CHAPTER 2

Preliminaries

In this chapter, we collect the necessary concepts involving hyperbolic geometry and group theory. See [1] and [3] for more details.

2.1 The hyperbolic plane

Before we proceed to the hyperbolic plane, let's start with the more familiar euclidean plane. The *euclidean plane* is the plane $\mathbb{R}^2$ equipped with the metric d_{euc} defined as follows: Let γ be a curve in $\mathbb{R}^2$ parametrized by

$$t \mapsto (x(t), y(t)), \quad a \leq t \leq b.$$

The *euclidean length* l_{euc} of γ is given by

$$l_{\text{euc}}(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

The *euclidean distance* d_{euc} between the points P and Q, denoted by $d_{\text{euc}}(P, Q)$, is the

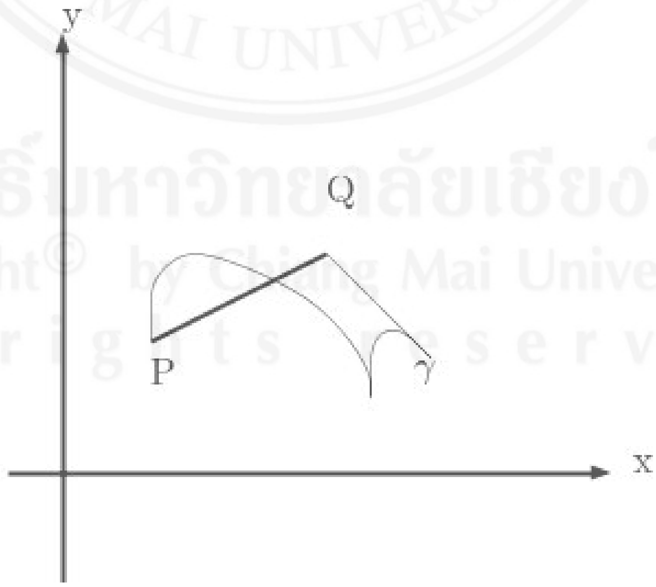

Figure 2.1: The euclidean plane and a curve γ

infimum of the lengths of all curves joining P and Q. That is

$$d_{\text{euc}}(P,Q) = \inf\{l_{\text{euc}}(\gamma); \gamma \text{ goes from } P \text{ to } Q\}.$$

We will now define a term for shortest curve in general metric spaces. A *geodesic* is a curve γ such that for each $P, Q \in \gamma$ with Q sufficiently closed to P, the part of γ between P and Q is the shortest curve joining them.

Proposition 2.1.1. *$d_{euc}(P,Q)$ is equal to $l_{euc}([P,Q])$ where $[P,Q]$ is the line segment joining P and Q. That is, euclidean geodesics are line segments and lines.*

So, we also obtain the following corollary.

Corollary 2.1.2. *The euclidean distance from $P_0 = (x_0, y_0)$ to $P_1 = (x_1, y_1)$ is*

$$d_{euc}(P,Q) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

Alternatively, we can consider the euclidean plane as $\mathbb{C}$ and rewrite d_{euc} between two points $P = z_0$ and $Q = z_1$ as $d_{\text{euc}}(P,Q) = |z_1 - z_0|$.

We now define the hyperbolic plane in a similar manner as the euclidean plane. The *hyperbolic plane* is the metric space consisting of the open upper half-plane

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\} = \{z \in \mathbb{C} : \text{Im}(x) > 0\}$$

together with the metric d_{hyp} defined below.
Similarly to the euclidean case, we start by defining the *hyperbolic length* of a curve γ parametrized by

$$t \mapsto (x(t), y(t)), \quad a \leq t \leq b$$

as

$$l_{\text{hyp}}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

The *hyperbolic distance* d_{hyp} between the points P and Q, denoted by $d_{\text{hyp}}(P,Q)$, is the infimum of the lengths of all curves joining P and Q. That is

$$d_{\text{hyp}}(P,Q) = \inf\{l_{\text{hyp}}(\gamma); \gamma \text{ goes from } P \text{ to } Q\}.$$

From the definition, we continue to the shortest curves in hyperbolic plane.

