CHAPTER 3

Main Results

In this chapter, we present the farthest vertex which can be joined with $\frac{u}{n}$ and $\frac{u+\frac{1}{k}}{n}$ in $\widehat{\mathcal{F}}_{u,n}$ and show that this result relates to some continued fractions. Moreover, we give some properties of the suborbital subgraph $\widehat{\mathcal{F}}_{u,n}$.

3.1 Suborbital Graphs of $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$

In this section, we give proofs of properties of the suborbital subgraph $\widehat{\mathcal{F}}_{u,n}$. We first prove the existence of an integer k such that $u^2 + ku + 1 \equiv 0 \pmod{n}$.

Lemma 3.1.1. *If $(u, n) = 1$, then there exists an integer k such that $u^2 + ku + 1 \equiv 0 \pmod{n}$.*

Proof. From $(u, n) = 1$, there exists an integer x such that $ux \equiv 1 \pmod{n}$. So $ux(-u^2 - 1) \equiv -u^2 - 1 \pmod{n}$. Taking $k = x(-u^2 - 1)$, then $u^2 + ku + 1 \equiv 0 \pmod{n}$ is satisfied. □

Corollary 3.1.2. *Let u, n be relatively prime positive integers, and let k be integer such that $u^2 + ku + 1 \equiv 0 \pmod{n}$. Then*

$$\varphi = \begin{pmatrix} -u & (u^2 + ku + 1)/n \\ -n & u + k \end{pmatrix}$$

is an element of $\widehat{\Gamma}_0(n)$ and also if $k = 0$ and $k = 1$ then φ is an elliptic element of order 2 and 3, respectively.

Proof. By Lemma 3.1.1, we obtain that φ is an element of $\widehat{\Gamma}_0(n)$. If $k = 0$, then

$$\varphi = \begin{pmatrix} -u & (u^2 + 1)/n \\ -n & u \end{pmatrix}$$

and we have

$$\begin{pmatrix} -u & (u^2+1)/n \\ -n & u \end{pmatrix} \begin{pmatrix} -u & (u^2+1)/n \\ -n & u \end{pmatrix} = \begin{pmatrix} u^2-u^2-1 & (-u^3-u+u^3+u)/n \\ un-un & -u^2-1+u^2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence φ is an elliptic element of order 2. If $k=1$, then

$$\varphi = \begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix}$$

and we observe that

$$\begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix} \begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix} \begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix}$$
$$= \begin{pmatrix} u^2-u^2-u-1 & (-u^3-u^2-u+u^3+2u^2+2u+1)/n \\ un-un-n & -u^2-u-1+u^2+2u+1 \end{pmatrix} \begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix}$$
$$= \begin{pmatrix} -u-1 & (u^2+u+1)/n \\ -n & u \end{pmatrix} \begin{pmatrix} -u & (u^2+u+1)/n \\ -n & u+1 \end{pmatrix}$$
$$= \begin{pmatrix} u^2+u-u^2-u-1 & (-(u+1)(u^2+u+1)+(u+1)(u^2+u+1))/n \\ un-un & -u^2-u-1+u^2+u \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore φ is an elliptic element of order 3. $\square$

Now we show the proof of the main theorem in this thesis.

Theorem 3.1.3. *Let u, n be relatively prime positive integers. In $\widehat{\mathcal{F}}_{u,n}$, we have the following results:*

(i) The farthest vertex which can be joined with $\frac{u}{n}$ is

$$\frac{u+\frac{1}{k}}{n}$$

when k is the unique integer such that $1 \leqslant k \leqslant n$ and $u^2+ku+1 \equiv 0 \pmod{n}$. The nearest vertex does not exist.

(ii) The farthest vertex which can be joined with $\frac{u+\frac{1}{k}}{n}$ is

$$\frac{u+\frac{1}{k-\frac{1}{k}}}{n}$$

when k is the unique integer such that $1 \leqslant k \leqslant n$ and $u^2 + ku + 1 \equiv 0 \pmod{n}$. The nearest vertex does not exist.

