CHAPTER 3

Main Results

Throughout this section, we assume that X is a finite set and Y a nonempty subset of X. We consider the subset of $S(X,Y)$ as follow.

Let $|X| = a$, $|Y| = b$ and $|X \setminus Y| = c$. For each of cardinals r, s, t such that $2 \leq r \leq a'$, $2 \leq s \leq b'$ and $1 \leq t \leq c'$, we define

$$S(r,s,t) = \{\alpha \in S(X,Y) : \ |X\alpha| < r, \ |Y\alpha| < s \text{ and } |X\alpha \setminus Y| < t\}.$$

In [4], the ideals of $S(X,Y)$ are precisely the sets $K(Z)$ for some nonempty subset Z of $S(X,Y)$ defined by

$$K(Z) = \{\alpha \in S(X,Y) : \ |X\alpha| \leq |X\beta|, \ |Y\alpha| \leq |Y\beta| \text{ and } |X\alpha \setminus Y| \leq |X\beta \setminus Y| \text{ for some } \beta \in Z\}.$$

Then we see that $Z \subseteq K(Z)$ and $Z_1 \subseteq Z_2$ implies that $K(Z_1) \subseteq K(Z_2)$. For convenience, if $Z = \{\alpha_1, ..., \alpha_n\}$ is a finite set, we write $K(\alpha_1, ..., \alpha_n)$ in stead of $K(Z)$.

3.1 Some Properties of $K(Z)$

In [4] the ideals of $S(X,Y)$ are of the form $K(Z)$ for some nonempty subset Z of $S(X,Y)$. The following proposition gives the relation between $K(Z)$ and $S(r,s,t)$.

Proposition 3.1.1. *Let $\beta \in S(X,Y)$ with $|X\beta| = r$, $|Y\beta| = s$ and $|X\beta \setminus Y| = t$. Then $K(\beta) = S(r+1, s+1, t+1)$.*

Proof. By the definition, we have

$$\begin{aligned} K(\beta) &= \{\alpha \in S(X,Y) : |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \text{ and } |X\alpha \setminus Y| \leq |X\beta \setminus Y|\}, \\ &= \{\alpha \in S(X,Y) : |X\alpha| \leq r, |Y\alpha| \leq s \text{ and } |X\alpha \setminus Y| \leq t\}, \\ &= \{\alpha \in S(X,Y) : |X\alpha| < r+1, |Y\alpha| < s+1 \text{ and } |X\alpha \setminus Y| < t+1\}, \\ &= S(r+1, s+1, t+1). \end{aligned}$$

□

Proposition 3.1.2. *Let $\beta, \gamma \in S(X,Y)$. Then $K(\beta, \gamma) = K(\beta) \cup K(\gamma)$.*

Proof. Let $\alpha \in K(\beta,\gamma)$. Then $\alpha \in S(X,Y)$ with $|X\alpha| \leq |X\delta|, |Y\alpha| \leq |Y\delta|$ and $|X\alpha \setminus Y| \leq |X\delta \setminus Y|$ for some $\delta \in \{\beta,\gamma\}$. If $\delta = \beta$, then $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$. If $\delta = \gamma$, then $|X\alpha| \leq |X\gamma|, |Y\alpha| \leq |Y\gamma|$ and $|X\alpha \setminus Y| \leq |X\gamma \setminus Y|$. So $\alpha \in K(\beta)$ or $\alpha \in K(\gamma)$. Thus $\alpha \in K(\beta) \cup K(\gamma)$ and then $K(\beta,\gamma) \subseteq K(\beta) \cup K(\gamma)$. Conversely, assume that $\alpha \in K(\beta) \cup K(\gamma)$. Then $\alpha \in K(\beta)$ or $\alpha \in K(\gamma)$. If $\alpha \in K(\beta)$, then $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$. Thus $\alpha \in K(\beta,\gamma)$. If $\alpha \in K(\gamma)$, then $|X\alpha| \leq |X\gamma|, |Y\alpha| \leq |Y\gamma|$ and $|X\alpha \setminus Y| \leq |X\gamma \setminus Y|$. Thus $\alpha \in K(\beta,\gamma)$ and then $K(\beta) \cup K(\gamma) \subseteq K(\beta,\gamma)$. Hence $K(\beta,\gamma) = K(\beta) \cup K(\gamma)$. □

By Mathematical Induction and Proposition 3.1.2, we have the following corollary.

Corollary 3.1.3. *Let $\beta_1, \beta_2, ..., \beta_n \in S(X,Y)$. Then*

$$K(\beta_1, \beta_2, ..., \beta_n) = K(\beta_1) \cup K(\beta_2) \cup ... \cup K(\beta_n).$$

Proposition 3.1.4. *Let $\alpha, \beta \in S(X,Y)$. Then $K(\alpha) = K(\beta)$ if and only if $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

Proof. Assume that $K(\alpha) = K(\beta)$. Since $\beta \in K(\beta)$, we have $\beta \in K(\alpha)$. Then $|X\beta| \leq |X\alpha|$, $|Y\beta| \leq |Y\alpha|$ and $|X\beta \setminus Y| \leq |X\alpha \setminus Y|$. Since $\alpha \in K(\alpha)$, we obtain that $\alpha \in K(\beta)$. Then $|X\alpha| \leq |X\beta|$, $|Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$. Thus $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$. Conversely, we may assume that $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.

$$\begin{aligned} \gamma \in K(\alpha) &\Leftrightarrow \gamma \in S(X,Y),\ |X\gamma| \leq |X\alpha|,\ |Y\gamma| \leq |Y\alpha| \text{ and } |X\gamma \setminus Y| \leq |X\alpha \setminus Y|; \\ &\Leftrightarrow \gamma \in S(X,Y),\ |X\gamma| \leq |X\beta|,\ |Y\gamma| \leq |Y\beta| \text{ and } |X\gamma \setminus Y| \leq |X\beta \setminus Y|; \\ &\Leftrightarrow \gamma \in K(\beta). \end{aligned}$$

□

Proposition 3.1.5. *Let $\alpha \in S(X,Y)$. Then $K(\alpha)$ is a principal ideal of $S(X,Y)$.*

Proof. Let $S = S(X,Y)$ and $\alpha \in S$. We prove that $K(\alpha) = S\alpha S$. Let $\beta \in K(\alpha)$. Then $|X\beta| \leq |X\alpha|$, $|Y\beta| \leq |Y\alpha|$ and $|X\beta \setminus Y| \leq |X\alpha \setminus Y|$. By Lemma 2.3.5, we have $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S$. Thus $\beta \in S\alpha S$. Conversely, let $\gamma \in S\alpha S$. Then $\gamma = \theta\alpha\eta$ for some $\theta, \eta \in S$. Again by Lemma 2.3.5, we obtain that $|X\gamma| \leq |X\alpha|$, $|Y\gamma| \leq |Y\alpha|$ and $|X\gamma \setminus Y| \leq |X\alpha \setminus Y|$. So $\gamma \in K(\alpha)$. Hence $K(\alpha)$ is a principal ideal of $S(X,Y)$. □

Theorem 3.1.6. *The ideals of $S(X,Y)$ are precisely the set $K(\alpha_1) \cup ... \cup K(\alpha_n)$, where $\alpha_1, ..., \alpha_n \in S(X,Y)$.*

Proof. By Corollary 3.1.3, we obtain that

$$K(\alpha_1) \cup ... \cup K(\alpha_n) = K(\alpha_1, \alpha_2, ..., \alpha_n).$$

Then $K(\alpha_1) \cup ... \cup K(\alpha_n)$ is an ideal of $S(X,Y)$ by Theorem 2.3.7.