Theorem 2.1.3. *The unique shortest curve joining P and Q in $\mathbb{H}^2$ is the circle arc centered on the x-axis or the vertical line segment passing through P and Q.*

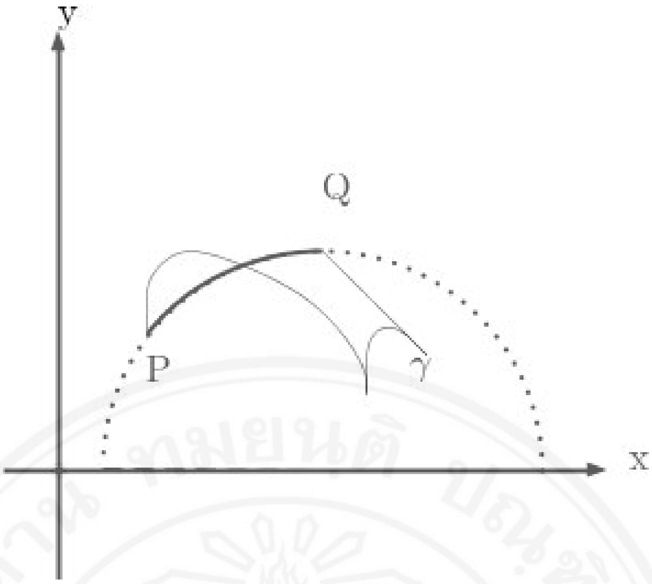

Figure 2.2: The hyperbolic plane and a curve γ

2.2 Isometries

An *isometry* on a metric space (X, d) is a bijection $T : X \to X$ such that

$$d(x, y) = d(T(x), T(y))$$

for every $x, y \in X$. In other words, an isometry is a bijection that also preserves distance between each pair of points.

Some examples of isometries in the euclidean plane $(\mathbb{C}, d_{\text{euc}})$ are :

a *translation* along z_0, $T(z) = z + z_0$.

a *rotation* of angle θ around the origin, $T(z) = e^{i\theta} z$.

a *reflection* across a line passing through the origin and making an angle of θ with the x-axis, $T(z) = e^{2i\theta}\bar{z}$. See Figure 2.3 for images of each isometry acting on a line segment in $\mathbb{C}$. In fact, every euclidean isometry can be written as a composition of these

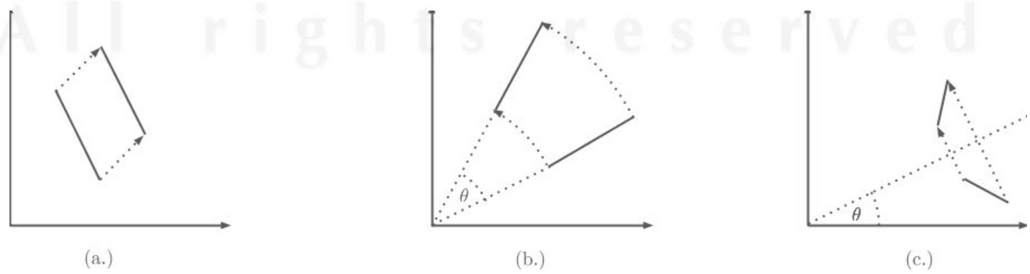

Figure 2.3: (a.) Translation (b.) Rotation (c.) Reflection

three isometries. Let T be an isometry in the euclidean plane, then either

$$T(z) = e^{i\theta} z + z_0$$

or

$$T(z) = e^{2i\theta} \bar{z} + z_0$$

where $\theta \in \mathbb{R}$ is an angle and z_0 is a point in $\mathbb{C}$.

