CHAPTER 2

Preliminaries

The discussion in this chapter is presented in four sections, including the hyperbolic plan $\mathbb{H}^2$, the action of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$, suborbital graphs of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$ and continued fraction.

2.1 The Hyperbolic Plane $\mathbb{H}^2$

This section contains basic properties of the hyperbolic plane and some isometries of the hyperbolic plane. See [2] for more details.

Definition 2.1.1. The *hyperbolic plane* is the metric space consisting of the upper half-plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

with the metric d_{hyp} defined below. We first define the *hyperbolic length* of a curve γ

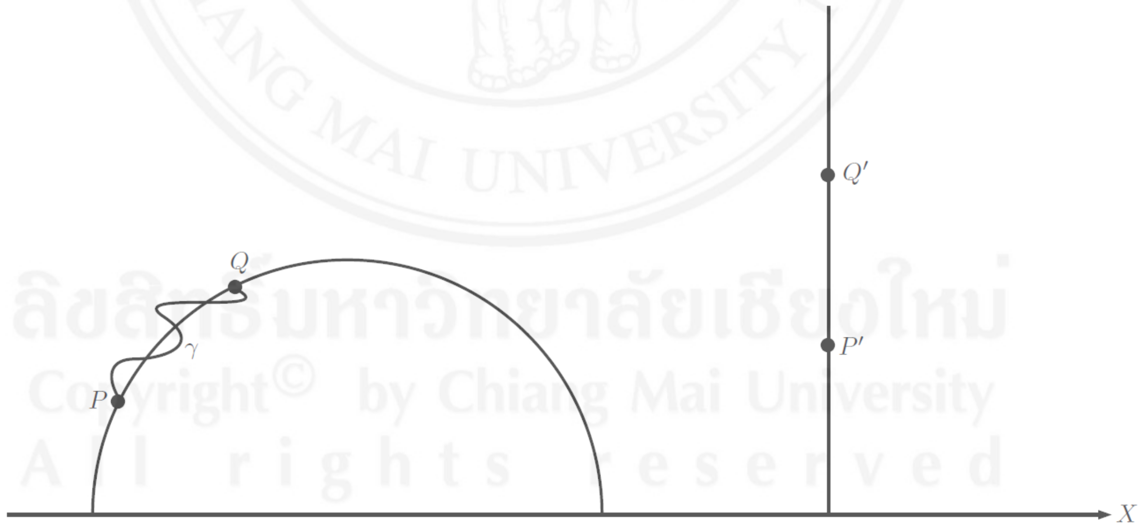

Figure 2.1: The Hyperbolic Plane

parametrized by the differentiable vector valued function

$$t \mapsto (x(t), y(t)),\ a \leqslant t \leqslant b,$$

as

$$l_{hyp}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

The *hyperbolic distance* between two points P and Q is the infimum of the hyperbolic lengths of all piecewise differentiable curves γ going from P to Q, namely

$$d_{hyp}(P, Q) = \inf \{l_{hyp}(\gamma) \mid \gamma \text{ goes from } P \text{ to } Q\}.$$

We call the curve of shortest length *geodesic.* It can be shown that the geodesic connecting any two points is either the semi-circle arcs centered on the X-axis or the straight line perpendicular to the X-axis containing both points.

An *isometry* of the hyperbolic plane is a bijective mapping $\varphi : \mathbb{H}^2 \cup \partial\mathbb{H}^2 \to \mathbb{H}^2 \cup \partial\mathbb{H}^2$ which preserves distance. Namely, for any P, Q in $\mathbb{H}^2$,

$$d_{hyp}(\varphi(P), \varphi(Q)) = d_{hyp}(P, Q).$$

Some isometries of the hyperbolic plane

The *homothety* is a mapping defined by

$$\varphi(x, y) = (\lambda x, \lambda y) \text{ for some } \lambda > 0.$$

The *horizontal translation* is a mapping defined by

$$\varphi(x, y) = (x + x_0, y) \text{ for some } x_0 \in \mathbb{R}.$$

The *reflection across the Y-axis* is a mapping defined by

$$\varphi(x, y) = (-x, y).$$

The *inversion across the unit circle* is a mapping defined by

$$\varphi(x, y) = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}),$$

see Figure 2.2.