Let I be an ideal of $S(X,Y)$. By Theorem 2.3.7, we get that $I = K(Z)$ for some $\emptyset \neq Z \subseteq S(X,Y)$. Since $S(X,Y)$ is a finite set, we can let $Z = \{\alpha_1, ..., \alpha_n\}$. By Corollary 3.1.3, we get that

$$\begin{aligned} I &= K(Z), \\ &= K(\alpha_1, ..., \alpha_n), \\ &= K(\alpha_1) \cup ... \cup K(\alpha_n). \end{aligned}$$ □

If X is a finite set with n elements and Y a nonempty subset of X with m elements, then we define

$$J_{r,s,t} = \{\alpha \in S(X,Y) : |X\alpha| = r,\ |Y\alpha| = s \text{ and } |X\alpha \setminus Y| = t\}$$

and

$$J_k = \{\alpha \in S(X,Y) : |X\alpha| = k\}$$

where $1 \leqslant r \leqslant n,\ 1 \leqslant s \leqslant m,\ 0 \leqslant t \leqslant n-m$ and $1 \leqslant k \leqslant n$. Thus $J_{r,s,t}$ is a $\mathscr{J}$-class and J_k is a union of $\mathscr{J}$-classes.

Example 1. Let $X = \{1,2,3\}$ and $Y = \{1,2\}$. Then $S(X,Y)$ has 12 elements, namely

$$1_X = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ X_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix},\ \beta_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix},\ \beta_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix},\ \beta_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix},\ \beta_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix},\ \text{and } \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}.$$

So $K(1_X) = S(X,Y) = K(\alpha),\ K(X_1) = J_1 = K(X_2),\ K(\beta_1) = K(\beta_6) = J_1 \cup J_{2,1,0}$, $K(\beta_2) = K(\beta_3) = K(\beta_4) = K(\beta_5) = J_1 \cup J_{2,1,0} \cup J_{2,2,0}$ and $K(\mu_1) = K(\mu_2) = J_1 \cup J_{2,1,0} \cup J_{2,1,1}$. Thus there are only five principal ideals of $S(X,Y)$:

$$K(1_X),\ K(X_1),\ K(\beta_1),\ K(\beta_2), \text{ and } K(\mu_1).$$

But $K(\beta_2) \cup K(\mu_1) = K(\beta_2, \mu_1) = J_1 \cup J_{2,1,0} \cup J_{2,1,1} \cup J_{2,2,0}$ is not a principal ideal of $S(X,Y)$.

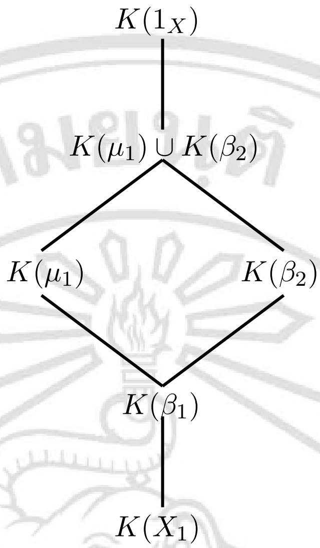

Figure 1

3.2 The Lattice of Ideals of $S(X,Y)$

Let $\mathscr{I}$ be the set of all ideals of $S(X,Y)$. Then $(\mathscr{I}, \subseteq)$ is a partially ordered set with the following properties.

Proposition 3.2.1. *J_1 is the minimum ideal of $S(X,Y)$ and is a right zero semigroup.*

Proof. We prove that $J_1 = S(2,2,1)$. It is clear that $J_1 \subseteq S(2,2,1)$. Let $\alpha \in S(2,2,1)$. Then $|X\alpha| < 2$, $|Y\alpha| < 2$, $|X\alpha \setminus Y| < 1$ and thus $|X\alpha| = 1 = |Y\alpha|$. So α is the constant map and $\alpha \in J_1$. Therefore, J_1 is an ideal of $S(X,Y)$. To show that J_1 is the minimum ideal, let I be an ideal of $S(X,Y)$ and $\beta \in J_1$. Then there exists $\emptyset \neq Z \subseteq S(X,Y)$ such that $I = K(Z)$. Let $\gamma \in Z$. Then $|X\gamma|, |Y\gamma| \geq 1$ and $|X\gamma \setminus Y| \geq 0$, so $|X\beta| = 1 \leq |X\gamma|$, $|Y\beta| = 1 \leq |Y\gamma|$ and $|X\beta \setminus Y| = 0 \leq |X\gamma \setminus Y|$. Thus $\beta \in I$, that is $J_1 \subseteq I$ as required. Hence J_1 is the minimum ideal of $S(X,Y)$. Next, we show that J_1 is a right zero semigroup. Let $\lambda, \mu \in J_1$. Then $|X\lambda| = 1 = |X\mu|$ and let $X\lambda = \{a\}$, $X\mu = \{b\}$ where $a, b \in Y$. Let $x \in X$. Thus $x(\lambda\mu) = (x\lambda)\mu = a\mu = b = x\mu$. Hence $\lambda\mu = \mu$. □

Proposition 3.2.2. *$(\mathscr{I}, \subseteq)$ is a complete lattice.*

Proof. Let $\emptyset \neq \mathscr{S} \subseteq \mathscr{I}$ be such that $\mathscr{S} = \{I_i \mid i \in \Omega\}$ for some nonempty index set Ω. We show that the least upper bound of $\mathscr{S}$ is $\bigcup_{i\in\Omega} I_i$. By Theorem 2.2.1, $\bigcup_{i\in\Omega} I_i \in \mathscr{I}$. Since $I_j \subseteq \bigcup_{i\in\Omega} I_i$ for all $j \in \Omega$, we obtain that $\bigcup_{i\in\Omega} I_i$ is an upper bound of $\mathscr{S}$. Let A be an upper bound of $\mathscr{S}$ and $a \in \bigcup_{i\in\Omega} I_i$. Then $a \in I_k$ for some k in Ω. Since A is an upper bound of $\mathscr{S}$, we have $a \in I_k \subseteq A$. Thus $\bigcup_{i\in\Omega} I_i \subseteq A$. So $\bigcup_{i\in\Omega} I_i$ is the least upper bound of $\mathscr{S}$. Now, we show that the greatest lower bound of $\mathscr{S}$ is $\bigcap_{i\in\Omega} I_i$. Since $J_1 \subseteq I_i$ for all $i \in \Omega$, we have $J_1 \subseteq \bigcap_{i\in\Omega} I_i$. Then $\bigcap_{i\in\Omega} I_i \neq \varnothing$. So $\bigcap_{i\in\Omega} I_i \in \mathscr{I}$ by Theorem 2.2.2. Since $\bigcap_{i\in\Omega} I_i \subseteq I_j$ for all j in Ω, we obtain that $\bigcap_{i\in\Omega} I_i$ is a lower bound of $\mathscr{S}$. Let B be a lower bound of $\mathscr{S}$ and $b \in B$. Since $B \subseteq I_i$ for all i in Ω, we obtain $b \in I_i$ for all i in Ω. Thus $b \in \bigcap_{i\in\Omega} I_i$. Hence $B \subseteq \bigcap_{i\in\Omega} I_i$. So $\bigcap_{i\in\Omega} I_i$ is the greatest lower bound of $\mathscr{S}$. Hence $(\mathscr{I}, \subseteq)$ is a complete lattice. □

Lemma 3.2.3. *If $|Y| = 1$ and $J_{2,s,t} \neq \emptyset$, then $J_1 \cup J_{2,s,t}$ is an ideal of S(X,Y) if and only if $s = 1 = t$.*

Proof. Assume that $|Y| = 1$ and $J_{2,s,t} \neq \emptyset$. Suppose that $J_1 \cup J_{2,s,t}$ is an ideal of $S(X,Y)$. Let $\alpha \in J_{2,s,t}$. Since $Y\alpha \subseteq Y$, we have $1 \leq |Y\alpha| \leq |Y| = 1$, so $|Y\alpha| = 1$ which implies that $s = |Y\alpha| = 1$. Since $|X\alpha| = 2$ and $Y = Y\alpha$, we have $|X\alpha \setminus Y| = |X\alpha \setminus Y\alpha| = 2 - 1 = 1$, that is $t = 1$.

