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MAXIMAL AND MINIMAL IDEALS IN TRANSFORMATION SEMIGROUPS WITH INVARIANT SETS

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***Abstract*—Let X be a set and $T(X)$ denote the semigroup (under composition) of transformations from X into itself. For a fixed nonempty subset Y of X, let**

$S(X,Y)=\{\alpha\in T(X): Y\alpha\subseteq Y\}$.

Then $S(X,Y)$ is a semigroup of total transformations of X which leave a subset Y of X invariant. In this paper, existence and uniqueness of maximal and minimal ideals of $S(X,Y)$ are proved. Moreover, we present a maximal congruence on $S(X,Y)$ when X is a finite set.

Keywords—maximal ideals, minimal ideals, transformation semigroups with invariant sets.

I. INTRODUCTION

Let X be a set and $\varnothing\neq Y\subseteq X$. The semigroup we consider is $S(X,Y)$ consists of all mappings in $T(X)$ which leave $Y\subseteq X$ invariant. K. D. Magill [5] introduced and studied the semigroup $S(X,Y)$ in 1996. In fact, if $Y=X$, then $S(X,Y)=T(X)$. So we may regard $S(X,Y)$ as a generalization of $T(X)$. In 2005, S. Nenthein, P. Youngkhong, and Y. Kemprasit [7] showed that $S(X,Y)$ is a regular semigroup if and only if $X=Y$ or Y contains exactly one element, and $\operatorname{Reg} S(X,Y)=\{\alpha\in S(X,Y): X\alpha\cap Y=Y\alpha\}$ is the set of all regular elements of $S(X,Y)$. Moreover, they counted the numbers of regular elements in $S(X,Y)$ for a finite set X. The numbers were given in terms of the cardinalities of X and Y. Later in 2013, W. Choomanee, P. Honyam and J. Sanwong [1] studied left regular, right regular and intra-regular elements of $S(X,Y)$ and consider the relationships between these elements. Moreover, they counted the number of left regular elements of $S(X,Y)$ when X is a finite set.

As far back in 1952, Malcev [6] determined ideals of $T(X)$. In 2011 P. Honyam and J. Sanwong [4] characterized when $S(X,Y)$ is isomorphic to $T(Z)$ for some set Z and prove that every semigroup A can be embedded in $S(A^1,A)$. Then they described Green's relations and ideals on $S(X,Y)$ and applied these results to obtain its group $\mathrm{H}-$classes and ideals.

In this paper, we determine maximal and minimal ideals of $S(X,Y)$. We also present a maximal congruence on $S(X,Y)$ when X is a finite set.

II. PRELIMINARIES AND NOTATIONS

In this section, we list some known results, definitions and notations that will be used throughout this paper.

Let X be a set and Y a nonempty subset of X. Then $S(X,Y)$ is a semigroup with identity 1_X, the identity map on X. Green's relation on $S(X,Y)$ are given by P. Honyam and J. Sanwong [4], which are needed in characterizing ideals on $S(X,Y)$.

Lemma 2.1. [4] *Let $\alpha,\beta\in S(X,Y)$. Then*

(1) $\alpha\mathrm{L}\beta$ if and only if $X\alpha=X\beta$ and $Y\alpha=Y\beta$;

(2) $\alpha\mathrm{R}\beta$ if and only if $\pi_\alpha=\pi_\beta$ and $\pi_\alpha(Y)=\pi_\beta(Y)$

(3) $\alpha\mathrm{J}\,\beta$ if and only if $|X\alpha|=|X\beta|,|Y\alpha|=|Y\beta|$ and
$|X\alpha\setminus Y|=|X\beta\setminus Y|$

Let p be any cardinal number such that

$$p'=\min\{q:q>p\}.$$

Note that p' always exists since the cardinals are well-ordered and when p is finite we have $p'=p+1=$ the successor of p.