Now, let's move on to the isometries of the hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$. Some of them are:

a *homothety* with ratio $\lambda > 0$, $T(z) = \lambda z$.

a *horizontal translation* along $a \in \mathbb{R}$ which is just a kind of translation, namely $T(z) = z + a$.

a *reflection* across the y-axis, $T(z) = -\bar{z}$.

a *standard inversion* or inversion across the unit circle, $T(z) = \frac{1}{\bar{z}}$. See Figure 2.4 for images of each isometry acting on a line segment in $\mathbb{C}$.

Moreover, every isometries on hyperbolic plane can be written as a composition of these maps which is either in the form

$$T(z) = \frac{az + b}{cz + d}$$

or

$$T(z) = \frac{c\bar{z} + d}{a\bar{z} + b}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

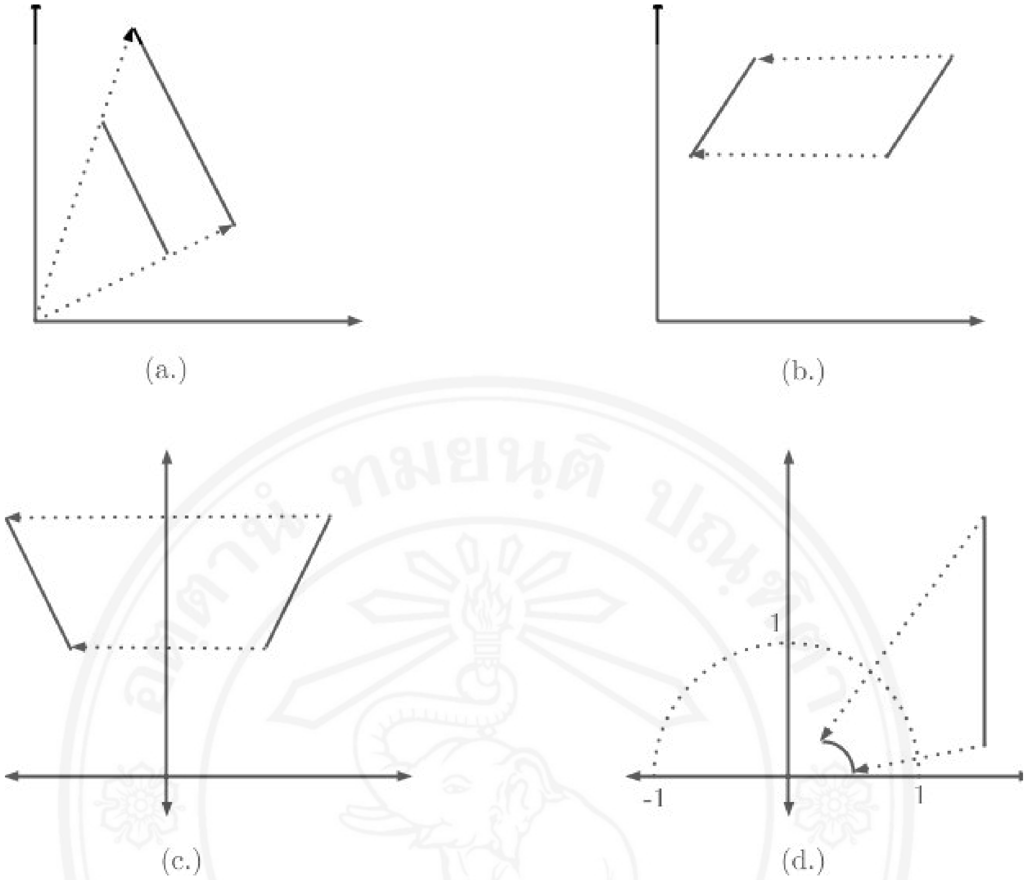

Figure 2.4: (a.) Homothety (b.) Horizontal translation (c.) Reflection across y-axis (d.) Standard inversion

2.3 Groups and group actions

Let G be a set and let $*$ be an operator that combines any two elements a, b of G to form a new element, denoted by $a * b$ or simply ab and also satisfies the following requirements :

1. $a * b \in G \; \forall a, b \in G$.
2. $(a * b) * c = a * (b * c) \; \forall a, b, c \in G$.
3. There is an identity element $e \in G$ such that $e * a = a * e = a \; \forall a \in G$.
4. For each $a \in G$, there is an element $b \in G$ such that $a * b = b * a = e$ where e is the identity element.