Conversely, assume that $s = 1 = t$. First, we show that $J_1 \cup J_{2,1,1} = S(3,2,2)$. Let $\alpha \in J_1 \cup J_{2,1,1}$. Then $\alpha \in J_1$ or $\alpha \in J_{2,1,1}$. If $\alpha \in J_1$, then $|X\alpha| = 1 < 3, |Y\alpha| = |X\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 1 - 1 = 0 < 2$. Thus $\alpha \in S(3,2,2)$. If $\alpha \in J_{2,1,1}$, then $|X\alpha| = 2 < 3, |Y\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 1 < 2$. Thus $\alpha \in S(3,2,2)$. For the other containment, let $\alpha \in S(3,2,2)$. Then $|X\alpha| \leq 2, |Y\alpha| \leq 1$ and $|X\alpha \setminus Y| \leq 1$. If $|X\alpha| = 1$, then $\alpha \in J_1$. If $|X\alpha| = 2$, then $|X\alpha \setminus Y| = |X\alpha \setminus Y\alpha| = 2 - 1 = 1$. Then $\alpha \in J_{2,1,1}$. Hence $J_1 \cup J_{2,1,1} = S(3,2,2)$ is an ideal of $S(X,Y)$. □

Since J_1 is the minimum ideal of $S(X,Y)$, we define a minimal ideal in $S(X,Y)$ as follows. An ideal I of $S(X,Y)$ is a minimal ideal if J is an ideal such that $J_1 \subseteq J \subseteq I$, then either $J = J_1$ or $J = I$.

Theorem 3.2.4. *If $|Y| = 1$, then $J_1 \cup J_{2,1,1}$ is the unique minimal ideal of $S(X,Y)$.*

Proof. Suppose that $Y = \{a\}$. By Lemma 3.2.3, we have $J_1 \cup J_{2,1,1}$ is an ideal of $S(X,Y)$. Next, we show that $J_1 \cup J_{2,1,1}$ is a minimal ideal of $S(X,Y)$. Let J be an ideal of $S(X,Y)$ such that $J_1 \subseteq J \subseteq J_1 \cup J_{2,1,1}$. Suppose that $J \subsetneq J_1 \cup J_{2,1,1}$. It is clear that $J_1 \subseteq J$. By our supposition, there exists $\alpha \in J_{2,1,1}$ but $\alpha \notin J$. We show that $J \subseteq J_1$ by supposing this is false, so $J \nsubseteq J_1$. Then there exists $\beta \in J$ such that $\beta \notin J_1$. Since $\alpha, \beta \in J_{2,1,1}$, we can write

$$\alpha = \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix}$$

where $Y \subseteq A, b \in X \setminus Y$ and

$$\beta = \begin{pmatrix} B & X \setminus B \\ a & c \end{pmatrix}$$

where $Y \subseteq B, c \in X \setminus Y$. Let $\gamma, \theta \in S(X, Y)$ be defined by

$$\gamma = \begin{pmatrix} A & X \setminus A \\ a & v \end{pmatrix}, \ \theta = \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix}$$

where $v \in X \setminus B$. Consider

$$\begin{aligned} \gamma\beta\theta &= \begin{pmatrix} A & X \setminus A \\ a & v \end{pmatrix} \begin{pmatrix} B & X \setminus B \\ a & c \end{pmatrix} \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix} \\ &= \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix} = \alpha. \end{aligned}$$

Then $\alpha = \gamma\beta\theta \in J$ which is a contradiction. So $J = J_1$. Hence $J_1 \cup J_{2,1,1}$ is a minimal ideal of $S(X, Y)$. Finally, we show that $J_1 \cup J_{2,1,1}$ is the unique minimal ideal of $S(X, Y)$. Let $M = J_1 \cup J_{2,1,1}$ and N be a minimal ideal of $S(X, Y)$. We show that $M = N$. Since N is an ideal of $S(X, Y)$, we get that $N = K(Z)$ for some $\emptyset \neq Z \subseteq S(X, Y)$. Since $J_1 \subsetneq N$, there exists $\alpha \in N$ with $|X\alpha| \geq 2$. Since $\alpha \in N = K(Z)$, we obtain that $|X\alpha| \leq |X\beta|$, $|Y\alpha| \leq |Y\beta|$ and $|X\alpha \backslash Y| \leq |X\beta \backslash Y|$ for some $\beta \in Z$. Let $\gamma \in J_{2,1,1}$. Then $|X\gamma| = 2$ and so $|X\gamma| \leq |X\alpha| \leq |X\beta|$. Since $|Y| = 1$, we have $|Y\gamma| = 1 = |Y\alpha| \leq |Y\beta|$ and $|X\gamma \backslash Y| = 1 \leq |X\alpha \backslash Y| \leq |X\beta \backslash Y|$. Then $\gamma \in K(Z) = N$. Thus $J_{2,1,1} \subseteq N$ and so $J_1 \cup J_{2,1,1} \subseteq N$ which implies by the minimality of N that $M = N$. □

However, minimal ideals of a semigroup S does not always exist.

Example 2. Let $\mathbb{N}$ be the set of all natural numbers. Then $\mathbb{N}$ is a semigroup under the addition. Thus I is an ideal of $\mathbb{N}$ if and only if $I = \{n \in \mathbb{N} : n \geq a\}$ for some $a \in \mathbb{N}$. We show that $\mathbb{N}$ does not contain minimal ideals. Assume that M is a minimal ideal of $\mathbb{N}$.

Then $M = \{m \in \mathbb{N} : m \geq a\}$ for some $a \in \mathbb{N}$. But there exists $M' = \{m' \in \mathbb{N} : m' \geq a+1\}$ such that $M' \subseteq M$ and $M' \neq M$ since $a \in M$ and $a \notin M'$ which is a contradiction. Then $\mathbb{N}$ does not contain minimal ideals.

If $X = Y$, then $S(X,Y) = T(X)$, and [5] has been characterized the ideals of $T(X)$ for any set X. So we will consider only the case where Y is a nonempty proper subset of X.

Lemma 3.2.5. *If* $|Y| > 1$ *and* $J_{2,s,t} \neq \emptyset$, *then* $J_1 \cup J_{2,s,t}$ *is an ideal of* $S(X,Y)$ *if and only if* $s = 1, t = 0$.

Proof. Assume that $|Y| > 1$ and $J_{2,s,t} \neq \emptyset$.

Suppose that $J_1 \cup J_{2,s,t}$ is an ideal. Let $\alpha \in J_{2,s,t}$. Then $|X\alpha| = 2, |Y\alpha| = s$ and $|X\alpha \setminus Y| = t$. Since $|X\alpha| = 2$ and $1 \leq |Y\alpha| \leq |X\alpha| = 2$, we have $1 \leq s \leq 2$, so $0 \leq |X\alpha \setminus Y| \leq 1$. Thus $0 \leq t \leq 1$. So there are four possible cases: $s = 2$ and $t = 0$; $s = 2$ and $t = 1$; $s = 1 = t$; or $s = 1$ and $t = 0$.

If $s = 2$ and $t = 1$, then $|X\alpha| = 2 = |Y\alpha|$. Since $Y\alpha \subseteq X\alpha$, we obtain that $X\alpha = Y\alpha$ and thus $t = |X\alpha \setminus Y| = |Y\alpha \setminus Y| = 0$ which is a contradiction.