To describe ideals of $S(X,Y)$, we let $|X|=a,\ |Y|=b$ and $|X\setminus Y|=c$. For each cardinals r,s,t such that $2\le r\le a',\ 2\le a\le b'$ and $1\le t\le c'$, we define

$$S(r,s,t)=\{\alpha\in S(X,Y):|X\alpha|<r,\ |Y\alpha|< s \text{ and } |X\alpha\setminus Y<t|\}.$$

Theorem 2.2. [4] *The set $S(r,s,t)$ is an ideal of $S(X,Y)$.*

To obtain ideals of $S(X,Y)$, we need the following notation. Let Z be a nonempty subset of $S(X,Y)$, and let

$$K(Z)=\{\alpha\in S(X,Y):|X\alpha|\le|X\beta|,|Y\alpha|\le|Y\beta| \text{ and } |X\alpha|\setminus Y\le|X\alpha\setminus Y| \text{ for some } \beta\in Z\}.$$

Then we see that $Z\subseteq K(Z)$ and $Z_1\subseteq Z_2$ implies that $K(Z_1)\subseteq K(Z_2)$.

Theorem 2.3. [4] *The ideals of $S(X,Y)$ are precisely the set $K(Z)$ for some nonempty subset Z of $S(X,Y)$.*

Let $G(A)$ be the group of permutation on the set A. Define

$G(X,Y)=\{\alpha\in G(X):\alpha|_Y\in G(Y)\}$,

Where $Y\subseteq X$ and $\alpha|_Y$ is the restriction of α on the set Y. Then $G(X,Y)$ is a subgroup of the permutation group $G(A)$.

If X is a finite set with n elements and Y a nonempty subset of X with m elements, then we define

$J_{r,s,t} = \{\alpha \in S(X,Y) : |X\alpha| = r, |Y\alpha| = s \text{ and } |X\alpha \setminus Y| = t\}$

and

$J_k = \{\alpha \in S(X,Y) : |X\alpha| = k\}$

where $1 \le r \le n$, $1 \le s \le m$, $0 \le t \le n-m$ and $1 \le k \le n$. Thus $J_{r,s,t}$ is a J $-$class of $S(X,Y)$, J_1 is the set of all constant maps with image in Y and $J_n = G(X,Y)$.

The following convenient notation will be used in this paper: given $\alpha \in S(X,Y)$ we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix},$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denote $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

With the above notation, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \varnothing; B_j, C_k \subseteq X \setminus Y; \{a_i\} \subseteq Y; \{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, I is a nonempty set, but J or K can be empty.

III. Results

From now on, we let X be a finite set with n elements and Y a nonempty subset of X with m elements. Let $\mathscr{I}$ be the set of all ideals of $S(X,Y)$. Then $(\mathscr{I}, \subseteq)$ is a partially ordered set with the following property.

Theorem 3.1. J_1 is a minimum ideal of $S(X,Y)$.

Proof: We prove that $J_1 = S(2,2,1)$. It is clear that $J_1 \subseteq S(2,2,1)$. Let $\alpha \in S(2,2,1)$. Then $|X\alpha| < 2, |Y\alpha| < 2, |X\alpha \setminus Y| < 1$ and thus $|X\alpha| = 1 = |Y\alpha|$. So α is a constant map and $\alpha \in J_1$. Therefore, J_1 is an ideal of $S(X,Y)$. To show that J_1 is a minimum ideal, let I be an ideal of $S(X,Y)$ and $\beta \in J_1$. Then there exists $\varnothing \neq Z \subseteq S(X,Y)$ such that $I = K(Z)$. Let $\gamma \in Z$. Then $|X\gamma|, |Y\gamma| \ge 1$, so $|X\beta| = 1 \le |X\gamma|, |Y\beta| = 1 \le |Y\gamma|$ and $|X\beta \setminus Y| = 0 \le |X\gamma \setminus Y|$. Thus $\beta \in I$, i.e., $J_1 \subseteq I$ as required. +

Lemma 3.2. If $|Y| = 1$ and $J_{2,s,t} \neq \varnothing$, then $J_1 \cup J_{2,s,t}$ is an ideal of $S(X,Y)$ if and only if $s = 1 = t$.