Proof of (i). The existance of an integer k such that $u^2 + ku + 1 \equiv 0 \pmod{n}$ is due to Lemma 3.1.1. Now we can assume that $1 \leqslant k \leqslant n$. To see this, if $k > n$, we choose k_1 such that $k \equiv k_1 \pmod{n}$. We have $ku + 1 \equiv k_1 u + 1 \pmod{n}$, this gives

$$u^2 + k_1 u + 1 \equiv u^2 + ku + 1 \equiv 0 \pmod{n}.$$

Now we show the uniqueness of k. Let m be another integer such that $1 \leqslant m \leqslant n$ and $u^2 + mu + 1 \equiv 0 \pmod{n}$. Hence $mu \equiv -u^2 - 1 \equiv ku \pmod{n}$, so we have $(k - m)u \equiv 0 \pmod{n}$. Because u and n are relatively prime, $k - m \equiv 0 \pmod{n}$. Thus, $k = m$ since $|k - m| < n$.

Now suppose there exists an edge $\frac{u}{n} \to \frac{x}{y}$ in $\widehat{\mathcal{F}}_{u,n}$ and $\frac{u}{n} < \frac{x}{y}$. We can write $\frac{x}{y}$ in the form

$$\frac{x}{y} = \frac{u}{n} + \frac{nx}{ny} - \frac{uy}{ny} = \frac{u + \frac{nx - uy}{y}}{n}.$$

With this and the fact that $uy < nx$, we can replace $\frac{x}{y}$ with $\frac{u + \frac{t}{s}}{n}$ where $\frac{t}{s}$ is in $\mathbb{Q}^+$. So we have

$$\frac{u}{n} \to \frac{u + \frac{t}{s}}{n} = \frac{su + t}{sn}.$$

Theorem 2.3.2 gives numerical informations when this edge exists. From this point onward, we aim to analyze each of the cases, $(i) - (iv)$, of this theorem.

Case(i): In this case, we have $su + t \equiv u^2 \pmod{n}$ and $u(sn) - n(su + t) = n$, which implies $t = -1$. Therefore, $su - 1 \equiv u^2 \pmod{n}$. Since $u^2 + ku + 1 \equiv 0 \pmod{n}$, we have $su - 1 \equiv -ku - 1 \pmod{n}$, that is $su \equiv -ku \pmod{n}$. Since $(u, n) = 1$, we have $s \equiv -k \pmod{n}$. In other words, $s = -k - nz$ for some z in $\mathbb{N} \cup \{0\}$. Thus,

$$\frac{t}{s} = \frac{1}{nz + k}.$$

Next, we find the largest value of $\frac{t}{s}$ by defining a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$,

$$f(z) = \frac{u + \frac{1}{nz+k}}{n}.$$

The derivative of f is $f'(z) = \frac{-1}{(nz+k)^2}$ which is negative for every non-negative z. This implies that the maximum occurs at $z = 0$, that is

$$\frac{u + \frac{1}{k}}{n} = \frac{uk + 1}{kn}.$$

It remains to show that $\frac{u+\frac{1}{k}}{n} = \frac{uk+1}{kn}$ is a vertex in $\widehat{\mathcal{F}}_{u,n}$. To see this, we show that it is an irreducible fraction. It is true that $(ku+1, k) = (ku+1-ku, k) = (1, k) = 1$ and since $u^2+ku+1 = ny$ for some y in $\mathbb{Z}$, we have $(ku+1, n) = (ku+1-ny, n) = (-u^2, n) = 1$. Thus, $(ku+1, kn) = 1$. We conclude that $\frac{u+\frac{1}{k}}{n}$ is a vertex in $\widehat{\mathcal{F}}_{u,n}$ and is the farthest one being joined with $\frac{u}{n}$. We also see that

$$\lim_{z\to\infty} \frac{u+\frac{1}{nz+k}}{n} = \frac{u}{n}.$$

This implies that there is no such nearest point being joined with the vertex $\frac{u}{n}$.

Case(ii): In this case, we obtain that $su+t \equiv -u^2 \pmod{n}$ and $u(sn)-n(us+t) = -n$, which implies $t = 1$. Thus, $su+1 \equiv -u^2 \pmod{n}$ and we know $u^2+ku+1 \equiv 0 \pmod{n}$. This implies that $su+1 \equiv ku+1 \pmod{n}$, that is $su \equiv ku \pmod{n}$. The fact that $(u, n) = 1$ implies that $s \equiv k \pmod{n}$. Therefore, $s = nz+k$ for some z in $\mathbb{N} \cup \{0\}$. Hence

$$\frac{t}{s} = \frac{1}{nz+k}.$$

This case is done by using a similar argument to that of the first case.