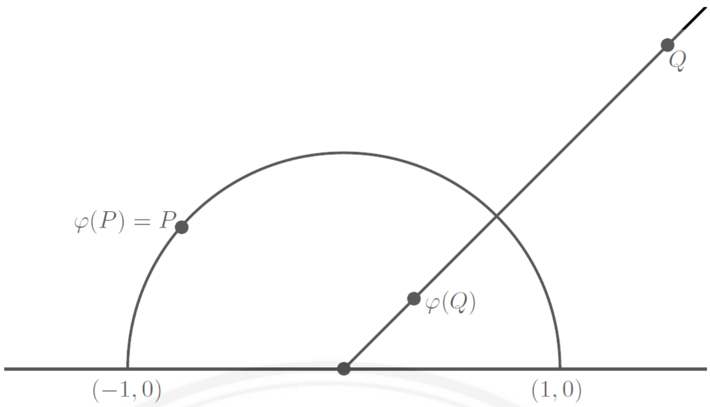

Figure 2.2: The Inversion across the unit circle

The following theorem gives a characterization of all isometries of the hyperbolic plane.

Theorem 2.1.1. [2] *The isometries of the hyperbolic plane are exactly the maps of the form*

$$\varphi(z) = \frac{az+b}{cz+d} \quad \textit{with } a, b, c, d \in \mathbb{R} \textit{ and } ad - bc = 1$$

or

$$\varphi(z) = \frac{c\bar{z}+d}{a\bar{z}+b} \quad \textit{with } a, b, c, d \in \mathbb{R} \textit{ and } ad - bc = 1.$$

We denote the set of all isometries by $\mathrm{Isom}(\mathbb{H}^2)$. The set of all orientation preserving isometries is denoted by $\mathrm{Isom}^+(\mathbb{H}^2)$. $\mathrm{Isom}^-(\mathbb{H}^2)$ is used for the set of all orientation reversing isometries. By Theorem 2.1.1,

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R}) = \{z \mapsto \varphi(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R},\ ad - bc = 1\}.$$

An element of $\mathrm{PSL}_2(\mathbb{R})$ can be classified by its trace. If

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the trace of φ is $\mathrm{tr}(\varphi) = |a + d|$. φ is elliptic, parabolic, or hyperbolic if $\mathrm{tr}(\varphi) < 2$, $\mathrm{tr}(\varphi) = 2$ or $\mathrm{tr}(\varphi) > 2$, respectively.

Proposition 2.1.2. *(i) An elliptic element has one fixed points in* $\mathbb{H}^2$.
(ii) A parabolic element has one fixed points in $\partial\mathbb{H}^2$.
(iii) A hyperbolic element has two fixed points in $\partial\mathbb{H}^2$.

2.2 The Action of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$

In this section, we recall a definition of group action on a space, we include some results on the extended modular group $\widehat{\Gamma}$ on the set of extended rational numbers $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. See [5] for more details.

Definition 2.2.1. Let G be a group and Ω be a nonempty set. A mapping $G \times \Omega \to \Omega$, denoted $(g, \alpha) \mapsto g \cdot \alpha$, is called an *action* of G on Ω, if it satisfies the following conditions:

i) $1 \cdot \alpha = \alpha$ for all $\alpha \in \Omega$.

ii) $g \cdot (h \cdot \alpha) = (gh) \cdot \alpha$ for all $\alpha \in \Omega$ and for all $g, h \in G$.

For any $\alpha \in \Omega$, the orbit of α, denoted by $[\alpha]$, under the group G acting on Ω is

$$[\alpha] = \{g \cdot \alpha \mid g \in G\}.$$

The subgroup $G_\alpha = \{g \in G \mid g \cdot \alpha = \alpha\}$ is called the *stabilizer* of α.