Proof: Assume that $|Y| = 1$ and $J_{2,s,t} \neq \varnothing$. Suppose that $J_1 \cup J_{2,s,t}$ is an ideal of $S(X,Y)$. Let $\alpha \in J_{2,s,t}$. Since $Y\alpha \subseteq Y$, we have $1 \le |Y\alpha| \le |Y| = 1$, so $|Y\alpha| = 1$ which implies that $s = |Y\alpha| = 1$. Since $|X\alpha| = 2$ and $Y = Y\alpha$, we have $|X\alpha \setminus Y| = |X\alpha \setminus Y\alpha| = 2 - 1 = 1$, that is $t = 1$.

Conversely, assume that $s = 1 = t$. First we show that $J_1 \cup J_{2,1,1} = S(3,2,2)$. Let $\alpha \in J_1 \cup J_{2,1,1}$. Then $\alpha \in J_1$ or $\alpha \in J_{2,1,1}$. If $\alpha \in J_1$, then $|X\alpha| = 1 < 3, |Y\alpha| \le |X\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 1 - 1 = 0 < 2$. thus $\alpha \in S(3,2,2)$. If $\alpha \in J_{2,1,1}$, then $|X\alpha| = 2 < 3, |Y\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 1 < 2$. Thus $\alpha \in S(3,2,2)$. For the other containment, let $\alpha \in S(3,2,2)$. Then $|X\alpha| \le 2, |Y\alpha| \le 1$ and $|X\alpha \setminus Y| \le 1$. If $|X\alpha| = 1$, then $\alpha \in J_1$. if $|X\alpha| = 2$, then $|X\alpha \setminus Y| = |X\alpha \setminus Y\alpha| = 2 - 1 = 1$. Then $\alpha \in J_{2,1,1}$. Hence $J_1 \cup J_{2,1,1} = S(3,2,2)$ is an ideal. +

Since J_1 is the minimum ideal of $S(X,Y)$, we define a minimum ideal in $S(X,Y)$ as follows. An ideal $J_1 Ø I$ of $S(X,Y)$ is a minimal ideal if J is an ideal such that $J_1 ØjJ \subseteq I$, then $J = I$.

Theorem 3.3. If $|Y| = 1$, then $J_1 \cup J_{2,1,1}$ is the unique minimal ideal of $S(X,Y)$.

Proof: Suppose that $Y = \{a\}$. By Lemma 3.2, we have $J_1 \cup J_{2,1,1}$ is an ideal of $S(X,Y)$. Next, we show that $J_1 \cup J_{2,1,1}$ is a minimal ideal of $S(X,Y)$. Let J be an ideal of $S(X,Y)$ such that $J_1 \subseteq jJ \subseteq J_1 \cup J_{2,1,1}$. Suppose that J Ø $J_1 \cup J_{2,1,1}$. It is clear that $J_1 \subseteq J$. By assumption, we have exists $\alpha \in J_{2,1,1}$ but $\alpha \notin J$. We show that $J \subseteq J_1$ by supposing this is false, so $J \not\subset J_1$. Then there exists $\beta \in J$, but $\beta \notin J_1$. Since $\alpha, \beta \in J_{2,1,1}$, we can write

$$\alpha = \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix}$$

Where $Y \subseteq A$, $a \in Y$, $b \in X \setminus Y$ and

$$\beta = \begin{pmatrix} B & X \setminus B \\ a & c \end{pmatrix}$$

Where $Y \subseteq B$, $c \in X \setminus Y$. Let $\gamma, \theta \in S(X,Y)$ be defined by

$$\gamma = \begin{pmatrix} A & X \setminus A \\ u & v \end{pmatrix}, \quad \theta = \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix},$$