Case(iii): We have $su+t \equiv u^2 \pmod{n}$ and $u(sn)-n(su+t) = -n$, which implies $t = 1$. Then, $su+1 \equiv u^2 \pmod{n}$. Since $u^2+ku+1 \equiv 0 \pmod{n}$, $su+1 \equiv -ku-1 \pmod{n}$. Hence, $su+1+ku+1 = nz$ for some z in $\mathbb{N}\cup\{0\}$, that is $s = \frac{nz-ku-2}{u}$. Thus,

$$\frac{t}{s} = \frac{u}{nz-ku-2}.$$

We again find the greatest value of $\frac{t}{s}$ by defining the function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$,

$$f(z) = \frac{u+\frac{u}{nz-ku-2}}{n}.$$

And the derivative of f is $f'(z) = \frac{-u}{(nz-ku-2)^2} < 0$, so the greatest value of the function f is at $z = 0$, that is

$$\frac{u-\frac{1}{k+\frac{2}{u}}}{n}.$$

But $\frac{u-\frac{1}{k+\frac{2}{u}}}{n}$ is nearer to $\frac{u}{n}$ than $\frac{u+\frac{1}{k}}{n}$. Therefore, the farthest vertex which can be joined with $\frac{u}{n}$ is $\frac{u+\frac{1}{k}}{n}$. Now we know that

$$\lim_{z\to\infty} \frac{u+\frac{u}{nz-ku-2}}{n} = \frac{u}{n}.$$

Hence, the nearest vertex does not exist.

Case(iv): We obtain that $su+t \equiv -u^2 \pmod{n}$ and $u(sn)-n(su+t) = n$, which implies $t=-1$. Thus, $su-1 \equiv -u^2 \pmod{n}$. Because $u^2+ku+1 \equiv 0 \pmod{n}$, $su-1 \equiv ku+1 \pmod{n}$, that is $-su+1 \equiv -ku-1 \pmod{n}$. So $-su+1+ku+1=nz$ for some z in $\mathbb{N} \cup \{0\}$, then $s = \frac{-(nz-ku-2)}{u}$. Therefore,

$$\frac{t}{s} = \frac{u}{nz-ku-2}.$$

The remaining proof is similar to the condition (iii).

Proof of (ii) By the above proof of existence, let k be such that $1 \leqslant k \leqslant n$ and $u^2+ku+1 \equiv 0 \pmod{n}$. We have shown that k is unique. From the proof in (i), we can suppose that

$$\frac{u+\frac{1}{k}}{n} < \frac{u+\frac{t}{s}}{n}$$

and

$$\frac{ku+1}{kn} = \frac{u+\frac{1}{k}}{n} \to \frac{u+\frac{t}{s}}{n} = \frac{us+t}{sn}$$

where $\frac{t}{s}$ is in $\mathbb{Q}^+$. We start working on each case as in the proof of (i).

Case(i): In this case, we have $us+t \equiv u^2k+u \pmod{n}$ and $ns(ku+1)-kn(su+t) = n$, which implies $s=kt+1$. Thus, $u+kut+t \equiv u^2k+u \pmod{n}$ and we get $t(ku+1) \equiv u^2k \pmod{n}$. We observe that $-u^2t \equiv u^2k \pmod{n}$. Moreover, $-t \equiv k \pmod{n}$ since $(u,n)=1$. So, $t=-nz-k$ for some z in $\mathbb{N}\cup\{0\}$, that is $s = 1-k(nz+k)$. Therefore,

$$\frac{t}{s} = \frac{nz+k}{k(nz+k)-1}.$$

Next, we find the largest value of $\frac{t}{s}$ by defining a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ by

$$f(z) = \frac{u+\frac{nz+k}{k(nz+k)-1}}{n}.$$

Since the derivative of f is $f'(z) = \frac{-1}{(k(nz+k)-1)^2} < 0$, then f has a maximum at $z=0$, that is

$$\frac{u+\frac{k}{k^2-1}}{n} = \frac{(k^2-1)u+k}{(k^2-1)n}.$$

Now we will show $((k^2-1)u+k, (k^2-1)n) = 1$. Let us suppose that

$$((k^2-1)u+k, k^2-1) = a,$$

then a divides k^2-1, which implies a divides $(k^2-1)u$. Since a divides $(k^2-1)u+k$, we have a divides k. Because a divides k^2-1, a divides -1 and so $a = \pm 1$.

