CHAPTER 1

Introduction

For an integer $q \geq 3$, $H(\lambda_q)$ is the group generated by $T_1(z) = z + \lambda_q$ and $T_2(z) = -1/z$ where $\lambda_q = 2cos(\pi/q)$. The groups $H(\lambda_q)$ are called *Hecke groups*. When $q = 3, 4$ and 6, the corresponding Hecke groups are the modular group $\Gamma = H(1)$, $H(\sqrt{2})$ and $H(\sqrt{3})$ respectiovely. These groups are the only Hecke groups whose elements are entirely known, see [4]. When $m = 1, 2$ or 3, a mapping T is in $H(\sqrt{m})$ if and only if T is in one of the following forms:

(i) $T(z) = \frac{az+b\sqrt{m}}{c\sqrt{m}z+d}, a, b, c, d \in \mathbb{Z}, ad - bcm = 1$

(ii) $T(z) = \frac{a\sqrt{m}z+b}{cz+d\sqrt{m}}, a, b, c, d \in \mathbb{Z}, adm - bc = 1.$

This is not true in general. For instance, when $m = 5$, a result in [6] stated that T in the above forms is in $H(\sqrt{5})$ if and only if $\frac{a}{c\sqrt{5}}$ or $\frac{a\sqrt{5}}{c}$ is a finite $\sqrt{5}$-fraction. In [5], the groups $\mathcal{H}(\sqrt{m})$ were defined for each square-free positive integer m. $\mathcal{H}(\sqrt{m})$ is the group containing all mappings of the form

(i) $T(z) = \frac{az+b\sqrt{m}}{c\sqrt{m}z+d}, a, b, c, d \in \mathbb{Z}, ad - bcm = 1$

(ii) $T(z) = \frac{a\sqrt{m}z+b}{cz+d\sqrt{m}}, a, b, c, d \in \mathbb{Z}, adm - bc = 1.$

When $m = 1, 2$ and 3, $\mathcal{H}(\sqrt{m})$ is, in fact, the Hecke groups $\Gamma, H(\sqrt{2})$ and $H(\sqrt{3})$ mentioned above.
In [3], the authors looked into suborbital graphs for the modular group Γ on $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and their circuits. A necessary and sufficient condition for a suborbital graph to contain a triangle was then given. Later in 2000, Keskin, see [4], worked on the suborbital graphs for the Hecke groups $H(\sqrt{m})$ on $\sqrt{m}\widehat{\mathbb{Q}}$ and also gave a necessary and sufficient condition for a suborbital graph to contain a circuit for $m = 1, 2, 3$.

We aim to determine the basic properties that the groups $\mathcal{H}(\sqrt{m})$ share with the three Hecke groups $\Gamma, H(\sqrt{2})$ and $H(\sqrt{3})$. Furthermore, we investigate the suborbital graphs for $\mathcal{H}(\sqrt{m})$ on $\sqrt{m}\widehat{\mathbb{Q}}$ by extending techniques of proof from [4]. We then find the conditions on m for the suborbital graphs of $\mathcal{H}(\sqrt{m})$ to share the same properties with the suborbital graphs of the Hecke groups as in [4].

The thesis consists of four chapters. Chapter 1 serves as an introduction to the concepts and the aim of the research. Chapter 2 provides the necessary preliminaries and fundamental theorems used later on. Chapter 3 states and proves the main results of the research involving the suborbital graphs and the existances of circuits in them which will be concluded in Chapter 4.