We call $(G, *)$ a group. If, moreover, $H \subseteq G$ and H is also a group, we say that H is a subgroup of G, denoted by $H \preceq G$.

Now, let $(G, *)$ be a group and let X be a set. The group action of G on X (we may also say that G acts on X) is a function from $G \times X$ to X sending (g, x) to $g.x$ $\forall g \in G, x \in X$ satisfying these two requirements :

1. $(g * h).x = g.(h.x) \quad \forall g, h \in G, x \in X.$
2. $e.x = x \quad \forall x \in X$ where e is the identity element of G.

An equivalence relation on X is naturally defined from a group action G on X by

$$x \approx y \leftrightarrow \exists g \in G, g.x = y$$

$\forall x, y \in X$. We call the equivalence classes of this relation containing an element $x \in X$ *orbits* of x under G. An action is called *transitive* if the relation defined above is trivial, that is, $\forall x, y \in X \exists g \in G$ such that $g.x = y$.

2.4 Suborbital graphs

Let G be a group of transformation that acts on X naturally by $g.x = g(x)$ $\forall g \in G, x \in X$. Then G acts on $X \times X$ by

$$g(a, b) = (g(a), g(b)) \quad \forall g \in G, (a, b) \in X.$$

The orbits of this action are called *suborbitals* of G. We denote the orbit containing (a, b) by $O(a, b)$. A *suborbital graph* $\mathcal{G}(a, b)$ is a graph with the elements of X as its vertices and there is a directed edge from x to y if $(x, y) \in O(a, b)$. We denote the directed edge from x to y by $x \to y$ or $y \leftarrow x$.

We can see that $O(b, a)$ is also a suborbital such that $O(b, a) = O(a, b)$ or $O(b, a)$ is $O(a, b)$ with reversed arrows. In case of equality, the graph consists of pairs of oppositely directed edges. We may replace each pair with an undirected edge for convinience and we called the graph *self-paired*. Otherwise, we call $\mathcal{G}(a, b)$ and $\mathcal{G}(b, a)$ *paired suborbital graphs*.

We see that $O(a, a)$ is the identity relation on X. $\mathcal{G}(a, a)$, is a self-paired suborbital graph called the *trivial suborbital graph*.

2.5 The modular group

A *Möbius transformation* is a rational funtion of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. There are many groups of Möbius transformations. A modular group is also a group that can be identified as a group of Möbius transformations. But to give its definition, let's start from introducing another group known as the group $\mathrm{SL}(2, \mathbb{Z})$. The group $\mathrm{SL}(2, \mathbb{Z})$ is a group of all matrices in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{Z}, ad - bc = 1.$$

Its quotient group by the centre $\{\pm 1\}$ is called the *modular group* $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. Thus, Γ consists of the pairs of matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{Z}, ad - bc = 1.$$

We will omit the symbol $\pm$ and identify the matrices with their negative. So, we can represent each element $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ as a Möbius tranformation $T : \mathbb{H}^2 \to \mathbb{H}^2$ with

$$T(z) = \frac{az + b}{cz + d} \quad \forall z \in \mathbb{H}^2.$$

In [3], G.A. Jones, D. Singerman and K. Wicks investigated the modular group's natural action on $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and here are some of their results: Let Γ acts on $\widehat{\mathbb{Q}}$ naturally, that is for each $T \in \Gamma$ such that $T(z) = \frac{az+b}{cz+d}$,

$$T.\frac{x}{y} = T(\frac{x}{y}) = \frac{ax + by}{cx + dy} \quad \forall \frac{x}{y} \in \widehat{\mathbb{Q}}.$$

Lemma 2.5.1. *The action of* Γ *on* $\widehat{\mathbb{Q}}$ *is transitive.*

As for suborbital graphs for the action of Γ on $\widehat{\mathbb{Q}}$, from the fact that the action is transitive, each suborbital contains a pair $(\infty, \frac{u}{n})$ with $n > 0$ and $(u, n) = 1$. We can denote the suborbital by $O_{u,n}$ and the corresponding suborbital by $G_{u,n}$.

