $S_{2,1} \sim A_{2,1} \sim \bar{u}^2 \sim T_{2,1}$, $S_{4,1} \sim A_{4,1} \sim \bar{u}^4 \sim u_4 \sim T_{4,1}$, $S_{1,2} \sim \bar{u}^1 \sim B_{1,2} \sim u_1 \sim T_{1,2}$, $S_{2,2} \sim \bar{u}^2 \sim B_{2,2} \sim u_2 \sim T_{2,2}$, $S_{3,1} \sim A_{3,1} \sim \bar{u}^3 \sim T_{3,1}$, $S_{2,3} \sim A_{2,3} \sim \bar{u}^2 \sim T_{2,3}$, $S_{3,2} \sim \bar{u}^3 \sim B_{3,2} \sim u_3 \sim T_{3,2}$, $S_{4,2} \sim A_{4,2} \sim \bar{u}^4 \sim u_4 \sim T_{4,2}$.

- Let $\sim_{short}$ and $\sim_{long}$ denote short and long paths along each of the three additional cycles, respectively. 9 communicating $\{C_{y,w}, D_{y,w}\}$- pairs are $C_{2,1} \sim_{long} \sim D_{2,1}$, $C_{3,2} \sim_{long} \sim D_{3,2}$, $C_{2,3} \sim_{long} \sim D_{2,3}$, $C_{1,1} \sim_{short} \sim D_{1,1}$, $C_{4,1} \sim_{short} \sim D_{4,1}$, $C_{1,2} \sim_{short} \sim D_{1,2}$, $C_{2,2} \sim_{short} \sim D_{2,2}$, $C_{3,3} \sim_{short} \sim D_{3,3}$, and $C_{4,3} \sim_{short} \sim D_{4,3}$.

- 4 communicating $\{S_{u_i}, T_{u_i}\}$-pairs are $S_{u_1} \sim \bar{u}^1 \sim B_{1,1} \sim \bar{u}^1 \sim A_{1,2} \sim T_{u_1}$, $S_{u_2} \sim \bar{u}^2 \sim B_{2,1} \sim \bar{u}^2 \sim A_{2,2} \sim \bar{u}^2 \sim B_{2,3} \sim T_{u_2}$, $S_{u_3} \sim \bar{u}^3 \sim B_{3,1} \sim \bar{u}^3 \sim A_{3,2} \sim T_{u_3}$, and $S_{u_4} \sim \bar{u}^4 \sim B_{4,1} \sim \bar{u}^4 \sim A_{4,2} \sim T_{u_4}$.

- 9 communicating $\{A_{i,j}, B_{i,j}\}$-pairs are $A_{1,1} \sim u^1 \sim B_{1,1}$, $A_{2,1} \sim u^2 \sim B_{2,1}$, $A_{4,1} \sim u^4 \sim B_{4,1}$, $A_{1,2} \sim u^1 \sim B_{1,2}$, $A_{2,2} \sim u^2 \sim B_{2,2}$, $A_{3,1} \sim u^3 \sim B_{3,1}$, $A_{2,3} \sim u^2 \sim B_{2,3}$, $A_{3,2} \sim u^3 \sim B_{3,2}$, and $A_{4,2} \sim u^4 \sim B_{4,2}$.

By supposing 34 such pairs of communicating sources exist, it is easy to see that the 3SAT instance is also satisfied by assigning *TRUE* to each corresponding literal in each clause gadget and *FALSE* to the others.

Lemma 5.5.1 (Simultaneous 3-Pairs Communication) *In the configuration of sources in the wireless mobility model in* Figure 5.21 *there are precisely two ways that source pairs* $\{a,a'\},\{b,b'\},$ *and* $\{c,c'\}$ *can communicate simultaneously without sharing a source.*

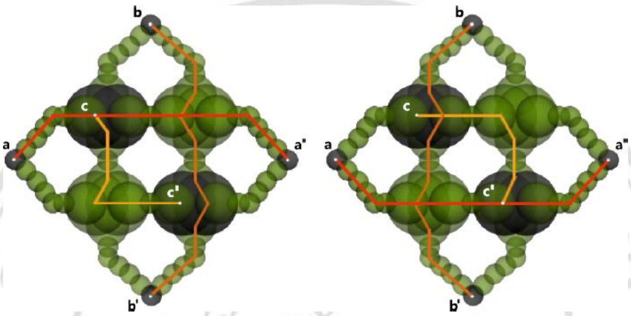

Figure 5.21 Two possible ways of simultaneous communication without sharing a source