If $s = 2$ and $t = 0$, then $J_1 \cup J_{2,s,t} = J_1 \cup J_{2,2,0}$. Let $\beta \in J_{2,2,0}$ be such that

$$\beta = \begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix}$$

where $a, b \in Y$ and $a \neq b$. Let $\gamma \in S(X,Y)$ defined by

$$\gamma = \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix}.$$

So

$$\gamma\beta = \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix} \notin J_1 \cup J_{2,2,0}.$$

Then $J_1 \cup J_{2,2,0}$ is not an ideal which is a contradiction.

If $s = 1 = t$, then $J_1 \cup J_{2,s,t} = J_1 \cup J_{2,1,1}$. Let $\lambda \in J_{2,1,1}$. So $|X\lambda| = 2$ and $Y\lambda \subseteq Y$, thus we can write

$$\lambda = \begin{pmatrix} A & X \setminus A \\ u & v \end{pmatrix}$$

where $u \in Y \subseteq A$ and $v \in X \setminus Y$. Since $|Y| > 1$, there exists $u \neq w \in Y$ and define $\mu \in S(X,Y)$ by

$$\mu = \begin{pmatrix} Y & X \setminus Y \\ u & w \end{pmatrix}.$$

So

$$\lambda\mu = \begin{pmatrix} A & X \setminus A \\ u & w \end{pmatrix} \notin J_1 \cup J_{2,1,1}.$$

Thus $J_1 \cup J_{2,1,1}$ is not an ideal which is a contradiction. Therefore, $s = 1$ and $t = 0$.

Conversely, assume that $s = 1$ and $t = 0$. We show that $J_1 \cup J_{2,1,0} = S(3,2,1)$. Let $\alpha \in J_1 \cup J_{2,1,0}$. Then $\alpha \in J_1$ or $\alpha \in J_{2,1,0}$. If $\alpha \in J_1$, then $|X\alpha| = 1 < 3$, $|Y\alpha| \leq |X\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 1 - 1 = 0 < 1$. Thus $\alpha \in S(3,2,1)$. If $\alpha \in J_{2,1,0}$, then $|X\alpha| = 2 < 3$, $|Y\alpha| = 1 < 2$ and $|X\alpha \setminus Y| = 0 < 1$. Thus $\alpha \in S(3,2,1)$. For the other containment, let $\alpha \in S(3,2,1)$. Then $|X\alpha| \leq 2, |Y\alpha| = 1$ and $|X\alpha \setminus Y| = 0$. If $|X\alpha| = 1$, then $\alpha \in J_1$. If $|X\alpha| = 2, |Y\alpha| = 1$ and $|X\alpha \setminus Y| = 0$, then $\alpha \in J_{2,1,0}$. □

Theorem 3.2.6. *If $|Y| > 1$, then $J_1 \cup J_{2,1,0}$ is the unique minimal ideal of $S(X,Y)$.*

Proof. Suppose that $|Y| > 1$. By Lemma 3.2.5, we have $J_1 \cup J_{2,1,0}$ is an ideal of $S(X,Y)$. To show that $J_1 \cup J_{2,1,0}$ is a minimal ideal of $S(X,Y)$, let J be an ideal of $S(X,Y)$ such that $J_1 \subseteq J \subsetneq J_1 \cup J_{2,1,0}$. It is clear that $J_1 \subseteq J$. By assumption, we have there exists $\alpha \in J_{2,1,0}$ but $\alpha \notin J$. We prove that $J \subseteq J_1$ by supposing this is false. Then there exists $\beta \in J$, but $\beta \notin J_1$. Since $\alpha, \beta \in J_{2,1,0}$, we can write

$$\alpha = \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix}$$

where $a, b \in Y \subseteq A$ and

$$\beta = \begin{pmatrix} B & X \setminus B \\ a' & c \end{pmatrix}$$

where $a', c \in Y \subseteq B$. Let $\gamma, \theta \in S(X,Y)$ be defined by

$$\gamma = \begin{pmatrix} A & X \setminus A \\ u & v \end{pmatrix}, \ \theta = \begin{pmatrix} a' & X \setminus \{a'\} \\ a & b \end{pmatrix}$$

where $u \in Y$ and $v \in X \setminus B$. So

$$\gamma\beta\theta = \begin{pmatrix} A & X \setminus A \\ u & v \end{pmatrix} \begin{pmatrix} B & X \setminus B \\ a' & c \end{pmatrix} \begin{pmatrix} a' & X \setminus \{a'\} \\ a & b \end{pmatrix}$$

$$= \begin{pmatrix} A & X \setminus A \\ a & b \end{pmatrix} = \alpha.$$

Then $\alpha = \gamma\beta\theta \in J$ which is a contradiction. Hence $J_1 \cup J_{2,1,0}$ is a minimal ideal of $S(X,Y)$. Now, we show that $J_1 \cup J_{2,1,0}$ is the unique minimal ideal of $S(X,Y)$. Let $M = J_1 \cup J_{2,1,0}$ and N be a minimal ideal of $S(X,Y)$. Since N is an ideal of $S(X,Y)$, we get that $N = K(Z)$ for some $\emptyset \neq Z \subseteq S(X,Y)$. Since $J_1 \subsetneq N$, there exists $\alpha \in N$ with $|X\alpha| \geq 2$. Since $\alpha \in N = K(Z)$, we obtain that $|X\alpha| \leq |X\beta|$, $|Y\alpha| \leq |Y\beta|$ and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$ for some $\beta \in Z$. Let $\gamma \in J_{2,1,0}$. Then $|X\gamma| = 2$ and so $|X\gamma| = 2 \leq |X\alpha| \leq |X\beta|$. Since $\gamma \in J_{2,1,0}$, we have $|Y\gamma| = 1 \leq |Y\alpha| \leq |Y\beta|$ and $|X\gamma \setminus Y| = 0 \leq |X\alpha \setminus Y| \leq |X\beta \setminus Y|$. Then $\gamma \in K(Z) = N$. Thus $J_{2,1,0} \subseteq N$ and so $J_1 \cup J_{2,1,0} \subseteq N$ which implies by the minimality of N that $M = N$. □

Example 3. Let $X = \{1,2,3,4\}$, $Y = \{1,2\}$. Consider the ideals of $S(X,Y)$ in the following diagram.

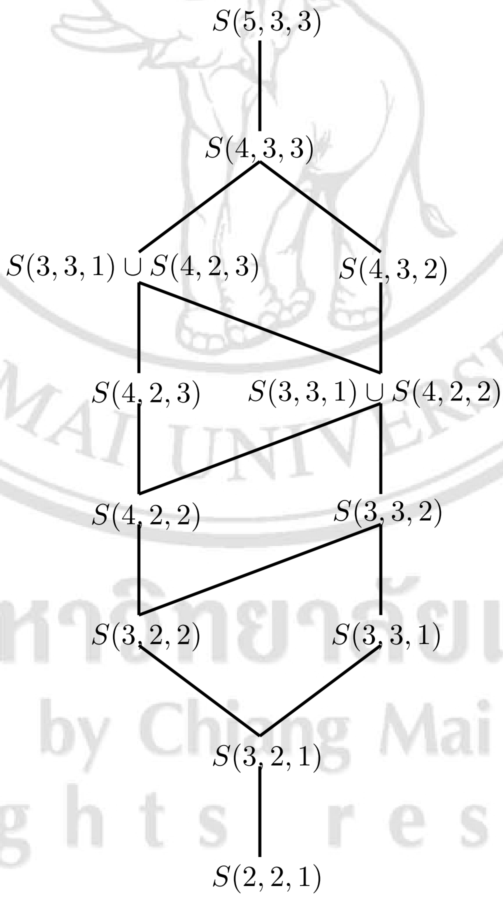

Figure 2

From example 3, it is clear that $S(3,2,1) = J_1 \cup J_{2,1,0}$ is the unique minimal ideal of $S(X,Y)$.