Next, we assume that $((k^2-1)u+k, n) = b$, then b divides $(k^2-1)u+k$. So we have $k(ku+1)-u=(k^2-1)u+k\equiv 0 \pmod{b}$. Since $u^2+ku+1\equiv 0 \pmod{n}$ and b divides n, $u^2+ku+1\equiv 0 \pmod{b}$. As we have shown $k(ku+1)-u\equiv 0 \pmod{b}$ and $-u^2\equiv ku+1 \pmod{b}$, then $k(-u^2)-u\equiv 0 \pmod{b}$.

Suppose b does not divide u. As $(u,n)=1$ and b divides n, so $(u,b)=1$. Moreover, $-ku-1\equiv 0 \pmod{b}$. Since we have $u^2\equiv -ku-1 \pmod{b}$, then b divides u^2, which gives a contradiction. Hence, b divides u. Thus, $b=1$ since $(u,n)=1$ and b divides n. Therefore, $((k^2-1)u+k,(k^2-1)n)=1$, that is

$$\frac{u+\frac{k}{k^2-1}}{n}=\frac{u+\frac{1}{k-\frac{1}{k}}}{n}$$

is a vertex in $\widehat{\mathcal{F}}_{u,n}$ and is also the farthest vertex which can be joined with $\frac{u+\frac{1}{k}}{n}$. Since

$$\lim_{z\to\infty}\frac{u+\frac{nz+k}{k(nz+k)-1}}{n}=\frac{u+\frac{1}{k}}{n},$$

there is no such a nearest vertex.

Case(ii): We obtain that $su+t\equiv -u^2k-u \pmod{n}$ and $ns(ku+1)-kn(su+t)=-n$, which implies $s=kt-1$. Thus, $kut-u+t\equiv -u^2k-u \pmod{n}$. We observe that $t(ku+1)\equiv -u^2k \pmod{n}$. Since $u^2+ku+1\equiv 0 \pmod{n}$, $u^2t\equiv u^2k \pmod{n}$. As $(u,n)=1$, $t\equiv k \pmod{n}$, that is $t=nz+k$ for some z in $\mathbb{N}\cup\{0\}$. So, $s=k(nz+k)-1$. Hence,

$$\frac{t}{s}=\frac{nz+k}{k(nz+k)-1}$$

and the remaining proof is the same as above proof.

Case(iii): We have $su+t\equiv u^2k+u \pmod{n}$ and $ns(ku+1)-kn(su+t)=-n$, which implies $s=kt-1$. So we have $kut-u+t\equiv u^2k+u \pmod{n}$, that is $t(ku+1)$ $-ku^2\equiv 2u \pmod{n}$. Since $u^2+ku+1\equiv 0 \pmod{n}$, $-tu^2-ku^2\equiv 2u \pmod{n}$. Moreover, $-tu-ku\equiv 2 \pmod{n}$ by $(u,n)=1$. Then, $tu+ku\equiv -2 \pmod{n}$, that is $t=\frac{nz-ku-2}{u}$ for some z in $\mathbb{N}\cup\{0\}$. Thus, $s=\frac{k(nz-ku-2)-u}{u}$, so

$$\frac{t}{s}=\frac{nz-ku-2}{k(nz-ku-2)-u}.$$

Now we find the greatest value of $\frac{t}{s}$ by defining a function $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}$,

$$f(z)=\frac{u+\frac{nz-ku-2}{k(nz-ku-2)-u}}{n}.$$

Since $f'(z)=\frac{-u}{(k(nz-ku-2)-u)^2}<0$, the greatest value of f is at $z=0$, that is

$$\frac{u+\frac{1}{k+\frac{1}{k+\frac{2}{u}}}}{n}.$$

But $\frac{u+\frac{1}{k+\frac{1}{k+\frac{2}{u}}}}{n}$ is nearer to $\frac{u+\frac{1}{k}}{n}$ than $\frac{u+\frac{1}{k-\frac{1}{k}}}{n}$, so the farthest one being joined with $\frac{u+\frac{1}{k}}{n}$ is $\frac{u+\frac{1}{k-\frac{1}{k}}}{n}$. As we have

$$\lim_{z\to\infty} \frac{u+\frac{nz-ku-2}{k(nz-ku-2)-u}}{n} = \frac{u+\frac{1}{k}}{n},$$

the nearest vertex does not exist.