Proof. We claim that a path $b \sim b'$ must contain neither source a nor source a'. This claim holds because source $a(a')$ only has one way in and one way out. Therefore, if path $b \sim b'$ contains source $a(a')$, source $a(a')$ would not be able to communicate with source $a(a')$ without sharing a source. Similar reasoning also shows that a path $a \sim a'$ must contain neither source b nor source b'.

Assume without loss of generality that, when the communication from source b arrives at the first group of six overlapping sources on the top left, it chooses three sources from the group to communicate. From this choosing, the communication can go either rightward or downward. Let us consider the former. When the communication arrives at the second group of six overlapping sources, it goes downward to the third group of six overlapping sources. At this point it can choose to continue downward or leftward. Either case is not possible because both source pairs $\{a,a'\}$, and $\{c,c'\}$ would not be able to communicate simultaneously without sharing a source. For the latter case, when the communication arrives at the second group of six overlapping sources, it can go rightward or downward. If it goes rightward to the third group of six overlapping sources, both source pairs $\{a, a'\}$, and $\{c, c'\}$ would not be able to communicate simultaneously without sharing a source. If it goes downward to b', we have the configuration on the right of Figure 5.21. By symmetry, the configuration on the left of Figure 5.21 immediately follows. Hence, the lemma is true. □

Lemma 5.5.2 (No Crossover) *If an instance is constructed according to Section 5, the communication path from* S_{u_i} *to* T_{u_i} *can exclusively be either* $S_{u_i} \sim A_{i,1} \sim u_i \sim B_{i,2}$ *or* $A_{i,2} \sim u^i \sim \ldots \sim A_{i,g_1}$ *or* $B_{i,g_1} \sim u^i \sim T_{u_i}$ *or* $S_{u_i} \sim \bar{u}^i \sim B_{i,1} \sim \bar{u}^i \sim A_{i,2}$ *or* $B_{i,2} \sim \bar{u}^i \sim \ldots \sim A_{i,g_2}$ *or* $B_{i,g_2} \sim T_{u_i}$ *for some* $g_1, g_2 \leq m$.

Proof. Consider row i in the instance and use induction on j in $A_{i,j}$. In the first variable gadget we have either $S_{u_i} \sim A_{i,1} \sim u^i \sim T_{u_i}$ or $S_{u_i} \sim \bar{u}^i \sim B_{i,1} \sim T_{u_i}$ by the implication of Lemma 5.5.1. Suppose the Lemma holds up to variable gadget $g_1 - 1 (g_2 - 1)$. By construction, no path in the consistency gadget intersects the other path in the consistency gadget except at S_{u_i} and T_{u_i} and, by the implication of Lemma 5.5.1, the communication in one path in the consistency gadget does not cross over to the other path in the consistency gadget via some paths in the variable gadget. Therefore, at the $g_1^{th} (g_2^{th})$ variable gadget, we have exclusively either $S_{u_i} \sim A_{i,1} \sim u^i \sim B_{i,2}$ or $A_{i,2} \sim u^i \sim \ldots \sim A_{i,g_1}$ or $B_{i,g_1} \sim u^i \sim T_{u_i}$ or $S_{u_i} \sim \bar{u}^i \sim B_{i,1} \sim \bar{u}^i \sim A_{i,2}$ or $B_{i,2} \sim \bar{u}^i \sim \ldots \sim A_{i,g_2}$ or $B_{i,g_2} \sim T_{u_i}$. The Lemma holds. □

Observe that each path contains exclusively either source u^i or source $\bar{u}^i$. We will use this fact to prove consistency of variable settings in a moment.