Definition 2.2.2. A group G is said to act *transitively* on Ω if, for all $\alpha, \beta \in \Omega$, there exists $g \in G$ such that $g(\alpha) = \beta$.

For instance, if $G = \widehat{\Gamma}$ and $\Omega = \mathbb{H}^2$, $\widehat{\Gamma}$ acts on $\mathbb{H}^2$ naturally defined as follows:

$$\text{for } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{\Gamma},\ z \in \mathbb{H}^2,\ T \cdot z = T(z) = \tfrac{az+b}{cz+d}.$$

Every element of the set $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$.

Lemma 2.2.1. [5] *$\widehat{\Gamma}$ acts transitively on $\widehat{\mathbb{Q}}$.*

Now we consider the imprimitivity of the action of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$, beginning with a general discussion of primitivity of permutation groups.

Definition 2.2.3. Let (G, Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called $G-$*invariant* if, whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$. An equivalence class is called *block*.

Definition 2.2.4. We call (G, Ω) *imprimitive* if Ω admits some $G-$invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$.

(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise, (G, Ω) is called *primitive*.

The following lemma gives a characterization of primitivity for (G, Ω), see [7] for more details.

Lemma 2.2.2. [7] *Let (G, Ω) be a transitive permutation group. (G, Ω) is primitive if and only if G_α, the stabilizer of $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

When (G, Ω) is $(\widehat{\Gamma}, \widehat{\mathbb{Q}})$, the stabilizer of ∞ is

$$\widehat{\Gamma}_\infty = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \rangle.$$

For any positive integer n, we define

$$\Gamma_0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{n} \}$$

and

$$\widehat{\Gamma}_0(n) = \langle \Gamma_0(n), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle.$$

We see that $\widehat{\Gamma}_\infty \lneqq \widehat{\Gamma}_0(n) \lneqq \widehat{\Gamma}$. In particular, $\widehat{\Gamma}_\infty$ is not a maximal subgroup of $\widehat{\Gamma}$, see [5]. Hence by Lemma 2.2.2, $(\widehat{\Gamma}, \widehat{\mathbb{Q}})$ is imprimitive.

Let v and w be elements in $\widehat{\mathbb{Q}}$ and write $v = \frac{r}{s}$, $w = \frac{x}{y}$, where $(r, s) = 1$ and $(x, y) = 1$. By transitivity of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$, one can find matrices

$$g = \begin{pmatrix} r & * \\ s & * \end{pmatrix} \text{ and } g' = \begin{pmatrix} x & * \\ y & * \end{pmatrix}$$

in $\widehat{\Gamma}$ such that $v = g(\infty)$ and $w = g'(\infty)$, when $*$ can be any number. Define the relation $\approx$ as follows:

$$v \approx w \text{ if and only if } g^{-1}g' \in \widehat{\Gamma}_0(n).$$

For any $h \in \widehat{\Gamma}$, $(hg)^{-1}(hg') = g^{-1}h^{-1}hg' = g^{-1}g' \in \widehat{\Gamma}_0(n)$, that is $h(v) \approx h(w)$. Thus $\approx$ is the $\widehat{\Gamma}-$invariant relation. We consider this relation in the following four cases :

Case 1 : If det $(g) = 1$ and det $(g') = 1$, then we get

$$g^{-1} = \begin{pmatrix} * & * \\ -s & r \end{pmatrix}$$

which implies that

$$g^{-1}g' = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix}.$$

Case 2 : If det $(g) = -1$ and det $(g') = -1$, then

$$g^{-1} = \begin{pmatrix} * & * \\ s & -r \end{pmatrix}$$

and we obtain that

$$g^{-1}g' = \begin{pmatrix} * & * \\ sx - ry & * \end{pmatrix}.$$

Case 3 : If det $(g) = 1$ and det $(g') = -1$, then we get the same result as case 1.

Case 4 : If det $(g) = -1$ and det $(g') = 1$, then the result is similar to case 2.