W here $u \in B \cap Y$, $v \in X \setminus B$. Consider

$$\gamma\beta\theta = \begin{pmatrix} A & X \setminus A \\ u & v \end{pmatrix}\begin{pmatrix} B & X \setminus B \\ a & c \end{pmatrix}\begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix}$$

$$= \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix} = \alpha.$$

Then $\alpha = \gamma\beta\theta \in J$, which is a contradiction. So $J = J_1$. Hence, $J_1 \cup J_{2,1,1}$ is a minimal ideal of $S(X,Y)$. Finally, we show that $J_1 \cup J_{2,1,1}$ is a unique minimal ideal of $S(X,Y)$. We show that $M = N$. Since N is an ideal of $S(X,Y)$, we get that $N = K(Z)$ for some $\varnothing \neq Z \subseteq S(X,Y)$. Since $J_1 Ø N$, there exists $\alpha \in N$ with $|X\alpha| \ge 2$. Since $\alpha \in N = K(Z)$, we obtain that $|X\alpha| \le |X\beta|, |Y\alpha| \le |Y\beta|$ and $|X\alpha \setminus Y| \le |X\beta \setminus Y|$ for some $\beta \in Z$. |Let $\gamma \in J_{2,1,1}$. Then $|X\gamma| = 2$ and so $|X\gamma| \le |X\alpha| \le |X\beta|$. Since $|Y| = 1$, we have $|Y\gamma| = 1 = |Y\alpha| \le |Y\beta|$ and $|X\gamma \setminus Y| = 1 \le |X\alpha \setminus Y| \le |X\beta \setminus Y|$. Then $\gamma \in K(Z) = N$. Thus $J_{2,1,1} \subseteq N$ and so $J_1 \cup J_{2,1,1} \subseteq N$ which implies that $M = N$.

Lemma 3.4. If $|Y| > 1$ and $J_{2,s,t} \neq \varnothing$, then $J_1 \cup J_{2,s,t}$ is an ideal of $S(X,Y)$ if and only if $s = 1, t = 0$.

Proof: Assume that $|Y|>1$ and $J_{2,s,t}\neq\varnothing$. Suppose that $J_1\cup J_{2,s,t}$ is an ideal. Let $\alpha\in J_{2,s,t}$. Then $|X\alpha|=2,\ |Y\alpha|=s$ and $|X\alpha\setminus Y|=t$. Since $|X\alpha|=2$ and $1\le|Y\alpha|\le|X\alpha|=2,$ we have $1\le s\le 2,$ so $0\le|X\alpha\setminus Y|\le 1.$ Thus $0\le t\le 1.$ So there are four possible cases: $s=2$ and $t=0; s=2$ and $t=1; s=1=t;$ or $s=1$ and $t=0$.

If $s=2$ and $t=1,$ then $|X\alpha|=2=|Y\alpha|.$ Since $Y\alpha\subseteq X\alpha,$ we obtain that $X\alpha=Y\alpha$ and thus $t=|X\alpha\setminus Y|=|Y\alpha\setminus Y|=0$ which is a contradiction.

If $s=2$ and $t=0,$ then $J_1\cup J_{2,s,t}=J_1\cup J_{2,2,0}$. Let $\beta\in J_{2,2,0}$. So $|X\alpha|=2=|Y\beta|$ and $Y\beta\subseteq Y,$ thus we can write

$$\beta=\begin{pmatrix}A & B\\ a & b\end{pmatrix}$$

where $A\cap Y\neq\varnothing\neq B\cap Y;$ $a,b\in Y.$ Since $\varnothing\neq Y\ Ø\ X,$ there exists $c\in B$ and define $\gamma\in S(X,Y)$ by

$$\gamma=\begin{pmatrix}C & X\setminus C\\ a & c\end{pmatrix}$$

where $Y\subseteq C.$ So

$$\gamma\beta=\begin{pmatrix}C & X\setminus C\\ a & b\end{pmatrix}\notin J_1\cup J_{2,2,0}.$$

Then $J_1\cup J_{2,2,0}$ is not an ideal is a contradiction.