Now, we will define the maximal ideal of $S(X,Y)$. Let I be a proper ideal of $S(X,Y)$. I is said to be a maximal ideal if J is an ideal such that $I \subseteq J \subseteq S(X,Y)$, then either $J = I$ or $J = S(X,Y)$.

Theorem 3.2.7. *Let S be a semigroup with identity* 1. *If S contains a proper ideal, then S has a maximal ideal.*

Proof. Suppose that S contains a proper ideal, say A.
Let $\mathcal{A} = \{B \subseteq S \mid B$ is a proper ideal of S containing $A\}$. Since $A \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$. Let $\subseteq$ be a partial ordering of the set $\mathcal{A}$. In order to apply Zorn's Lemma, we must show that every nonempty chain $\mathscr{C} = \{I_j | j \in J\}$ of $\mathcal{A}$ has an upper bound in $\mathcal{A}$. Let $I = \bigcup_{j \in J} I_j$. By Theorem 2.2.1, I is an ideal of S. Now, we show that I is a proper ideal. We assume that I is not a proper ideal. Then $1 \in S = I$. Thus $1 \in I = \bigcup_{j \in J} I_j$. We obtain that $1 \in I_{j_0}$ for some $j_0 \in J$. So $I_{j_0} = S$, i.e., I_{j_0} is not a proper ideal. This leads to a contradiction. Hence I is a proper ideal of S. Since $A \subseteq I_j$ for all $j \in J$, we have $A \subseteq \bigcup_{j \in J} I_j = I$. And since $I_j \subseteq \bigcup_{j \in J} I_j = I$ for all $j \in J$, we obtain that I is an upper bound of $\mathscr{C}$. Then every nonempty chain has an upper bound in $\mathcal{A}$. By Zorn's Lemma, we obtain that $\mathcal{A}$ contains a maximal element, say M. That is, if N is an element of $\mathcal{A}$ and $M \subseteq N$, then $M = N$. Next, we prove that M is a maximal ideal of S. Since $M \in \mathcal{A}$, we have $M \neq S$. If P is an ideal of S such that $M \subseteq P \subseteq S$. Suppose that $P \neq S$. So $P \in \mathcal{A}$ and $M \subseteq P$. We get $M = P$ by the maximality of M. □

The following example shows that the condition "S is a semigroup with identity" is necessary.

Example 4. Let S be a left zero semigroup and $|S| > 1$. Suppose that S contains an identity, say e. Let $e \neq a \in S$. Then $ea = a$ and so $e = a$ which is a contradiction. Thus S does not contain an identity. Let I be an ideal of S and $i \in I$. Then $a = ai \in I$ for all $a \in S$. So $I = S$. Hence S has no a maximal ideal.

Example 5. Let $S = \{x \in \mathbb{R} : x > 1\}$. Then S is a semigroup under the multiplication. We first prove that I is an ideal of S if and only if $I = [a, \infty)$ or $I = (a, \infty)$ for some $a \in S$.

Proof. Let I be an ideal of S. Since I has a lower bound, we obtain that the greatest lower bound of I exists, say a. We consider the following two cases.

Case 1: $a \in I$. We show that $I = [a, \infty)$. Let $x \in I$. Since a is the greatest lower bound of I, we have $a \leq x$ and so $x \in [a, \infty)$. For the other containment, let $y \in [a, \infty)$. If $y = a$, then $y \in I$. If $y > a$, then choose $1 < z = \dfrac{y}{a} \in S$. Thus $y = az \in I$. So $I = [a, \infty)$.

Case 2: $a \notin I$. We show that $I = (a, \infty)$. Let $x \in I$. Since a is the greatest lower bound of I ,we have $a < x$ and so $x \in (a, \infty)$. For the other containment, let $y \in (a, \infty)$. Since a is a greatest lower bound of I, we have that there exists $k \in I$ such that $a < k < y$, then choose $1 < m = \dfrac{y}{k} \in S$. Thus $y = mk \in I$. So $I = (a, \infty)$.

Conversely, we assume that $I = [a, \infty)$ or $I = (a, \infty)$ for some $a \in S$.

Case 1: $I = [a, \infty)$. Let $x \in I$ and $y \in S$. Then $x \geq a$ and $y > 1$. Thus $xy \geq a$ and so $xy \in I$. Since S is commutative, we get that $yx \in I$. So I is an ideal of S.

Case 2: $I = (a, \infty)$. Let $x \in I$ and $y \in S$. Then $x > a$ and $y > 1$. Thus $xy > a$ and so $xy \in I$. Since S is commutative, $yx \in I$. So I is an ideal of S. $\square$

We see that the semigroup S in Example 5, has no an identity element. Since $[2, \infty)$ is an ideal of S such that $[2, \infty) \neq S$. Then S contains a proper ideal. Let I be a maximal ideal of S and a is the greatest lower bound of I. Then $I = [a, \infty)$ or $I = (a, \infty)$. If $I = [a, \infty)$, then we can choose $1 < b < a$. Thus $[a, \infty) \subsetneq [b, \infty)$ which contradicts the maximality of $I = [a, \infty)$. If $I = (a, \infty)$, we can choose $1 < b < a$. Thus $(a, \infty) \subsetneq (b, \infty)$ which is a contradiction.

Example 6. Let G be a group and I an ideal of G. It is clear that G contains an identity, say e. If $a \in I$, then there exists $a^{-1} \in G$ such that $e = aa^{-1} \in I$ and then $I = G$. Thus G has no a proper ideal. So G has no a maximal ideal.

Lemma 3.2.8. *$J_1 \cup J_2 \cup ... \cup J_k$ is an ideal of $S(X, Y)$ where $1 \leq k \leq n$.*

Proof. Let $\alpha \in J_1 \cup J_2 \cup ... \cup J_k$ and $\beta, \gamma \in S(X, Y)$. Then $\alpha \in J_{i_0}$ for some $1 \leq i_0 \leq k$ and thus $|X\alpha| = i_0$. Since $X\beta\alpha\gamma = (X\beta)\alpha\gamma \subseteq X\alpha\gamma$, we get that $|X\beta\alpha\gamma| \leq |X\alpha\gamma| \leq |X\alpha| = i_0$. Then $|X\beta\alpha\gamma| = p$ for some $1 \leq p \leq i_0 \leq k$. Hence $\beta\alpha\gamma \in J_p \subseteq J_1 \cup J_2 \cup ... \cup J_k$. $\square$

Lemma 3.2.9. *Let S be a semigroup with identity 1. If S has a maximal ideal, then it is unique.*

Proof. Suppose that S has a maximal ideal, say M. Let M' be a maximal ideal of S. It is clear that $M \cup M'$ is an ideal and $1 \notin M \cup M'$. Since $M \subseteq M \cup M'$ and M is a maximal ideal, we have $M \cup M' = M$. Similarly, we have $M \cup M' = M'$. So $M = M \cup M' = M'$. Therefore, S has the unique maximal ideal. □

If $|X| = |Y| = 1$, then $S(X,Y) = G(X,Y)$. Thus $S(X,Y) \setminus G(X,Y) = \emptyset$. So we consider the case $|X| > 1$.

