CHAPTER 2

Preliminaries

2.1 Elementary Concepts

In this thesis, we assume that X is a finite set and Y a nonempty subset of X and the cardinality of a set X is denoted by $|X|$.

Definition 2.1.1. A semigroup is a pair $(S, \cdot)$ in which S is a nonempty set and $\cdot$ is a binary associative operation on S, i.e., the equation $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all $x, y, z \in S$.

Definition 2.1.2. Let S be a semigroup.

(i) If there exists an element 1 of S such that

$$x1 = x = 1x \text{ for all } x \in S,$$

then 1 is called an identity element of S and S is called a *semigroup with identity* or a monoid.

(ii) If there exists an element 0 of S such that

$$x0 = 0 = 0x \text{ for all } x \in S,$$

then 0 is called a *zero element* of S and S is called a *semigroup with zero.*

A nonempty subset T of a semigroup S is called a *subsemigroup* of S if $xy \in T$ for all $x, y \in T$.

Definition 2.1.3. Let $A \neq \varnothing$. Then a relation R on A is an *equivalence relation* on A provided R is:

reflexive: $(a, a) \in R$ for all $a \in A$;

symmetric: if $(a, b) \in R$, then $(b, a) \in R$ for all $a, b \in A$;

transitive: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$.

Definition 2.1.4. Let S be a semigroup. A relation R on the set S is called *left compatible* (with the operation on S) if

$$(\text{for all } s,t,a \in S) \text{ if } (s,t) \in R, \text{ then } (as, at) \in R,$$

and *right compatible* if

$$(\text{for all } s,t,a \in S) \text{ if } (s,t) \in R, \text{ then } (sa, ta) \in R.$$

It is called *compatible* if R is left and right compatible. A left [right] compatible equivalence relation is called a *left [right] congruence*. A compatible equivalence relation is called a *congruence*.

Definition 2.1.5. A *partially ordered set* is a nonempty set A together with a relation R on A (called a partial ordering of A) which is reflexive and transitive and

antisymmetric: if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ for all $a, b \in A$.

If R is a partial ordering of A, then we usually write $a \leq b$ in place of $(a,b) \in R$. In this notation the conditions reflexive, transitive and antisymmetric become (for all $a, b, c \in A$):

$$a \leq a;$$
$$\text{if } a \leq b \text{ and } b \leq c, \text{ then } a \leq c;$$
$$\text{if } a \leq b \text{ and } b \leq a, \text{ then } a = b.$$

We write $a < b$ if $a \leq b$ and $a \neq b$.

Elements $a, b \in A$ are said to be *comparable*, provided $a \leq b$ or $b \leq a$. However, two given elements of a partially ordered set need not be comparable. A partial ordering of a set A such that any two elements are comparable is called a *linear ordering*.

Let $(A, \leq)$ be a partially ordered set. An element $a \in A$ is *maximal* in A if for every $c \in A$, if $a \leq c$, then $a = c$. An *upper bound* of a nonempty subset B of A is an element $d \in A$ such that $b \leq d$ for every $b \in B$. A nonempty subset B of A that is linearly ordered by $\leq$ is called a *chain* in A.

Theorem 2.1.1. [6] ***(Zorn's Lemma)*** *If A is a nonempty partially ordered set such that every chain in A has an upper bound in A, then A contains a maximal element.*

Definition 2.1.6. A partially ordered set $(L, \leq)$ is called a *join-semilattice* and a *meet-semilattice* if each two-element subset $\{a, b\} \subseteq L$ has a join (i.e. the least upper bound) and a meet (i.e. the greatest lower bound), denoted by $a \vee b$ and $a \wedge b$, respectively. $(L, \leq)$ is called a *lattice* if it is both a join-semilattice and a meet-semilattice.

Definition 2.1.7. A partially ordered set $(L, \leq)$ is a *complete lattice* if every nonempty subset A of L has both the greatest lower bound and the least upper bound.

2.2 Ideals and Green's Relations

Definition 2.2.1. A nonempty subset A of a semigroup S is called a *left ideal* of S if $SA \subseteq A$, a *right ideal* of S if $AS \subseteq A$, and an *(two-sided) ideal* of S if it is both a left and a right ideal.

Note that if S has the identity, then A is an ideal of S if SAS is contained in A.