If $s=1=t,$ then $J_1\cup J_{2,s,t}=J_1\cup J_{2,1,1}.$ Let $\lambda\in J_{2,1,1}$. So $|X\lambda|=2$ and $Y\lambda\subseteq Y,$ thus we can write

$$\lambda=\begin{pmatrix}A & X\setminus A\\ u & v\end{pmatrix}$$

where $Y\subseteq A;$ $u\in Y,\ v\in X\setminus Y.$ Since $|Y|>1,$ there exists $u\neq w\in Y$ and define $\mu\in S(X,Y)$ by

$$\mu=\begin{pmatrix}Y & X\setminus Y\\ u & w\end{pmatrix}$$

So

$$\lambda\mu=\begin{pmatrix}A & X\setminus A\\ u & w\end{pmatrix}\notin J_1\cup J_{2,1,1}.$$

Thus $J_1\cup J_{2,1,1}$ is not an ideal which is a contradiction. Therefore, $s=1$ and $t=0.$

Conversely, assume that $s=1$ and $t=0.$ We show that $J_1\cup J_{2,1,0}=S(3,2,1).$

Let $\alpha\in J_1\cup J_{2,1,0}$. Then $\alpha\in J_1$ or $\alpha\in J_{2,1,0}$. If $\alpha\in J_1,$ then $|X\alpha|=1<3,\ |Y\alpha|\le|X\alpha|=1<2$ and $|X\alpha\setminus Y|=1-1=0<1.$ Thus $\alpha\in S(3,2,1).$ For the other containment, let $\alpha\in S(3,2,1).$ Then $|X\alpha|\le 2,\ |Y\alpha|=1$ and $|X\alpha\setminus Y|=0.$ If $|X\alpha|=1,$ then $\alpha\in J_1.$ If $|X\alpha|=2,\ |Y\alpha|=1$ and $|X\alpha\setminus Y|=0,$ then $\alpha\in J_{2,1,0}$ +

Theorem 3.5. If $|Y|>1,$ then $J_1\cup J_{2,1,0}$ is the unique minimal ideal of $S(X,Y).$

Proof: Suppose that $|Y|>1.$ By Lemma 3.4, we have $J_1\cup J_{2,1,0}$ is an ideal of $S(X,Y).$ To show that $J_1\cup J_{2,1,0}$ is a minimal ideal of $S(X,Y),$ let J be an ideal of $S(X,Y)$ such that $J_1\subseteq J\ Ø\ J_1\cup J_{2,1,0}$. It is clear that $J_1\cup J.$ By assumption we have there exists $\alpha\in J_{2,1,0}$ but $\alpha\in J.$ We prove that $J\subseteq J_1$ by supposing this false. Then there exists $\beta\in J,$ but $\beta\in J_1.$ Since $\alpha,\beta\in J_{2,1,0},$ we can write

$$\alpha=\begin{pmatrix}A & X\setminus A\\ a & b\end{pmatrix}$$

where $Y\subseteq A;$ $a,b\in Y$ and

$$\beta=\begin{pmatrix}B & X\setminus B\\ a' & c\end{pmatrix}$$

where $Y\subseteq B;$ $a',c\in Y.$ Let $\gamma,\theta\in S(X,Y)$ be defined by

$$\gamma=\begin{pmatrix}A & X\setminus A\\ u & v\end{pmatrix},\quad \theta=\begin{pmatrix}a & X\setminus\{a\}\\ a & b\end{pmatrix}$$

where $u\in B\cap Y,\ v\in X\setminus B.$ so

$$\gamma\beta\theta=\begin{pmatrix}A & X\setminus A\\ u & v\end{pmatrix}\begin{pmatrix}B & X\setminus B\\ a' & c\end{pmatrix}\begin{pmatrix}a & X\setminus\{a\}\\ a & b\end{pmatrix}$$