From the above four cases, we can view this relation as

$$\tfrac{r}{s} \approx \tfrac{x}{y} \text{ if and only if } ry - sx \equiv 0 \pmod{n}.$$

Thus the block of ∞, the equivalence class of ∞, is given as

$$[\infty] = \{\frac{x}{y} \in \widehat{\mathbb{Q}} \mid y \equiv 0 \pmod{n}\}.$$

2.3 Suborbital Graphs of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$

In this section, we restate some basic definitions and properties of suborbital graph and prove our main theorem. In general set up, let G be a group acting on a set Ω transitively. Then G acts on $\Omega \times \Omega$ given by

$$g(\alpha, \beta) = (g(\alpha), g(\beta)) \ , \ g \in G, \alpha, \beta \in \Omega.$$

The orbits of this action are call *suborbitals* of G, that containing (α, β) being denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $\mathcal{G}(\alpha, \beta)$: its vertices are the elements of Ω and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$, denoted by $\gamma \to \delta$.

We can view a suborbital graph in the hyperbolic plane $\mathbb{H}^2$. The set of vertices is a subset of $\widehat{\mathbb{Q}} \subseteq \widehat{\mathbb{R}} = \partial\mathbb{H}^2$ the boundary of infinity of $\mathbb{H}^2$ and the edge connecting two vertices is the complete hyperbolic geodesic joining between them. Namely, the semicircle centered on $\partial\mathbb{H}^2$ with $X-$intercept at those two vertices.

The orbit $O(\beta, \alpha)$ is also a suborbital and it is either equal to or disjoint from

$O(\alpha, \beta)$. In the latter case $\mathcal{G}(\beta, \alpha)$ is just $\mathcal{G}(\alpha, \beta)$ with the arrows reversed and we call, in this case, $\mathcal{G}(\beta, \alpha)$ and $\mathcal{G}(\alpha, \beta)$ *paired suborbital graphs.* In the former case $\mathcal{G}(\alpha, \beta) = \mathcal{G}(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call *self paired.*

We now examine the suborbital graphs for the action $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$. Since $\widehat{\Gamma}$ acts transitively on $\widehat{\mathbb{Q}}$, each suborbital contains a pair $(\infty, \frac{u}{n})$ for some $\frac{u}{n} \in \widehat{\mathbb{Q}}$, where $n \geqslant 1$ and $(u, n) = 1$. We will denote this suborbital by $\widehat{O}_{u,n}$ and corresponding suborbital graph $\widehat{\mathcal{G}}(\infty, \frac{u}{n})$ by $\widehat{\mathcal{G}}_{u,n}$.

The following theorem gives a necessary and sufficient condition on when two vertices on suborbital graph $\widehat{\mathcal{G}}_{u,n}$ are connected numerically, see [5] for the detail of the proof.

Theorem 2.3.1. [5] *Let $\frac{r}{s}, \frac{x}{y} \in \widehat{\mathbb{Q}}$. There is an edge $\frac{r}{s} \to \frac{x}{y}$ in the suborbital graph $\widehat{\mathcal{G}}_{u,n}$ if and only if one of the following conditions is satisfied :*

(i) $x \equiv ur \pmod{n},\ y \equiv us \pmod{n}$ *and* $ry - sx = n$,

(ii) $x \equiv -ur \pmod{n},\ y \equiv -us \pmod{n}$ *and* $ry - sx = -n$,

(iii) $x \equiv ur \pmod{n},\ y \equiv us \pmod{n}$ *and* $ry - sx = -n$,

(iv) $x \equiv -ur \pmod{n},\ y \equiv -us \pmod{n}$ *and* $ry - sx = n$.