Theorem 3.2.10. *If $|X| > 1$, then $S(X,Y) \setminus G(X,Y)$ is the unique maximal ideal of $S(X,Y)$.*

Proof. Let $a \in Y$ and α be the constant map with $X\alpha = \{a\}$. Then $\alpha \in S(X,Y) \setminus G(X,Y)$, so $S(X,Y) \setminus G(X,Y) \neq \emptyset$. By Lemma 3.2.8, we have

$$S(X,Y) \setminus G(X,Y) = S(X,Y) \setminus J_n = J_1 \cup J_2 \cup ... \cup J_{n-1}$$

is an ideal of $S(X,Y)$. We show that $S(X,Y) \setminus G(X,Y)$ is a maximal ideal of $S(X,Y)$. Let I be an ideal of $S(X,Y)$ such that $S(X,Y) \setminus G(X,Y) \subseteq I \subsetneq S(X,Y)$. We prove that $I = S(X,Y) \setminus G(X,Y)$ by supposing this is not true. Then there exists $\alpha \in I$ but $\alpha \notin S(X,Y) \setminus G(X,Y)$, i.e., $\alpha \in G(X,Y)$. Since $G(X,Y)$ is a group, we obtain that $\alpha^{-1} \in G(X,Y)$ and $id_X = \alpha\alpha^{-1} \in I$. Thus $I = S(X,Y)$ which is a contradiction. Therefore, $I = S(X,Y) \setminus G(X,Y)$. So $S(X,Y) \setminus G(X,Y)$ is a maximal ideal of $S(X,Y)$. By Lemma 3.2.9, we obtain that $S(X,Y) \setminus G(X,Y)$ is the unique maximal ideal of $S(X,Y)$. □

Example 7. By Example 3. Consider the ideals of $S(X,Y)$ in the following diagram.

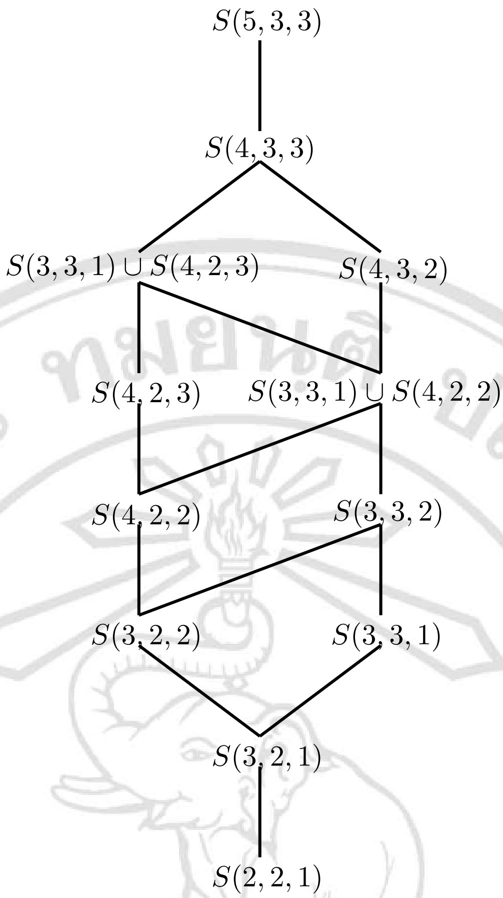

Figure 3

It is clear that $S(4,3,3) = J_1 \cup J_2 \cup J_3 \cup J_4 = S(X,Y) \setminus G(X,Y)$ is the unique maximal ideal of $S(X,Y)$.

By Theorem 3.2.10, we have that $S(X,Y) \setminus G(X,Y)$ is a maximal ideal of $S(X,Y)$ when X is a finite nonempty set and Y a subset of X with $|X| > 1$. However, if X is an infinite set and Y a subset of X, then $S(X,Y) \setminus G(X,Y)$ may not be a maximal ideal as shown in the following example.

Example 8. Let $X = \mathbb{N}$ and $Y = \{1,2\}$. Then we define $\alpha, \beta \in S(X,Y)$ by

$$x\alpha = \begin{cases} x & : \quad x \in Y, \\ 1 & : \quad x = 3, \\ x-1 & : \quad x \in X \setminus \{1,2,3\}, \end{cases}$$

and

$$x\beta = \begin{cases} x & : \quad x \in Y, \\ x+1 & : \quad x \in X \setminus Y. \end{cases}$$

Since $1\alpha = 3\alpha$, we have α is not injective. Since $X\beta = X \setminus \{3\}$, we have β is not surjective. Then $\alpha, \beta \in S(X,Y) \setminus G(X,Y)$. Since

$$x\beta\alpha = (x\beta)\alpha = \begin{cases} x\alpha & : \quad x \in Y, \\ (x+1)\alpha & : \quad x \in X \setminus \{1,2,3\}, \\ 4\alpha & : \quad x = 3, \end{cases}$$

$$= \begin{cases} x & : \quad x \in Y, \\ x & : \quad x \in X \setminus \{1,2,3\}, \\ 3 & : \quad x = 3, \end{cases}$$

we obtain that $x\beta\alpha = x$ for all $x \in X$. Then $\beta\alpha \in G(X,Y)$. Thus $\beta\alpha \notin S(X,Y) \setminus G(X,Y)$. So $S(X,Y) \setminus G(X,Y)$ is not an ideal of $S(X,Y)$.

Lemma 3.2.11. *Let X be a finite nonempty set and Y be a subset of X such that $|Y| = 1$. Then*

(1) *$J_k = \{\alpha \in S(X,Y) : |X\alpha| = k\}$ is a J-class,*

(2) *If I is an ideal of $S(X,Y)$ and $k = max\{|X\alpha\ | : \alpha \in I\}$, then $I = J_1 \cup J_2 \cup ... \cup J_k$.*

Proof. (1) Let $\alpha, \beta \in J_k$. Then $|X\alpha| = k = |X\beta|$, $|Y\alpha| = 1 = |Y\beta|$. Since $Y = Y\alpha \subseteq X\alpha$ and $Y = Y\beta \subseteq X\beta$, we get that $|X\alpha \setminus Y| = |X\alpha| - |X\alpha \cap Y| = |X\alpha| - |Y| = k - 1 = |X\beta| - |Y| = |X\beta| - |X\beta \cap Y| = |X\beta \setminus Y|$. Hence $\alpha \mathcal{J} \beta$. Now, let $\gamma \in S(X,Y) \setminus J_k$. Thus $|X\gamma| \neq k = |X\alpha|$. Then γ is not $\mathcal{J}$-related to α.

(2) Suppose that I is an ideal of $S(X,Y)$ and $k = max\{|X\alpha| : \alpha \in I\}$ and $Y = \{a_1\}$. If $X = Y = \{a_1\}$, then

$$S(X,Y) = I = \left\{ \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} \right\} = J_1$$

and $k = 1 = max\{|X\alpha| : \alpha \in I\}$. In the case $Y \subsetneq X$, let $\alpha \in I$ be such that $|X\alpha| = k$. Then we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_k \\ a_1 & a_2 & ... & a_k \end{pmatrix}$$

where $Y \subseteq A_1$, $A_i \subseteq X \setminus Y$ and $a_i \in X \setminus Y$ for all $2 \leq i \leq k$. Let $2 \leq t \leq k$ and define

$$\beta = \begin{pmatrix} a_2 & ... & a_t & X \setminus \{a_2, ..., a_t\} \\ b_2 & ... & b_t & a_1 \end{pmatrix}$$

where $b_i \in X \setminus Y$ for all $2 \leq i \leq t$. Consider

$$\gamma = \alpha\beta = \begin{pmatrix} A_2 & \dots & A_t & A_1 \cup \left(\underset{t<i\leq k}{\cup} A_i \right) \\ b_2 & \dots & b_t & a_1 \end{pmatrix}.$$

Then $\gamma \in I$ and $\gamma \in J_t$. Hence $I \cap J_t \neq \emptyset$ for all $2 \leq t \leq k$. Since $J_1 \subseteq I$ by Proposition 3.2.1, we obtain that $I \cap J_1 \neq \emptyset$.