$$=\begin{pmatrix}A & X\setminus A\\ a & b\end{pmatrix}=\alpha.$$

Then $\alpha=\gamma\beta\theta\in J,$ which is a contradiction. Hence, $J_1\cup J_{2,1,0}$ is a minimal ideal of $S(X,Y).$ Now, we show that $J_1\cup J_{2,1,0}$ is a unique minima ideal of $S(X,Y).$ Let $M=J_1\cup J_{2,1,0}$ and N be a minimal ideal of $S(X,Y).$ Since N is an ideal of $S(X,Y),$ we get that $N=K(Z)$ for some $\varnothing\neq Z\subseteq S(X,Y).$ Since $J_1\ Ø\ N,$ there exists $\alpha\in N$ with $|X\alpha|\ge 2.$ Since $\alpha\in N=K(Z),$ we obtain that $|X\alpha|\le|X\beta|,\ |Y\alpha|\le|Y\beta|$ and $|X\alpha\setminus Y|\le|X\beta\setminus Y|$ for some $\beta\in Z.$ Let $\gamma\in J_{2,1,0}$. Then $|X\gamma|=2$ and so

$|X\gamma| = 2 = |X\alpha| \le |X\beta|$. Since $|Y| > 1$, we have $|Y\gamma| = 1 \le |Y\alpha| \le |Y\beta|$ and $|X\alpha \setminus Y| = 0 \le |X\alpha \setminus Y| \le |X\beta \setminus Y|$. Then $\gamma \in K(Z) = N$. Thus $J_{2,1,0} \subseteq N$ and so $J_1 \cup J_{2,1,0} \in N$ which implies that $M = N$.

Lemma 3.6. $J_1 \cup J_2 \cup \ldots \cup J_k$ is an ideal of $S(X,Y)$ where $1 \le k \le n$.

Proof: Let $\alpha \in J_1 \cup J_2 \cup \ldots \cup J_k$ and $\beta, \gamma \in S(X,Y)$. Then $\alpha \in J_{i_0}$ for some $1 \le i_0 \le k$ and thus $|X\alpha| = i_0$. Since $X\beta\alpha\gamma = (X\beta)\alpha\gamma \subseteq X\alpha\gamma$, we get that $|X\beta\alpha\gamma| \le |X\alpha\gamma| \le |X\alpha| = i_0$. Then $|X\beta\alpha\gamma| = p$ for some $1 \le p \le i_0$. Hence $\beta\alpha\gamma \in J_p \subseteq J_1 \cup J_2 \cup \ldots \cup J_k$. +

An ideal $I \text{ Ø } S(X,Y)$ of $S(X,Y)$ is a maximal ideal if J is an ideal such that $I \subseteq J \text{ Ø } S(X,Y)$, then $I = J$.

Lemma 3.7. Let S be a semigroup with identity 1. If S has a maximal ideal, then it is unique.

Proof: suppose that S has a maximal ideal, say M. Let M' be a maximal ideal of S. It is clear that $M \cup M'$ is an ideal and $1 \notin M \cup M'$. Since $M \subseteq M \cup M'$ and M is a maximal ideal, we have $M \cup M' = M$. Similarly, we have $M \cup M' = M'$. So $M = M \cup M' = M'$ and therefore, S has a unique maximal ideal of S.

If $|X| = |Y| = 1$, then $S(X,Y) = G(X,Y)$. Thus $S(X,Y) \setminus G(X,Y) = \varnothing$. So we consider the case $|X| > 1$.

Theorem 3.8. If $|X| > 1$, then $S(X,Y) \setminus G(X,Y)$ is a unique maximal ideal of $S(X,Y)$.