The $\widehat{\Gamma}-$invariant equivalence relation $\approx$ on the congruence subgroup $\widehat{\Gamma}_0(n)$ as defined in the last section gives subgraphs of suborbital graph $\widehat{\mathcal{G}}_{u,n}$. Since $\widehat{\Gamma}$ acts transitively on $\widehat{\mathbb{Q}}$, it permutes the blocks transitively, so all of the subgraphs are isomorphic. For this, we determine the block of

$$[\infty] = \{\frac{x}{y} \in \widehat{\mathbb{Q}} \mid y \equiv 0 \pmod{n}\},$$

and denote $\widehat{\mathcal{F}}_{u,n}$ for the subgraph of $\widehat{\mathcal{G}}_{u,n}$ corresponding to $[\infty]$. Then Theorem 2.3.1 gives the following.

Theorem 2.3.2. [5] *Let $\frac{r}{s}, \frac{x}{y} \in [\infty]$. Then $\frac{r}{s} \to \frac{x}{y}$ in $\widehat{\mathcal{F}}_{u,n}$ if and only if one of the following conditions is satisfied :*

(i) $x \equiv ur \pmod{n}$ *and* $ry - sx = n$,

(ii) $x \equiv -ur \pmod{n}$ *and* $ry - sx = -n$,

(iii) $x \equiv ur \pmod{n}$ *and* $ry - sx = -n$,

(iv) $x \equiv -ur \pmod{n}$ *and* $ry - sx = n$.

Theorem 2.3.3. [5] *$\widehat{\Gamma}_0(n)$ permutes the vertices and the edges of $\widehat{\mathcal{F}}_{u,n}$ transitively.*

Definition 2.3.1. Let $v_1, v_2, ..., v_m$ be different vertices in $\widehat{\mathcal{F}}_{u,n}$. The path

$$v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_m \rightarrow v_1$$

is called a *directed circuit* in $\widehat{\mathcal{F}}_{u,n}$. If the above path has at least an arrow (not all) reversed, this path is called an *anti-directed circuit*. Moreover, if $m = 3$ then we call this path *triangle*.

Theorem 2.3.4. [5] *$\widehat{\mathcal{F}}_{u,n}$ contains a directed triangle if and only if*

$$u^2 \pm u + 1 \equiv 0 \pmod n.$$

Theorem 2.3.5. [5] *If $n > 1$, then $\widehat{\mathcal{F}}_{u,n}$ contains no anti-directed triangles.*

Theorem 2.3.6. [5] *If n is even, then $\widehat{\mathcal{F}}_{u,n}$ does not contain any directed triangle.*

Now we give examples of suborbital graphs of the extended modular group in Figure 2.3 and Figure 2.4. The first suborbital graph consists of isomorphic copies of $\widehat{\mathcal{F}}_{1,5}$, with unbroken and broken edges indicating the blocks $[\infty]$ and $[1]$ respectively.