We show that $I = J_1 \cup J_2 \cup ... \cup J_k$. Let $\alpha \in I$. Then $|X\alpha| = l \leq k$ which implies that $\alpha \in J_l \subseteq J_1 \cup J_2 \cup ... \cup J_k$. Hence $I \subseteq J_1 \cup J_2 \cup ... \cup J_k$. Let $\beta \in J_1 \cup J_2 \cup ... \cup J_k$. Then $\beta \in J_p$ for some $1 \leq p \leq k$. We can write

$$\beta = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ a_1 & b_2 & \dots & b_p \end{pmatrix}$$

where $Y \subseteq B_1$, $B_i \subseteq X \setminus Y$ and $b_i \in X \setminus Y$ for all $2 \leq i \leq p$. Since $I \cap J_t \neq \emptyset$ for all $1 \leq t \leq k$, we have $I \cap J_p \neq \emptyset$ which implies that there exists $\mu \in I \cap J_p$ such that

$$\mu = \begin{pmatrix} C_1 & C_2 & \dots & C_p \\ a_1 & c_2 & \dots & c_p \end{pmatrix}$$

where $Y \subseteq C_1$, $C_i \subseteq X \setminus Y$ and $c_i \in X \setminus Y$ for all $2 \leq i \leq p$. Let $\theta, \lambda \in S(X, Y)$ be defined by

$$\theta = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ a_1 & d_2 & \dots & d_p \end{pmatrix}, \ \lambda = \begin{pmatrix} c_2 & c_3 & \dots & c_p & X \setminus \{c_2, c_3, ..., c_p\} \\ b_2 & b_3 & \dots & b_p & a_1 \end{pmatrix}$$

where $Y \subseteq D_1$, $d_i \in C_i$ for all $2 \leq i \leq p$. Consider

$$\theta\mu\lambda = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ a_1 & d_2 & \dots & d_p \end{pmatrix} \begin{pmatrix} C_1 & C_2 & \dots & C_p \\ a_1 & c_2 & \dots & c_p \end{pmatrix} \begin{pmatrix} c_2 & c_3 & \dots & c_p & X \setminus \{c_2, c_3, ..., c_p\} \\ b_2 & b_3 & \dots & b_p & a_1 \end{pmatrix}$$

$$= \beta$$

Thus $\beta = \theta\mu\lambda \in I$. Hence $J_1 \cup J_2 \cup ... \cup J_k \subseteq I$. □

Theorem 3.2.12. *If $|Y| = 1$, then the lattice of ideals of $S(X, Y)$ forms a chain.*

Proof. Let I, J be an ideals of $S(X, Y)$. Choose $k = max\{|X\alpha| : \alpha \in I\}$ and $l = max\{|X\beta| : \beta \in J\}$. By Lemma 3.2.11, $I = J_1 \cup J_2 \cup ... \cup J_k$ and $J = J_1 \cup J_2 \cup ... \cup J_l$. If

$k \leq l$, then $I = J_1 \cup J_2 \cup ... \cup J_k \subseteq J_1 \cup J_2 \cup ... \cup J_l = J$. If $k > l$, then $J = J_1 \cup J_2 \cup ... \cup J_l \subseteq J_1 \cup J_2 \cup ... \cup J_k = I$. Thus $I \subseteq J$ or $J \subseteq I$. $\square$

Example 9. Let $X = \{1, 2, 3, ..., n\}$, $Y = \{1\}$. Consider the ideals of $S(X, Y)$ in the following chain.

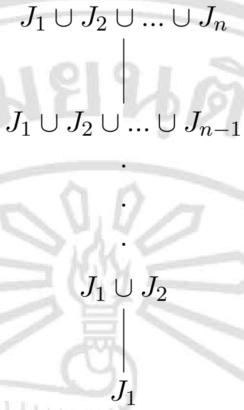

Figure 4

3.3 Minimal and Maximal congruences on $S(X, Y)$

Let ρ be a congruence on a semigroup S. We call ρ a maximal congruence if δ is a congruence on S with $\rho \subsetneq \delta \subseteq S \times S$ implies $\delta = S \times S$.

Suppose that X is a finite set where $|X| \geqslant 2$ and let $Q = T(X) \setminus G(X)$, the authors in [7] proved that $\sigma = (Q \times Q) \cup [G(X) \times G(X)]$ is the only maximal congruence on $T(X)$.

In this section, we determine maximal and minimal congruences on $S(X, Y)$.

Theorem 3.3.1. *Let $S = S(X, Y)$ and $G = G(X, Y)$. Then*

$$\rho = (S \setminus G \times S \setminus G) \cup (G \times G)$$

is a maximal congruence on S.

Proof. It is clear that ρ is an equivalence relation on S. Let $\alpha, \beta, \gamma \in S$ and $(\alpha, \beta) \in \rho$. Then $(\alpha, \beta) \in (S \setminus G) \times (S \setminus G)$ or $(\alpha, \beta) \in G \times G$. If $(\alpha, \beta) \in (S \setminus G) \times (S \setminus G)$, then $\gamma\alpha, \alpha\gamma, \gamma\beta, \beta\gamma \in S \setminus G$ since $S \setminus G$ is an ideal of $S(X, Y)$. Thus $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in (S \setminus G) \times (S \setminus G) \subseteq \rho$. If $(\alpha, \beta) \in G \times G$, we consider the following two cases.

Case 1: $\gamma \in S \setminus G$. Since $S \setminus G$ is an ideal, we have $(\alpha\gamma, \beta\gamma), (\gamma\alpha, \gamma\beta) \in (S \setminus G) \times (S \setminus G) \subseteq \rho$.

Case 2: $\gamma \in G$. Then $\alpha, \beta, \gamma \in G$. Since G is a group, we obtain that $\gamma\alpha, \alpha\gamma \in G$ and $\gamma\beta, \beta\gamma \in G$. Thus $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in G \times G \subseteq \rho$.

Next, we show that ρ is a maximal congruence on S. Let δ be a congruence on S such that $\rho \subsetneq \delta \subseteq S \times S$. Since $\rho \subsetneq \delta$, there exists $(\alpha, \beta) \in \delta \setminus \rho$ with $\alpha \in S \setminus G$ and $\beta \in G$. Let k be the order of β. Then $id_X = \beta^k \ \delta \ \alpha^k$ where $\alpha^k \in S \setminus G$ since $S \setminus G$ is an ideal. Now, let $(\lambda, \mu) \in S \times S$. So $\lambda \ \delta \ \alpha^k \lambda$ and $\mu \ \delta \ \alpha^k \mu$ where $\alpha^k \lambda, \alpha^k \mu \in S \setminus G$. So $\alpha^k \lambda \ \rho \ \alpha^k \mu$. Since $\rho \subseteq \delta$, we obtain $\alpha^k \lambda \ \delta \ \alpha^k \mu$. Thus $\lambda \ \delta \ \mu$ and $\delta = S \times S$ as required. □

Let ρ be a congruence on a semigroup S. We call ρ a minimal congruence if δ is a congruence on S with $1_S \subsetneq \delta \subseteq \rho$ implies $\delta = \rho$.

Let I be a proper ideal of a semigroup S. Then a Rees congruence on S induced by I is

$$\rho_I = (I \times I) \cup 1_{S(X,Y)}.$$

On $S(X,Y)$, if $Y = \{a\}$, then $J_1 = \left\{ \begin{pmatrix} X \\ a \end{pmatrix} \right\}$ and hence $\rho_{J_1} = 1_{S(X,Y)}$. We recall that if $|Y| = 1$, then $S(X,Y)$ has a zero element. In this case, we will use 0 to denote the zero element of $S(X,Y)$.