Theorem 2.2.1. [5] *Let $I = \{I_j : j \in J\}$ be a family of ideals of S. Then $\bigcup\limits_{j \in J} I_j$ is an ideal of S.*

Theorem 2.2.2. [5] *Let $I = \{I_j : j \in J\}$ be a family of ideals of S. If $\bigcap\limits_{j \in J} I_j \neq \emptyset$, then $\bigcap\limits_{j \in J} I_j$ is an ideal of S.*

We note that if A and B are ideals of a semigroup S, then $A \cup B$ and $A \cap B$ are ideals of S.

For any semigroup S, the notation S^1 means S itself if S contains the identity element, otherwise, we let $S^1 = S \cup \{1\}$ and define the binary operation on S^1 by

$$1 \cdot s = s = s \cdot 1 \text{ for all } s \in S,\ 1 \cdot 1 = 1 \text{ and}$$
$$a \cdot b = ab \text{ for all } a, b \in S.$$

Then S^1 becomes a semigroup with the identity element 1.

For any element a in S,

the smallest left ideal of S containing a is $Sa \cup \{a\} = S^1a$,

the smallest right ideal of S containing a is $aS \cup \{a\} = aS^1$, and

the smallest ideal of S containing a is $SaS \cup aS \cup Sa \cup \{a\} = S^1aS^1$,

which we call the *principal left ideal*, *principal right ideal* and *principal ideal generated by* a, respectively.

An ideal I such that $I \subseteq S$ and $I \neq S$ is called a *proper ideal* of S.

Definition 2.2.2. Let I be a proper ideal of a semigroup S. Then

$$\rho_I = (I \times I) \cup 1_S$$

is a congruence on S where $1_S = \{(a,a) | a \in S\}$. Note that $x\rho_I y$ if and only if either $x = y$ or both x and y belong to I. The relation ρ_I is called a *Rees congruence.*

Lemma 2.2.3. *Let S be a semigroup and I be an ideal of S. Suppose that δ is a congruence on S such that $1_S \subsetneq \delta \subseteq \rho_I$. If $x\delta y$ for all $x, y \in I$, then $\delta = \rho_I$.*

Proof. Suppose that δ is a congruence on S such that $1_S \subsetneq \delta \subseteq \rho_I$ and $x\delta y$ for all $x, y \in I$. Let $(a,b) \in \rho_I$. Then $a = b$ or $a, b \in I$. If $a = b$, then $(a,b) \in \delta$. If $a, b \in I$, then $(a,b) \in \delta$ by our supposition. □

In 1951, J.A. Green defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on S by the rules that, for $a, b \in S$,

$a\mathcal{L}b$ if and only if $S^1a = S^1b$,
$a\mathcal{R}b$ if and only if $aS^1 = bS^1$ and
$a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$.

Then he defined the equivalence relations

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R},$$

and obtained that the composition of $\mathcal{L}$ and $\mathcal{R}$ is commutative. This follows that $\mathcal{D}$ is the join $\mathcal{L} \vee \mathcal{R}$, that is, $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L} \cup \mathcal{R}$. Moreover, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. But, in commutative semigroups, we have $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ are called Green's relations on S.

Theorem 2.2.4. [5] *Let S be a semigroup and $a, b \in S$. Then*
(1) *$a\mathcal{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$;*
(2) *$a\mathcal{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$;*
(3) *$a\mathcal{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.*

2.3 Transformation Semigroups

In this section, we list some known results, definitions and notations about transformation semigroups that will be used throughout this thesis.

2.3.1 The Semigroups $T(X)$

Let X be a nonempty set and $T(X)$ denote the set of all transformations from X into itself. Then $T(X)$ is a semigroup under the composition of maps, that is, if $\alpha, \beta \in T(X)$, then $\alpha\beta \in T(X)$ is defined by

$$x(\alpha\beta) = (x\alpha)\beta \text{ for all } x \in X,$$

and it is called the *full transformation semigroup* on X.

For a nonempty subset A of X, we let id_A denote the identity map on A. Then it is clear that id_X is the identity element of $T(X)$.

Definition 2.3.1. Let S be a semigroup with identity 1. An element $a \in S$ is called a *unit* of S if there exists $b \in S$ such that $ab = 1 = ba$.

Lemma 2.3.1. [4] *Let S be a semigroup with identity* 1 *and*

$$G = \{x \in S \ : x \text{ is a unit of } S\}.$$

Then G is a maximal subgroup of S having 1 *as the identity.*

We call the subgroup G of S (in lemma 2.3.1) the *group of unit* of S.

Let $G(X)$ be the set of all injective maps from X onto X. Then $G(X)$ is a group of units of $T(X)$.