Case(iv): We obtain that $su+t \equiv -u^2k-u \pmod{n}$ and $ns(ku+1)-kn(su+t) = n$, which implies $s = kt+1$. Therefore, $u+kut+t \equiv -u^2k-u \pmod{n}$ and we have $t(ku+1) \equiv -ku^2-2u \pmod{n}$. As $u^2+ku+1 \equiv 0 \pmod{n}$, $-tu^2 \equiv -ku^2-2u \pmod{n}$. Since $(u,n) = 1$, $-tu \equiv -uk-2 \pmod{n}$, that is $t = \frac{-(nz-ku-2)}{u}$ for some z in $\mathbb{N}\cup\{0\}$. Moreover, $s = \frac{-(k(nz-ku-2)-u)}{u}$. Thus,

$$\frac{t}{s} = \frac{nz-ku-2}{k(nz-ku-2)-u}.$$

The proof is similar to the proof for the condition (iii). $\square$

Example :

Let $u = 1$ and $n = 5$, then we have $1^2+3+1 \equiv 0 \pmod{5}$, that is $k = 3$. By Theorem 3.1.3 and calculation, we get the following graph showing the farthest vetex in the right hand side that join with a previous vertex.

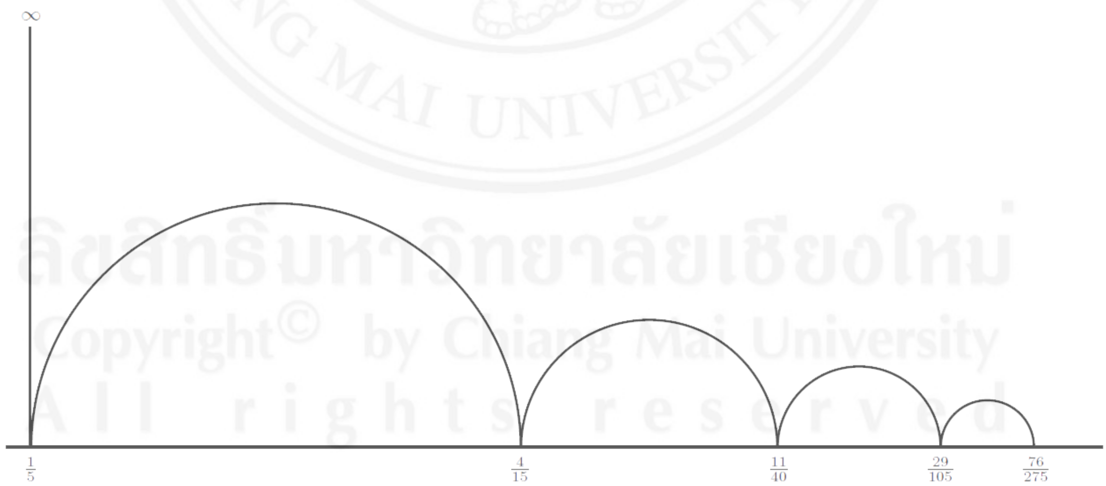

Figure 3.1: $\widehat{\mathcal{F}}_{1,5}$

Since we have φ in Corollary 3.1.2, the following corollary shows that φ is the element in $\widehat{\Gamma}$ that maps $(\infty, \frac{u}{n})$ to $(\frac{u}{n}, \frac{u+\frac{1}{k}}{n})$. It also maps $(\frac{u}{n}, \frac{u+\frac{1}{k}}{n})$ to $(\frac{u+\frac{1}{k}}{n}, \frac{u+\frac{1}{k-\frac{1}{k}}}{n})$.

Corollary 3.1.4. *If* $(u,n)=1$ *and* $u^2+ku+1\equiv 0 \pmod{n}$, *then*

$$\varphi(\frac{u}{n}) = \frac{u+\frac{1}{k}}{n},\ \ \varphi(\frac{u+\frac{1}{k}}{n}) = \frac{u+\frac{1}{k-\frac{1}{k}}}{n}.$$

Proof.