Lemma 3.3.2. *Let $|Y| = 1$ and δ be a congruence on $S(X,Y)$. If $0 \ \delta \ \alpha$ for some $\alpha \in J_2$, then $0 \ \delta \ \beta$ for all $\beta \in J_2$.*

Proof. Let $Y = \{a\}$ and $0 \ \delta \ \alpha$ for some $\alpha \in J_2$. Let $\beta \in J_2$. Then we can write

$$\beta = \begin{pmatrix} A_1 & A_2 \\ a & b \end{pmatrix}$$

where $Y \subseteq A_1$, $b \in X \setminus Y$ and $A_2 \subseteq X \setminus Y$. Since $\alpha \in J_2$, we can write

$$\alpha = \begin{pmatrix} B_1 & B_2 \\ a & c \end{pmatrix}$$

where $Y \subseteq B_1$, $c \in X \setminus Y$ and $B_2 \subseteq X \setminus Y$. Thus

$$0 = \begin{pmatrix} A_1 & A_2 \\ a & d \end{pmatrix} 0 \begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix} \delta \begin{pmatrix} A_1 & A_2 \\ a & d \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ a & c \end{pmatrix} \begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix} = \beta$$

where $d \in B_2$. Therefore, $0 \ \delta \ \beta$. □

Proposition 3.3.3. *Let $S = S(X,Y)$ and $|X| = 2, |Y| = 1$. Then the Rees congruence $\rho_{J_1 \cup J_2}$ is a minimal congruence on S.*

Proof. Let $X = \{a, b\}, Y = \{a\}$ and δ be a congruence on S such that $1_S \subsetneq \delta \subseteq \rho_{J_1 \cup J_2}$. We note that in this case

$$J_1 \cup J_2 = \left\{ 0 = \begin{pmatrix} X \\ a \end{pmatrix},\ id_X = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\}.$$

So

$$\rho_{J_1 \cup J_2} = \{(\alpha, \alpha) : \alpha \in S(X,Y)\} \cup \{(0, id_X), (id_X, 0)\}.$$

Since $1_S \subsetneq \delta$, there is $(\alpha, \beta) \in \delta$ such that $\alpha \neq \beta$. Since $\delta \subseteq \rho_{J_1 \cup J_2}$, it follows that $(\alpha, \beta) = (0, id_X)$ or $(\alpha, \beta) = (id_X, 0)$, hence $\delta = \rho_{J_1 \cup J_2}$. □

Theorem 3.3.4. *Let $S = S(X,Y)$ and $|X| > 2, |Y| = 1$. Then the Rees congruence $\rho_{J_1 \cup J_2}$ is a minimal congruence on S.*

Proof. Let $Y = \{a\}$ and δ be a congruence on S such that $1_S \subsetneq \delta \subseteq \rho_{J_1 \cup J_2}$. Since $1_S \subsetneq \delta$, we obtain that there exists $(\alpha, \beta) \in \delta$ but $\alpha \neq \beta$. Since $\delta \subseteq \rho_{J_1 \cup J_2}$, we have $\alpha \in J_1$ and $\beta \in J_2$; $\alpha \in J_2$ and $\beta \in J_1$ or $\alpha, \beta \in J_2$. If $\alpha \in J_1$ and $\beta \in J_2$, then $\alpha\ \delta\ \gamma$ for all $\gamma \in J_2$ by Lemma 3.3.2. Thus $\delta = \rho_{J_1 \cup J_2}$. If $\alpha, \beta \in J_2$, then we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 \\ a & b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 \\ a & c \end{pmatrix}$$

where $Y \subseteq A_1$, $Y \subseteq B_1$, $A_2, B_2 \subseteq X \setminus Y$ and $b, c \in X \setminus Y$. Since $\alpha \neq \beta$, there are two cases to consider.

Case 1: $b \neq c$. We choose

$$\lambda = \begin{pmatrix} \{a,b\} & X \setminus \{a,b\} \\ a & c \end{pmatrix}.$$

Then $\lambda \in S(X,Y)$ and

$$0 = \begin{pmatrix} X \\ a \end{pmatrix} = \alpha\lambda\ \delta\ \beta\lambda = \begin{pmatrix} B_1 & B_2 \\ a & c \end{pmatrix} = \beta.$$

By Lemma 3.3.2, we get that $0\ \delta\ \gamma$ for all $\gamma \in J_2$. Thus $\delta = \rho_{J_1 \cup J_2}$.

Case 2: $b = c$. Then $A_1 \neq B_1$ or $A_2 \neq B_2$. If $A_1 \neq B_1$ and there exists $u \in A_1 \setminus B_1$, then define $\mu \in S(X,Y)$ by

$$\mu = \begin{pmatrix} a & X \setminus \{a\} \\ a & u \end{pmatrix}.$$

So

$$0 = \begin{pmatrix} X \\ a \end{pmatrix} = \mu\alpha\ \delta\ \mu\beta = \begin{pmatrix} a & X \setminus \{a\} \\ a & c \end{pmatrix}.$$

By Lemma 3.3.2, we get that $0 \ \delta \ \gamma$ for all $\gamma \in J_2$. Thus $\delta = \rho_{J_1 \cup J_2}$. Now, if $A_1 \neq B_1$ and there exists $v \in B_1 \setminus A_1$, then define $\theta \in S(X,Y)$ by

$$\theta = \begin{pmatrix} a & X \setminus \{a\} \\ a & v \end{pmatrix}.$$

So

$$\begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix} = \theta\alpha \ \delta \ \theta\beta = \begin{pmatrix} X \\ a \end{pmatrix} = 0,$$

thus by Lemma 3.3.2, we get that $0 \ \delta \ \gamma$ for all $\gamma \in J_2$. Thus $\delta = \rho_{J_1 \cup J_2}$. The case $A_2 \neq B_2$ can be prove in the same way. Therefore, we conclude that $\rho_{J_1 \cup J_2}$ is a minimal congruence on S.

□

Theorem 3.3.5. *Let $S = S(X,Y)$ and $|Y| > 2$. Then the Rees congruence ρ_{J_1} is a minimal congruence on S.*

Proof. Let δ be a congruence on S such that $1_S \subsetneq \delta \subseteq \rho_{J_1}$ (This is possible since $|Y| \geq 2$). Since $1_S \subsetneq \delta$, there exists $(\alpha, \beta) \in \delta$ but $\alpha \neq \beta$. Then $(\alpha, \beta) \in \rho_{J_1}$ and we can write α and β by

$$\alpha = \begin{pmatrix} X \\ a \end{pmatrix}, \ \beta = \begin{pmatrix} X \\ b \end{pmatrix}$$

for some $a, b \in Y$ such that $a \neq b$. Let $(\lambda, \mu) \in \rho_{J_1}$. So $\lambda = \mu$ or $\lambda, \mu \in J_1$. If $\lambda = \mu$, then $(\lambda, \mu) \in \delta$. If $\lambda, \mu \in J_1$, then

$$\lambda = \begin{pmatrix} X \\ u \end{pmatrix} \text{ and } \mu = \begin{pmatrix} X \\ v \end{pmatrix}$$

for some $u, v \in Y$ such that $u \neq v$. Since $\alpha\delta\beta$, we obtain that

$$\lambda = \begin{pmatrix} X \\ u \end{pmatrix} = \begin{pmatrix} X \\ a \end{pmatrix} \begin{pmatrix} a & X\setminus\{a\} \\ u & v \end{pmatrix} \delta \begin{pmatrix} X \\ b \end{pmatrix} \begin{pmatrix} a & X\setminus\{a\} \\ u & v \end{pmatrix} = \begin{pmatrix} X \\ v \end{pmatrix} = \mu.$$

Hence $\delta = \rho_{J_1}$. □

Example 10. Let $X = \{1,2,3\}$ and $Y = \{1\}$. Then

$$J_1 \cup J_2 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \right\}.$$

So

$$\rho_{J_1\cup J_2} = [(J_1\cup J_2)\times(J_1\cup J_2)]\cup\left\{\left(\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right),\left(\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\right)\right\}.$$

Example 11. Let $X = \mathbb{N}$ and $Y = \{1\}$. Then

$$I = J_1 \cup J_2 = \left\{\begin{pmatrix}X\\1\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}A_1 & A_2\\1 & n\end{pmatrix} : 1 \in A_1, \text{ and } n \in \mathbb{N}\setminus\{1\}\right\}$$

So

$$\rho_I = (I \times I) \cup \{(\alpha, \alpha) : |X\alpha| \geq 3\}.$$