Proof Let $a \in Y$ and α be the constant map with $X\alpha = \{a\}$. Then $\alpha \in S(X,Y) \setminus G(X,Y)$, so $S(X,Y) \setminus G(X,Y) \neq \varnothing$. By Lemma 3.6, we have $S(X,Y) \setminus G(X,Y) = S(X,Y) \setminus J_n = J_1 \cup J_2 \cup \ldots \cup J_{n-1}$ Is an ideal of $S(X,Y)$. We show that $S(X,Y) \setminus G(X,Y)$ is a maximal ideal of $S(X,Y)$. Let I be an ideal of $S(X,Y)$ such that $S(X,Y) \subseteq I \text{ Ø } S(X,Y)$. We prove that $I = S(X,Y) \setminus G(X,Y)$ by supposing this is not true. Then there exist $\alpha \in I$ but $\alpha \notin S(X,Y) \setminus G(X,Y)$, i.e., $\alpha \in G(X,Y)$. Since $G(X,Y)$ is a group, we obtain that $\alpha^{-1} \in G(X,Y)$ and $1_X = \alpha\alpha^{-1} \in I$. Thus $I = S(X,Y)$ which is a contradiction. Therefore $I = S(X,Y) \setminus G(X,Y)$. So $S(X,Y) \setminus G(X,Y)$ is a maximal ideal of $S(X,Y)$. By Lemma 3.7, we obtain that $S(X,Y) \setminus G(X,Y)$ is a unique maximal ideal of $S(X,Y)$.

Let ρ be a congruence on a semigroup S. We recall that ρ is a maximal congruence if δ is a congruence on S with $\rho \text{ Ø } \delta \subseteq S \times S$ implies $\delta = S \times S$.

Theorem 3.9 Let $S = S(X,Y)$ and $G = G(X,Y)$. Then $\rho = (S \setminus G \times S \setminus G) \cup (G \times G)$ is a maximal congruence on S.

Proof It is clear that ρ is a equivalence relation on S. Let $\alpha, \beta, \gamma \in S$ and $(\alpha, \beta) \in \rho$. Then $(\alpha, \beta) \in (S \setminus G \times S \setminus G)$ or $(\alpha, \beta) \in G \times G$. If $(\alpha, \beta) \in (S \setminus G \times S \setminus G)$, then $\gamma\alpha, \alpha\gamma, \gamma\beta, \beta\gamma \in S \setminus G$ since $S \setminus G$ is an ideal of $S(X,Y)$. Thus $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in (S \setminus G) \times (S \setminus G) \subseteq \rho$. If $(\alpha, \beta) \in G \times G$, we consider two cases.

Case 1: $\gamma \in S \setminus G$. Since $S \setminus G$ is an ideal, we have $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in (S \setminus G) \times (S \setminus G) \subseteq \rho$.

Case 2: $\gamma \in G$. Then $\alpha, \beta, \gamma \in G$ and G is a group, so we obtain that $\gamma\alpha, \alpha\gamma \in G$ and $\gamma\beta, \beta\gamma \in G$. Thus $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in G \times G \subseteq \rho$.

Next, we show that ρ is a maximal congruence on S. Let δ be a congruence on S such that $\rho \text{ Ø } \delta \subseteq S \times S$. Since $\rho \text{ Ø } \delta$, there exists $(\alpha, \beta) \in (\delta \setminus \rho)$ with $\alpha \in S \setminus G$ and $\beta \in G$. Let k be the order of β. Then $1_X = \beta^k \delta \alpha^k$ where $\alpha^k \in S \setminus G$ since $S \setminus G$ is an ideal. Now, let $(\lambda, \mu) \in S \times S$. So $\lambda\delta\alpha^k\lambda$ and $\mu\delta\alpha^k\mu$ where $\alpha^k\lambda, \alpha^k\mu \in S \setminus G$ so $\alpha^k\lambda\rho\alpha^k\mu$. Since $\rho \subseteq \delta$, we have $\alpha^k\lambda\delta\alpha^k\mu$. Thus $\lambda\delta\mu$ and $\delta = S \times S$ as required.

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