Theorem 2.5.2. $\frac{r}{s} \to \frac{x}{y}$ *in* $G_{u,n}$ *if and only if either*

1. $x \equiv ur \pmod{n}$, $y \equiv us \pmod{n}$ *and* $ry - sx = n$ *or*

2. $x \equiv -ur \pmod{n}$, $y \equiv -us \pmod{n}$ *and* $ry - sx = -n$.

Corollary 2.5.3. $G_{u,n}$ *is self-paired if and only if* $u^2 \equiv -1 \pmod{n}$.

2.6 The Farey graph

G.A. Jones, D. Singerman and K. Wicks worked on the Farey graph $G_{1,1}$ with $\widehat{\mathbb{Q}}$ as its vertex set. From the above corollary, it is self-paired. We call $G_{1,1}$ the *Farey graph* and denote it by F.

Lemma 2.6.1. *Let $\frac{r}{s}, \frac{x}{y} \in \mathbb{Q}$ be reduced rationals. Then $\frac{r}{s}$ and $\frac{x}{y}$ are adjacent in F if and only if $ry - sx = \pm 1$.*

Corollary 2.6.2. *No edges of F cross in $\mathbb{H}^2$ if we represent each edge by a hyperbolic geodesic.*

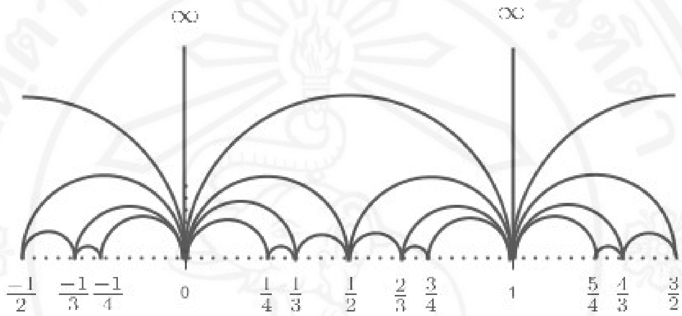

Figure 2.5: $F = G_{1,1}$

Next, let $F_{u,n}$ be the subgraph of $G_{u,n}$ whose vertices form the block

$$[\infty] = \{\frac{x}{y} \in \widehat{\mathbb{Q}} : y \equiv 0 (\mathrm{mod}\ n)\}.$$

Then

Theorem 2.6.3. *$\frac{r}{s} \to \frac{x}{y}$ in $F_{u,n}$ if and only if either*

1. *$x \equiv ur$ (mod n) and $ry - sx = n$ or*

2. *$x \equiv -ur$ (mod n) and $ry - sx = -n$.*

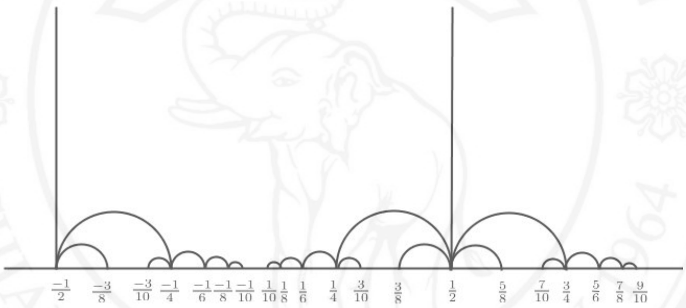

Figure 2.6: $F_{1,2}$