CHAPTER 3

Main Results

In this chapter, we present the characterizations of left regular elements, right regular elements, intra-regular elements, completely regular elements and unit regular elements on $Fix(X,Y)$. Moreover, we count the numbers of left regular, right regular and intra-regular elements and determine the maximal congruence on $Fix(X,Y)$ when X is a finite set.

3.1 The action of $\mathcal{H}(\sqrt{m})$ on $\sqrt{m}\widehat{\mathbb{Q}}$

$\mathcal{H}(\sqrt{m})$ acts on $\sqrt{m}\hat{\mathbb{Q}}$ naturally by $T.x = T(x)\ \forall T \in \mathcal{H}(\sqrt{m}) \forall x \in \sqrt{m}\hat{\mathbb{Q}}$. Then we have the following lemma:

Lemma 3.1.1. *$\mathcal{H}(\sqrt{m})$ acts on $\sqrt{m}\widehat{\mathbb{Q}}$ transitively if and only if m is prime or $m = 1$.*

Proof. Let m be prime or $m = 1$ and $(x/y)\sqrt{m} \in \sqrt{m}\widehat{\mathbb{Q}} \setminus \{\infty\}$ with $(x, y) = 1$. We will show that we can find $T \in \mathcal{H}(\sqrt{m})$ such that $T(\infty) = (x/y)\sqrt{m}$. Since $(x, y) = 1$, there are $a, b \in \mathbb{Z}$ such that $ax - by = 1$. If $m \mid y$, we may take

$$T(z) = \frac{xz + b\sqrt{m}}{(y/m)\sqrt{m}z + a}$$

as the element desired.

If $m \nmid y$, since m is prime or 1, we have $(mx, y) = 1$. Thus, there exist $a, b \in \mathbb{Z}$ such that $mxa - yb = 1$. Now we take

$$T(z) = \frac{x\sqrt{m}z + b}{yz + \sqrt{m}a},$$

and we have $T(\infty) = (x/y)\sqrt{m}$. Since the orbit of ∞ on $\mathcal{H}(\sqrt{m})$ is $\sqrt{m}\widehat{\mathbb{Q}}$, the action is transitive.

Conversely, let m be a composite number. Then there are different primes p, q such that $p \mid m$ and $q \mid m$. We will show that there is no such $T \in \mathcal{H}(\sqrt{m})$ that $T(\infty) = (p/q)\sqrt{m}$, and so the action is not transitive. Suppose that such $T \in \mathcal{H}(\sqrt{m})$ exists. Then either

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d}, a, b, c, d \in \mathbb{Z}, ad - bcm = 1$$

or

$$T(z) = \frac{a\sqrt{m}z + b}{cz + d\sqrt{m}}, a, b, c, d \in \mathbb{Z}, adm - bc = 1.$$

In the former case, $a = (cpm)/q$. Since $q \mid m$, we have $p \mid a$ and so $p \mid (ad - bcm) = 1$ which is impossible. As for the latter case, $c = (aq)/p$. Since $(p, q) = 1$, $p \mid a$ and $q \mid c$. Thus, $q \mid (adm - bc) = 1$ which is also impossible. Hence, such T doesn't exist. The action is not transitive. $\square$

From here on we only consider the case where m is a prime.

Definition 3.1.1. Let (G, X) be a transitive permutation group and R an equivalence relation on X. If, for each $(x, y) \in R$, we have $(g(x), g(y)) \in R\ \forall g \in G$, then R is G-invariant. Equivalence classes of a G-invariant relation are called blocks.

In [2], a G-invariant relation R was defined on X. Let H be a subgroup of G containing G_x, the stabilizer of x in G, for some $x \in X$. Then

$$R = \{(g(x), gh(x)) : g \in G, h \in H\}$$

is a G-invariant relation. In our case when $G = \mathcal{H}\sqrt{m}$ and $X = \sqrt{m}\hat{\mathbb{Q}}$, we have that $((r/s)\sqrt{m}, (x/y)\sqrt{m}) \in R$ if and only if $n \mid (ry - sx)/m$.

Lemma 3.1.2. *In* $(\mathcal{H}(\sqrt{m}), \sqrt{m}\widehat{\mathbb{Q}})$, *if* $(m, n) = 1$, *then* $((r/s)\sqrt{m}, (x/y)\sqrt{m}) \in R$ *if and only if* $n \mid (ry - sx)$. *There are a total of* $|\mathcal{H}(\sqrt{m}) : \mathcal{H}_0^m(n)|$ *blocks induced by* R *where* $\mathcal{H}_0^m(n) = \{T \in \mathcal{H}(\sqrt{m}) : c \equiv 0 (\text{mod } n)\}$.