$$\begin{aligned}
\varphi(\frac{u}{n}) &= \frac{-u(\frac{u}{n})+\frac{u^2+ku+1}{n}}{-n(\frac{u}{n})+u+k}\\
&= \frac{\frac{-u^2+u^2+ku+1}{n}}{-u+u+k}\\
&= \frac{ku+1}{nk}\\
&= \frac{\frac{ku+1}{k}}{n}\\
&= \frac{u+\frac{1}{k}}{n}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\frac{u+\frac{1}{k}}{n}) &= \frac{-u(\frac{u+\frac{1}{k}}{n})+\frac{u^2+ku+1}{n}}{-n(\frac{u+\frac{1}{k}}{n})+u+k}\\
&= \frac{\frac{-u^2}{n}-\frac{u}{kn}+\frac{u^2+ku+1}{n}}{-u-\frac{1}{k}+u+k}\\
&= \frac{\frac{k^2u-u+k}{kn}}{\frac{k^2-1}{k}}\\
&= \frac{u(k^2-1)+k}{n(k^2-1)}\\
&= \frac{u+\frac{k}{k^2+1}}{n}\\
&= \frac{u+\frac{1}{k-\frac{1}{k}}}{n}.
\end{aligned} \tag{3.1}$$

□

Next, we will use Theorem 3.1.3 to prove that $\widehat{\mathcal{F}}_{u,n}$ contains directed circuits if and only if it contains directed triangles. To prove this, we first prove the following theorem.

Theorem 3.1.5. *No edge of* $\widehat{\mathcal{G}}_{1,1}$ *cross in* $\mathbb{H}^2$.

Proof. Let $\frac{r}{s}\to\frac{x}{y}$ be an edge in $\widehat{\mathcal{G}}_{1,1}$. Then, there exists $\sigma\in\widehat{\Gamma}$ such that $\sigma(\infty)=\frac{r}{s}$ and $\sigma(1)=\frac{x}{y}$. Let $\gamma(z)=z+1$. So, we have $\sigma\gamma(\infty)=\frac{r}{s}$ and $\sigma\gamma(0)=\frac{x}{y}$. Since any element of $\widehat{\Gamma}$ preserves the geodesics, we can suppose that the edges $0\to\infty$ and $\frac{r}{s}\to\frac{x}{y}$ cross in $\mathbb{H}^2$. Thus, $\frac{r}{s}<0<\frac{x}{y}$, which contradicts to $ry-sx=\pm1$. □

Corollary 3.1.6. *No edge of $\widehat{\mathcal{F}}_{u,n}$ cross in $\mathbb{H}^2$.*

Proof. Suppose that the edges $\frac{r_1}{s_1 n} \to \frac{x_1}{y_1 n}$ and $\frac{r_2}{s_2 n} \to \frac{x_2}{y_2 n}$ cross in $\mathbb{H}^2$. Then, $r_1 y_1 n - x_1 s_1 n = \pm n$ and $r_2 y_2 n - x_2 s_2 n = \pm n$, so $r_1 y_1 - x_1 s_1 = \pm 1$ and $r_2 y_2 - x_2 s_2 = \pm 1$. Thus, $\frac{r_1}{s_1} \to \frac{x_1}{y_1}$ and $\frac{r_2}{s_2} \to \frac{x_2}{y_2}$ are the edges in $\widehat{\mathcal{G}}_{1,1}$ that cross in $\mathbb{H}^2$, a contradiction by Theorem 3.1.5. $\square$

By Theorem 2.3.5, we add the condition $n > 1$ into the following theorem. And now we have enough implements to prove the following theorem.

Theorem 3.1.7. *Let u, n be relatively prime postive integers and $n > 1$. $\widehat{\mathcal{F}}_{u,n}$ contains directed circuits if and only if it contains directed triangles.*

Proof. ($\Leftarrow$) It is obvious.
($\Rightarrow$) Assume that $\widehat{\mathcal{F}}_{u,n}$ contains a directed circuit of minimal length in the form

$$w_1 \to w_2 \to w_3 \to \dots \to w_p \to w_1.$$

We will prove by contradiction, then we suppose that $\widehat{\mathcal{F}}_{u,n}$ contains no directed triangles. Then, by Theorem 2.3.4, we have $u^2 \pm u + 1 \not\equiv 0 \pmod n$. Since (w_1, w_2) is in $\widehat{O}(\infty, \frac{u}{n})$, there exists T in $\widehat{\Gamma}$ such that $T(\infty) = w_1,\ T(\frac{u}{n}) = w_2$, that is $T^{-1}(w_1) = \infty, T^{-1}(w_2) = \frac{u}{n}$. Now we take $v_{i-1} = T^{-1}(w_i)$. Then, we obtain the circuit A

$$\infty \to v_1 = \frac{u}{n} \to v_2 \to \dots \to v_m \to \infty$$

such that $m = p - 1$. Since $u^2 \pm u + 1 \not\equiv 0 \pmod n$ and $v_m \to \infty$, then