2.3.2 Transformation Semigroups with Invariant Sets

Fix a nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of total transformations on X which leave a subset Y of X invariant and

$$E = \{\alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha\}$$

is the set of all regular elements of $S(X,Y)$.

As in Clifford and Preston [2] vol 2, p. 241, we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

With the above notation, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \varnothing$; $B_j, C_k \subseteq X \setminus Y$; $\{a_i\} \subseteq Y$, $\{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, I is a nonempty set, but J or K can be empty.

Lemma 2.3.2. [4] *The following statements are equivalent:*

(1) *E is a regular subsemigroup of $S(X,Y)$;*

(2) *$S(X,Y)$ is a regular semigroup;*

(3) *$X = Y$ or $|Y| = 1$.*

Lemma 2.3.3. [4] *$S(X,Y)$ has the zero element if and only if $|Y| = 1$.*

We note that for any $\alpha \in S(X,Y)$, the symbol π_α denotes the decomposition of X induced by the map α, namely $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$. For a nonempty subset Z of X, we denote $\pi_\alpha(Z)$ by $\pi_\alpha(Z) = \{x\alpha^{-1} : x \in X\alpha \cap Z\}$. Thus $\pi_\alpha(Y) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$.

Green's relations on $S(X,Y)$ are given by P. Honyam and J. Sanwong [4]. For convenience, we present them here.

Lemma 2.3.4. [4] *Let $\alpha, \beta \in S(X,Y)$. Then*

(1) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$;*

(2) *$\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$;*

(3) *$\alpha\mathcal{J}\beta$ if and only if $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

Lemma 2.3.5. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(X,Y)$ if and only if $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$.*

Let $G(X)$ be the group of permutations on the set X. Define

$$G(X,Y) = \{\alpha \in G(X) : \alpha\,|_Y \in G(Y)\},$$

where $Y \subseteq X$ and $\alpha\,|_Y$ is the restriction of α on the set Y. Then $G(X,Y)$ is a subgroup of the permutation group $G(X)$. Moreover, $G(X,Y)$ is a group of units of $S(X,Y)$.

Let X be a finite set with n elements and Y a nonempty subset of X with m elements, then we define

$$J_{r,s,t} = \{\alpha \in S(X,Y) : |X\alpha| = r,\ |Y\alpha| = s \text{ and } |X\alpha \setminus Y| = t\}$$

and

$$J_k = \{\alpha \in S(X,Y) : |X\alpha| = k\}$$

where $1 \leqslant r \leqslant n,\ 1 \leqslant s \leqslant m,\ 0 \leqslant t \leqslant n-m$ and $1 \leqslant k \leqslant n$. Thus $J_{r,s,t}$ is a $\mathcal{J}$-class of $S(X,Y)$ and J_1 is the set of all constant maps with image in Y.

Let p be any cardinal number and let

$$p' = min\{q : q > p\}$$

Note that p' always exists since the cardinals are well-ordered and when p is finite we have $p' = p+1$ is the successor of p.

As shown by Malcev, the ideals of $T(X)$ for any set X are precisely the sets:

$$T_r = \{\alpha \in T(X) : |X\alpha| < r\},$$

where $2 \leq r \leq |X|'$ (see also [2] vol 2, Theorem 10.59).

To describe ideals of $S(X,Y)$ for any set X and any nonempty subset Y of X, we let $|X| = a$, $|Y| = b$ and $|X \setminus Y| = c$. In addition, for each cardinals r, s, t such that $2 \leq r \leq a'$, $2 \leq s \leq b'$ and $1 \leq t \leq c'$, we define

$$S(r,s,t) = \{\alpha \in S(X,Y) : |X\alpha| < r,\ |Y\alpha| < s \ \text{ and } |X\alpha \setminus Y| < t\}.$$

Theorem 2.3.6. [4] *The set $S(r,s,t)$ is an ideal of $S(X,Y)$.*

To obtain ideals of $S(X,Y)$ we need the following notation. Let Z be a nonempty subset of $S(X,Y)$. Define

$$\begin{aligned} K(Z) = \{\alpha \in S(X,Y) : & |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \text{ and} \\ & |X\alpha \setminus Y| \leq |X\beta \setminus Y| \text{ for some } \beta \in Z\}. \end{aligned}$$

Then we see that $Z \subseteq K(Z)$ and $Z_1 \subseteq Z_2$ implies that $K(Z_1) \subseteq K(Z_2)$.

Theorem 2.3.7. [4] *The ideals of $S(X,Y)$ are precisely the sets $K(Z)$ for some nonempty subset Z of $S(X,Y)$.*