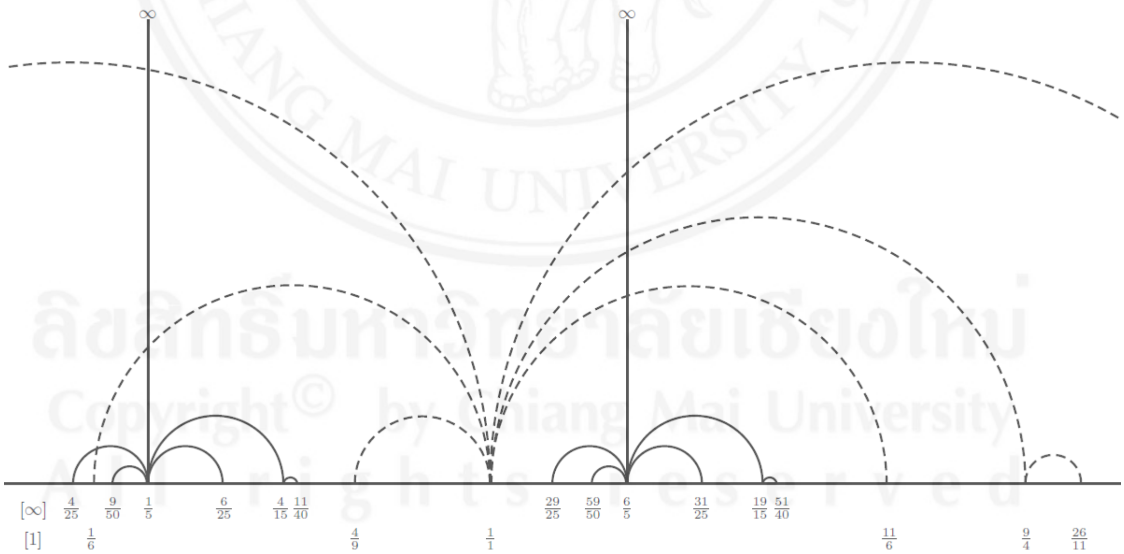

Figure 2.3: $\widehat{\mathcal{G}}_{1,5}$

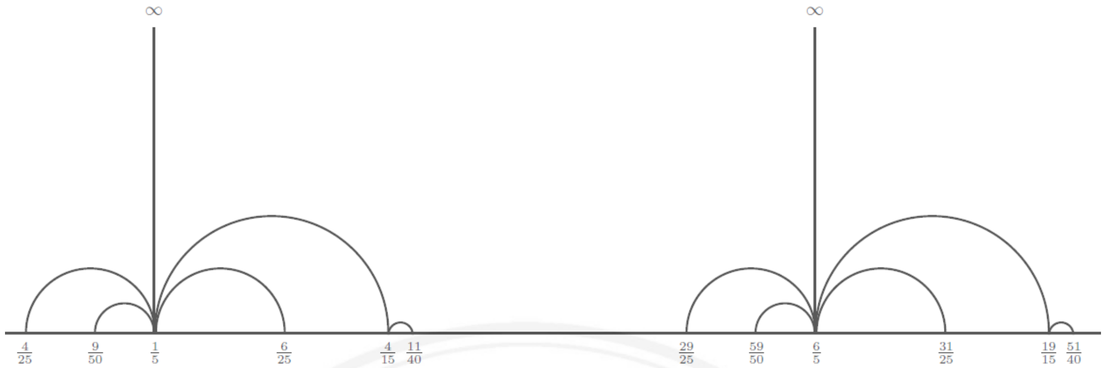

Figure 2.4: $\widehat{\mathcal{F}}_{1,5}$

2.4 Continued Fraction

In this section, we give a short introduction of a simple continued fraction as a composition of a sequence of Möbius maps.

Let $\{t_m\}$ be a sequence of Möbius transformation

$$t_m(z) = \frac{a_m}{b_m + z},\ a_m \neq 0$$

and let $T_m(z) = t_1 t_2 ... t_m(z),\ m \geqslant 1$ with T_0 the identity map. Note that $T_m(\infty) = T_{m-1}(0)$. If one computes $t_1(0), t_1t_2(0), t_1t_2t_3(0)$ and so on, form a continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots}}}} \tag{2.1}$$

and the value of the continued fraction 2.1 (when it exists) is equal to the limit of the sequence $\{T_m(0)\}$. If $a_i = 1$ for all $i = 1, 2, ...$ in continued fraction (2.1), this continued fraction is called *simple continued fraction*

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}} \tag{2.2}$$

A simple continued fraction can be shown as a composition of a sequence of Möbius maps of the form

$$t_m(z) = \frac{1}{b_m + z},$$

evaluated at $z = 0$. The following theorem gives a sufficient condition for convergence which was discovered by Ivan Śleszyński and Alfred Pringsheim in the late 19th century.

Theorem 2.4.1. [6](Śleszyński-Pringsheim theorem) *Let $|b_m| \geqslant 1 + |a_m|$ for all $m \in \mathbb{N}$. Then the continued fraction* (2.1) *converges to some valued v with $|v| \leqslant 1$.*

Example :

$$\cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\ddots}}}} \tag{2.3}$$

We observe that $|b_m| \geqslant 1 + |a_m|$ for all $m \in \mathbb{N}$ in the continued fraction 2.3 and we known that this continued fraction converges to the value $\frac{1}{e-2} - 1 \approx 0.3922$ which was calculated by Leonhard Euler.