CHAPTER 1

Introduction

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is a group of Möbius transformations under functional composition, that is

$$\Gamma = \{T : \mathbb{C} \to \mathbb{C} \mid T(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{Z},\ ad-bc=1\}.$$

As a convention, since $\Gamma \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ whenever we refer to a matrix for an element in $\mathrm{PSL}_2(\mathbb{Z})$, we really mean one of the corresponding matrices in

$$\mathrm{SL}_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1\}$$

under the quotient isomorphism. The modular group acts on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$ by conformal extension of Möbius transformation on the boundary at infinity of $\mathbb{H}^2$, $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. Indeed, it is a discrete subgroup of the group of all orientation preserving isometries on $\mathbb{H}^2$, known as, $\mathrm{PSL}_2(\mathbb{R})$.

The extended modular group $\widehat{\Gamma}$ is generated by Γ and the reflection $z \mapsto -\bar{z}$, so

$$\widehat{\Gamma} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=\pm 1\}.$$

Therefore $\widehat{\Gamma}$ includes orientation preserving and reversing isometries acting on $\mathbb{H}^2$.

In 1991, Jones, Singerman and Wicks determined in [4] that the edge conditions on adjacent vertices in suborbital graph $\mathcal{F}_{u,n}$ of the modular group Γ. Then Değer, Beşenk and Güler applied this property to prove some properties of $\mathcal{F}_{u,n}$ related to the continued fractions. Moreover, Akbas showed in [1] that a suborbital graph for the modular group is a forest if and only if it contains no triangles, and this proved a conjecture of Jones, Singerman and Wicks in [4] and gived another properties of suborbital graph $\mathcal{F}_{u,n}$ of the modular group Γ in 2001.

In 2013, Kader and Güler, see [5], used the notion of the imprimitive action for a $\widehat{\Gamma}$-invariant equivalence relation on $\widehat{\mathbb{Q}}$ by the congruence subgroup $\widehat{\Gamma}_0(n)$ to obtain suborbital graphs $\widehat{\mathcal{F}}_{u,n}$ and examined the edge conditions on adjacent vertices.

The purposes of this research are:

1. To investigate the farthest vertex from a given vertex on $\widehat{\mathcal{F}}_{u,n}$.
2. To show that the result from 1 is related to the continued fraction.
3. To find some properties of the suborbital subgraph $\widehat{\mathcal{F}}_{u,n}$.

This thesis is divided into four chapters. The first chapter draws the aims of this research. The second chapter deals with preliminaries and some useful results which will be used in the next chapters. The third chapter contains the main results of this research, consisting of the following two sections: suborbital graphs of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$ and continued fraction. Finally, the last chapter is the conclusion of this research.