Theorem 5.5.3 (*NP*-Completeness of MSSCP) MSSCP *is NP-complete.*

Proof. We first show that the MSSCP is in *NP*. Given a mobility model $M = \{S, D, U, L, R, V, C, O\}$, k pairs of distinct sources $\{s_1, s_1'\}, \{s_2, s_2'\}, \ldots \{s_k, s_k'\}$, a time $t \in \mathrm{N}$, and a certificate $< s_{i,t'}, x_1, x_2, \ldots, x_e, s_{i,t'}' >$, where $e \in \mathrm{N}$, $s_{i,t} = s_i$, $s_{i,t}' = s_i'$, and $x_j \in S$ for $1 \leq i \leq k$, $1 \leq j \leq |S| - 2$, and $1 \leq t' \leq t$, a polynomial time verification algorithm can check that, at each time step t', each sequence of sources $< s_{i,t'}, x_1, x_2, \ldots, x_e, s'_{i,t'} >$ has a legitimate path of overlapping communication coverages and that each sequence is composed of unique sources. Therefore, the problem is in *NP*.

We will next show that MSSCP is *NP*-hard. Given an instance of the 3SAT, we construct a corresponding instance of the MSSCP as described in Section 5.5 while setting D to $\{0000, 0001, 1000, 0100, 0010, 0110, 1001\}$ and setting U, L, V, O to ϕ. Observe that coverage $c_i \in C$ can be any size that enables the construction in Section 5.5. Additionally, we set all $r_i \in R$ to 0000, and designate $10m + n$ pairs of distinct sources to be $\{S_{C_i}, T_{C_i}\}$-pairs, $\{S_{i,j}, T_{i,j}\}$-pairs, $\{C_{y,w}, D_{y,w}\}$- pairs, $\{S_{u_i}, T_{u_i}\}$-pairs, and $\{A_{i,j}, B_{i,j}\}$-pairs. Finally, we set t to 1. We claim that there is a truth assignment for U that satisfies all the clauses in C in exactly 1 in 3 manner if and only if all $10m + n$ pairs of sources simultaneously communicate throughout the duration 1 of the model without sharing a source.

$\Longrightarrow$ Suppose there is a truth assignment $f : U \rightarrow \{TRUE, FALSE\}$ that satisfies all the clauses in C in exactly 1 in 3 manner. Therefore, exactly one variable in each clause is assigned *TRUE*. Suppose the variable′s corresponding literal is l_i, where $l \in \{u, \bar{u}\}$. In each clause gadget there are three parallel paths from S_{C_q} to T_{C_q} and each path contains a source named after each literal in the clause. Exactly one of these paths must be chosen for otherwise some sources would be shared. In this case we choose the path that contains the source l_i. Therefore, we have a path $S_{C_q} \sim l_i \sim T_{C_q}$ for $1 \le q \le m$.

Consider a variable gadget containing the literal l_i. The paths from $S_{i,j}$ to $T_{i,j}$, for some $i \in [1, n]$ and some $j \in [1, m]$, must necessarily be the paths that contain the corresponding l^i because otherwise a source l_i in the paths $S_{c_q} \sim l_i \sim T_{c_q}$ would be shared. Let X be a variable gadget category. If X is negative and $l_i = u_i$, the path is $S_{i,j} \sim u^i \sim B_{i,j} \sim T_{i,j}$. If X is negative and $l_i = \bar{u}_i$, the path is $S_{i,j} \sim A_{i,j} \sim \bar{u}^i \sim T_{i,j}$. If X is positive and $l_i = u_i$, the path is $S_{i,j} \sim A_{i,j} \sim u^i \sim T_{i,j}$. If X is positive and $l_i = \bar{u}_i$, the path is $S_{i,j} \sim \bar{u}^i \sim B_{i,j} \sim T_{i,j}$.

For variable gadgets containing the literal $l_h, h \neq i$ and $l \in \{u, \bar{u}\}$, the paths from $S_{h,j}$ to $T_{h,j}$, for some $h \in [1,n]$ and some $j \in [1,m]$, are chosen to contain sources either $\bar{u}^h$ and u_h or u^h and $\bar{u}_h$. Specifically, if X is negative and $l_h = u_h$, the path is $S_{h,j} \sim A_{h,j} \sim \bar{u}^h \sim u_h \sim T_{h,j}$. If X is negative and $l_h = \bar{u}_h$, the path is $S_{h,j} \sim u^h \sim B_{h,j} \sim \bar{u}_h \sim T_{h,j}$. If X is positive and $l_h = u_h$, the path is $S_{h,j} \sim \bar{u}^h \sim B_{h,j} \sim u_h \sim T_{h,j}$. If X is positive and $l_h = \bar{u}_h$, the path is $S_{h,j} \sim A_{h,j} \sim u^h \sim \bar{u}_h \sim T_{h,j}$. Observe that this particular path is chosen for such variable gadget to "disable" the path $S_{C_q} \sim l_h \sim T_{C_q}$ after the path $S_{C_q} \sim l_i \sim T_{C_q}$ has been chosen.