$$v_m > \frac{u+1}{n}. \tag{3.2}$$

Let v be the farthest vertex which can be joined with v_1 and $v_1 < v$. We will show that $v_2 = v$, so we assume that $v_2 < v$. If v is a vertex in A, we have

$$\infty \to v_1 \to v \to \dots \to v_m \to \infty$$

is shorter than A which contradicts to A being of the minimal length. If v is not a vertex in A, there are vertices v_s, v_{s+1} in A such that $v_s < v < v_{s+1}$. We observe that $v_1 \to v$ and $v_s \to v_{s+1}$ cross in $\widehat{\mathcal{F}}_{u,n}$ which gives a contradiction to Corollary 3.1.6. Therefore, $v_2 = v$.

By Theorem 3.1.3, $v_2 = \frac{u+\frac{1}{k}}{n}$ when k is the unique integer such that $1 \leqslant k \leqslant n$ and $u^2 + ku + 1 \equiv 0 \pmod n$. From Corollary 3.1.4, we have $\varphi(\infty) = v_1,\ \varphi(v_1) = v_2$ and in general,

$$\varphi\left(\frac{u + \frac{x}{y}}{n}\right) = \frac{u + \frac{y}{ky-x}}{n}.$$

For $k = 1$, we have $\varphi(v_1) = \frac{u+1}{n}$ and $\varphi(v_2) = \infty$. For $1 < k \leqslant n$, if x and y are positive integers and $\frac{x}{y} < 1$, then $\frac{y}{ky-x} < 1$ since $k \geqslant 2$ and $x < y$. Therefore, we can see that $\varphi^i(v_1) \leqslant \frac{u+1}{n}$ for positive integers i.

Next, we will show that $v_i = \varphi^{i-1}(v_1)$ for $1 \leqslant i \leqslant m$. First, we know that $v_1 = \varphi^0(v_1)$ and $v_2 = \varphi(v_1)$. Now suppose that $v_i = \varphi^{i-1}(v_1)$ for all $1 \leqslant i \leqslant t$ and we will show that $v_{t+1} = \varphi^t(v_1)$. Assume that $v_{t+1} < \varphi^t(v_1)$. By Theorem 2.3.3, we have

$$v_t = \varphi^{t-1}(v_1) \to \varphi^{t-1}(v_2) = \varphi^t(v_1)$$

is an edge in $\widehat{\mathcal{F}}_{u,n}$. If $\varphi^t(v_1)$ is not a vertex in A, then there are vertices v_r, v_{r+1} such that $v_r < \varphi^t(v_1) < v_{r+1}$ since $\varphi^t(v_1) < v_m$. Thus, the edges $v_r \to v_{r+1}$ and $v_t \to \varphi^t(v_1)$ cross, a contradiction. If $\varphi^t(v_1)$ is a vertex in A, then a circuit

$$\infty \to v_1 \to ... \to v_t \to \varphi^t(v_1) \to ... \to \infty$$

is shorter than A, which gives a contradiction. Assume that $v_{t+1} > \varphi^t(v_1)$. From above, we have

$$\varphi^{-(t-1)}(v_{t+1}) > \varphi^{-(t-1)}(\varphi^t(v_1)) = \varphi(v_1) = v_2.$$

By Theorem 2.3.3, $v_1 = \varphi^{-(t-1)}(v_t) \to \varphi^{-(t-1)}(v_{t+1})$, which contradicts to the choice of v_2. Hence, $v_i = \varphi^{i-1}(v_1)$ for $1 \leqslant i \leqslant m$, so $v_m = \varphi^{i-1}(v_1) \leqslant \frac{u+1}{n}$, which gives a contradiction to (3.2). Therefore, $\widehat{\mathcal{F}}_{u,n}$ contains directed triangles. □

Corollary 3.1.8. *If n is even, then $\widehat{\mathcal{F}}_{u,n}$ does not contain any directed circuit.*

Proof. It follows by Theorem 3.1.7 and Theorem 2.3.6. □

3.2 Continued Fraction

In this section, we show that the result from Theorem 3.1.3 is related to some continued fraction.