Consider three pairs of $\{C_{i,w}, D_{i,w}\}$ in each column *w* of the constructed instance. Sources $C_{i,w}$ and $D_{i,w}$ in each column of 3 variable gadgets can communicate through either a short path or a long path. If the corresponding literal l_i is assigned *TRUE*, a long path is used. Otherwise, a short path is used. Given the literal $l_i \in \{u_i, \bar{u}_i\}$, observe that there is exactly one previously chosen communicating path from $S_{i,j}$ to $T_{i,j}$ in each column of 3 variable gadgets that does not contain both the source $u^i(\bar{u}^i)$ and its corresponding source $u_i'(u_i)$ This path forces a long path between the corresponding $\{C_{i,w}, D_{i,w}\}$ pair.

Consider a $\{A_{i,j}, B_{i,j}\}$ pair and its corresponding variable gadget. Let us first also consider the case of literal l_i. If $l_i = u_i$, the path is $A_{i,j} \sim \bar{u}^i \sim B_{i,j}$. If $l_i = \bar{u}_i$, the path is $A_{i,j} \sim u^i \sim B_{i,j}$. We next consider the case of literal l_h. If $l_h = u_h$, the path is $A_{h,j} \sim u^h \sim B_{h,j}$. If $l_h = \bar{u}_h$, the path is $A_{h,j} \sim \bar{u}^h \sim B_{h,j}$.

Consider the paths from S_{u_i} to T_{u_i}. By Lemma 5.5.2, it suffices to consider only the first variable gadget in each row. Let $g_1, g_2 \leq m$ be some integers. If the variable u_i is set to *TRUE* (or $\bar{u}_i$ is set to *FALSE*), the corresponding path from source $S_{i,j}$ to source $T_{i,j}$ is chosen to be the lower path of the variable gadget. By Lemma 5.5.1 (right configuration in Figure 5.21), the path $S_{u_i} \sim A_{i,1} \sim u^i \sim B_{i,2}$ or $A_{i,2} \sim u^i \sim \ldots \sim A_{i,g_1}$ or $B_{i,g_1} \sim u^i \sim T_{u_i}$ is chosen. On the other hand, if the variable u_i is set to *FALSE* (or $\bar{u}_i$ is set to *TRUE*), the corresponding path from source $S_{i,j}$ to source $T_{i,j}$ is chosen to be the upper path of the variable gadget. By Lemma 5.5.1 (left configuration in Figure 5.21), the path $S_{u_i} \sim \bar{u}^i \sim B_{i,1} \sim \bar{u}^i \sim A_{i,2}$ or $B_{i,2} \sim \bar{u}^i \sim \ldots \sim A_{i,g_2}$ or $B_{i,g_2} \sim T_{u_i}$ is chosen. Thus, we have shown that all $10m+n$ pairs of sources simultaneously communicate throughout the duration 1 of the model without sharing a source.

$\Longleftarrow$ Conversely, suppose all $10m+n$ pairs of sources simultaneously communicate throughout the duration 1 of the model without sharing a source. In each clause gadget we have three parallel paths from S_{C_q} to T_{C_q}. Each path contains a source named after a literal in the corresponding clause. In each clause gadget exactly one path is chosen to communicate for some sources would be shared, otherwise, and let this particular path be path $S_{C_q} \sim l_i \sim T_{C_q}$, where $l \in \{u, \bar{u}\}$. We set each l_i to *TRUE* and the other literals in the same clause to *FALSE*. More precisely, l_i = *TRUE* if and only if $S_{C_q} \sim l_i \sim T_{C_q}$. We claim that these settings are consistent (i.e., both u_i and $\bar{u}_i$ are not set to the same truth setting).

To see why this is true, consider the two cases where source l_i is located on the lower path of the variable gadget. Observe that $S_{C_q} \sim l_i \sim T_{C_q}$ if and only if $S_{i,j} \sim A_{i,j} \sim l^i \sim T_{i,j}$ for otherwise source l_i would be shared. Similarly, we have $S_{C_q} \sim l_i \sim T_{C_q}$ if and only if $S_{i,j} \sim l^i \sim B_{i,j} \sim T_{i,j}$ for the two cases where source l_i is located on the upper path of the variable gadget.

Let $g_1, g_2 \leq m$ be some integers. Suppose for the purpose of contradiction that $S_{C_{q1}} \sim u_i \sim T_{C_{q1}}$ and $S_{C_{q2}} \sim \bar{u}_i \sim T_{C_{q2}}$ for some $q_1 \neq q_2$. Without loss of generality, consider the case where both u_i and $\bar{u}_i$ are located on the upper path of each variable gadget. It follows that $S_{i,q_1} \sim u^i \sim B_{i,q_1} \sim T_{i,q_1}$ and $S_{i,q_2} \sim \bar{u}^i \sim B_{i,q_2} \sim T_{i,q_2}$. Therefore, by Lemma 6.1, we have $S_{u_i} \sim \ldots \sim u^i \sim \ldots \sim T_{u_i}$ and $S_{u_i} \sim \ldots \sim \bar{u}^i \sim \ldots \sim T_{u_i}$, respectively, which in turn implies both source u^i and source u^i are in the path from S_{u_i} to T_{u_i}. By Lemma 5.5.2, this is a contradiction. (Similar reasoning applies to the cases where both u_i and $\bar{u}_i$ are located on the lower path and on different paths of each variable gadget.) Hence, both u_i and $\bar{u}_i$ are not set to the same truth setting. We conclude that there is a truth assignment for U that satisfies all the clauses in C in exactly 1 in 3 manner. □

We would like to note here that MSSCP, where k pairs of sources may or may not be distinct remains *NP*-complete. In [14] there is a similar problem to MSSCP but the question asks whether given users can communicate instead. The problem name is MULTI-USERS SIMULTANEOUS COMMUNICATION PROBLEM (MUSCP). We extend our result to the three dimensional equivalent of this problem.

Corollary 5.5.4 (*NP*-Completeness of MUSCP) MUSCP *is NP-complete.*

Proof. The proof follows directly from Theorem 5.5.3 by reducing it from MSSCP and making all k user locations identical to all k source locations at each time step (i.e., identical movement). □

Hence, Corollary 5.5.4 solves the three-dimensional version of the open problem of the same name in [14].

Note that in our *NP*-complete proof, we exclude the users, users' movement, and the obstacles in our instance construction. It seems that we prove on $M = \{S, D, \phi, \phi, R, V, C, \phi\}$ to be *NP*-complete. However, we can make a reduction from this model to another mobility model $M' = \{S', D', U', L', R', V', C', O'\}$ which if an instance of M is a YES instance, the corresponding instance of M' is also a YES instance, and if an instance of M is a No instance, the corresponding instance of M' is also a No instance. We can do the reduction by let $S = S', D = D', R = R', V = V', C = C'$. We can position users in U' in any location and define their movement in L' in any pattern because it has no impact on the sources' communications. We can lay the obstacles on the positions which do not overlap with any sources, at any time, so it also will not impact on the sources' communications. These builds remain the corresponding answer of the corresponding instance between the two models. Therefore, even our proof seem to be the proof to a particular case of MSSCP where $M = \{S, D, \phi, \phi, R, V, C, \phi\}$ is *NP*-complete, it also have the meaning as Theorem 5.5.3, which states "MSSCP *is NP-complete*".

5.6 Summary

In this chapter, we introduce a wireless mobility model on a three-dimensional grid and its communication protocol. This model is extended from the original two-dimensional model in [14] to make it more realistic. We then give the definition of MULTI-SOURCES SIMULTANEOUS COMMUNICATION PROBLEM (MSSCP). This problem modeled the real situation in which *k* pairs of sources communicate without sharing a source. We show that efficiently computing a solution to the MSSCP is very unlikely unless *P* = *NP*. We also extend the *NP*-completeness proof to MULTI-USERS SIMULTANEOUS COMMUNICATION PROBLEM (MUSCP) to show that this problem is intractable.