In this thesis, we work with special Möbius transformation

$$t_m(z) := t(z) = \frac{1}{k-z} = \frac{-1}{-k+z}.$$

To see some relations between continued fraction and hyperbolic path of suborbital graphs, by using Theorem 3.1.3, if $k \geqslant 2$, we can give the following infinite path

$$\frac{1}{0} \to \frac{u}{n} \to \frac{u+\frac{1}{k}}{n} \to \frac{u+\frac{1}{k-\frac{1}{k}}}{n} \to \frac{u+\frac{1}{k-\frac{1}{k-\frac{1}{k}}}}{n} \to ...$$

The above path gives rise to a continued fraction

$$\cfrac{1}{k-\cfrac{1}{k-\cfrac{1}{k-\cfrac{1}{\ddots}}}} \tag{3.3}$$

when $k \geqslant 2$. Now we have a continued fraction, this gives a problem about the convergence of a continued fraction. Since the above continued fraction is a special case of Theorem 2.4.1, we use this theorem to prove the following corollary.

Corollary 3.2.1. *The continued fraction* (3.3) *converges to* $\frac{k-\sqrt{k^2-4}}{2}$.

Proof. Since we have $a_m = -1$ and $b_m = -k$ where $k \geqslant 2$, then $|b_m| \geqslant 1 + |a_m|$. By Theorem 2.4.1, the continued fraction (3.3) converges to v with $|v| \leqslant 1$, that is $\lim_{m\to\infty} T_m(0) = v$. As we know

$$T_m(0) = \frac{1}{k - T_{m-1}(0)}\ ,$$

$T_m(0)(k - T_{m-1}(0)) = 1$ and since $\lim_{m\to\infty} T_m(0) = \lim_{m\to\infty} T_{m-1}(0)$, we have $v(k-v) = 1$. Moreover, $v^2 - kv + 1 = 0$ and

$$v = \frac{k \pm \sqrt{k^2-4}}{2}.$$

We observe that if $k = 2$ then $v = 1$. Because $|v| \leqslant 1$, if $k > 2$ then $v = \frac{k-\sqrt{k^2-4}}{2}$. $\square$

Example :

Since $1^2 + 3(1) + 1 \equiv 0 \pmod 5$, then we obtain a finite path in Figure 3.2 and if this path is infinite then it gives a continued fraction

$$\cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{\ddots}}}} \tag{3.4}$$

By Corollary 3.2.1, the above continued fraction converges to $\frac{3-\sqrt{5}}{2} \approx 0.671$.

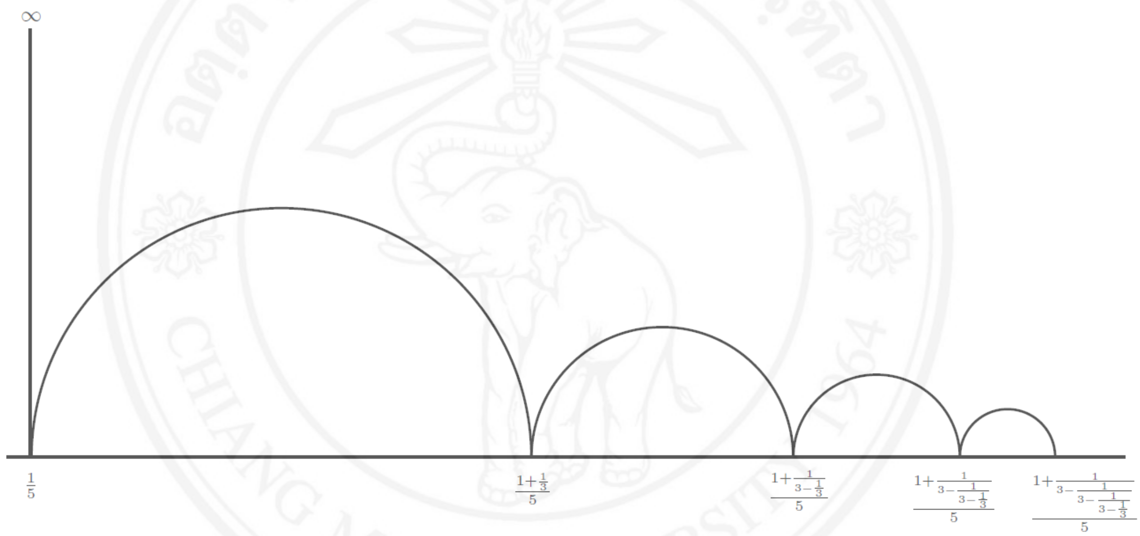

Figure 3.2: $\widehat{\mathcal{F}}_{1,5}$