3.2 Suborbital graph for $\mathcal{H}(\sqrt{m})$ on $\sqrt{m}\widehat{\mathbb{Q}}$

$\mathcal{H}(\sqrt{m})$ acts on $\sqrt{m}\widehat{\mathbb{Q}} \times \sqrt{m}\widehat{\mathbb{Q}}$. by

$$T(\alpha, \beta) = (T(\alpha), T(\beta)), T \in \mathcal{H}(\sqrt{m}), \alpha, \beta \in \sqrt{m}\widehat{\mathbb{Q}}$$

Recall that the orbits of this action are called suborbitals of $\mathcal{H}(\sqrt{m})$. We denote the orbit containing (α, β) by $O(\alpha, \beta)$. A suborbital graph $\mathcal{G}(\alpha, \beta)$ is a graph with the elements of $\sqrt{m}\widehat{\mathbb{Q}}$ as its vertices and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. We denote the directed edge from γ to δ by $\gamma \to \delta$ or $\delta \leftarrow \gamma$. That is, the vertices are the points on $\partial\mathbb{H}^2$ and we represent the edges as hyperbolic geodesics in $\mathbb{H}^2$.

From now on, we will work on the non-trivial suborbital graphs $\mathcal{G}(\alpha, \beta)$ with $\alpha \neq \beta$.

Since $\mathcal{H}(\sqrt{m})$ acts on $\sqrt{m}\widehat{\mathbb{Q}}$ transitively, each suborbital graph contains a pair $(\infty, (u/n)\sqrt{m})$ for some $(u/n)\sqrt{m} \in \sqrt{m}\widehat{\mathbb{Q}} \setminus \{\infty\}$. We can see that

$$O(\infty, (u/n)\sqrt{m}) = O(\infty, (v/n)\sqrt{m}) \text{ if and only if } n \mid (u-v).$$

Therefore, we may assume that each suborbital graph is in the form $O(\infty, (u/n)\sqrt{m})$ with $u \leq n$ where $(u, n) = 1$.

Now we give a necessary and sufficient condition for the connection of two vertices in $\mathcal{G}(\infty, (u/n)\sqrt{m})$.

Theorem 3.2.1. *If $(m, n) = 1$, then there exists an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, (u/n)\sqrt{m})$ if and only if $ry - sx = \pm n$ and either*

(i) $m \mid s$ and $x \equiv \pm ur (mod\ n)$, $y \equiv \pm us (mod\ n)$ or

(ii) $m \mid y$ and $x \equiv \pm mur (mod\ n)$, $y \equiv \pm mus (mod\ n)$.

Proof. Suppose that there exists an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, (u/n)\sqrt{m})$. Then there exists $T \in \mathcal{H}(\sqrt{m})$ such that $T(\infty) = (r/s)\sqrt{m}$ and $T((u/n)\sqrt{m}) = (x/y)\sqrt{m}$. If

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m} + d}, a, b, c, d \in \mathbb{Z}, ad - bcm = 1.$$

Then we have $a/mc = r/s$ and $(au+bn)/(mcu+dn) = x/y$. Since $ad-bcm = 1, (a, mc) = 1$. Thus, $a = ir, mc = is$ where $i = \pm 1$. So we have $m \mid s$. On the other hand, since

$$a(muc + dn) - mc(au + bn) = n$$

and

$$d(au + bn) - b(muc + dn) = u,$$

we have $(au + bn, muc + dn) = (n, u) = 1$. Thus, $jx = au + bn$ and $jy = muc + dn$ where $j = \pm 1$. Hence,

$$\begin{aligned} x &\equiv j(au + bn) \pmod n \\ &\equiv jau \pmod n \\ &\equiv ijur \pmod n \end{aligned}$$

So $x \equiv \pm ur(\text{mod } n)$. Similarly,

$$\begin{aligned} y &\equiv j(muc + dn) \pmod n \\ &\equiv jmuc \pmod n \\ &\equiv ijus \pmod n \end{aligned}$$

That is $y \equiv \pm us (\text{mod } n)$. Also, $ry - sx = (ij)[a(muc + dn) - mc(au + bn)] = (ij)(ad - bcm)n = \pm n$.

In the case that

$$T(z) = \frac{a\sqrt{m}z + b}{c + d\sqrt{m}}, a, b, c, d \in \mathbb{Z}, mad - bc = 1.$$

Then we have $a/c = r/s$ and $(mau + bn)/m(cu + dn) = x/y$. Since $(a, c) = (r, s) = 1$, $a = ir, c = is$ where $i = \pm 1$. On the other hand, since $(m, n) = 1$ and $mad - bc = 1$, we have $(m, mau + bn) = 1$. We also have

$$ma(cu + dn) - c(mau + bn) = n$$

and

$$d(mau + bn) - b(cu + dn) = u,$$

we have $(mau + bn, cu + dn) = (n, u) = 1$. Thus, $(m(cu + dn), mau + bn) = 1$. Then, $jx = mau + bn$ and $jy = m(cu + dn)$ where $j = \pm 1$. Hence,

$$\begin{aligned} x &\equiv j(mau + bn) \pmod n \\ &\equiv jmau \pmod n \\ &\equiv ijmur \pmod n. \end{aligned}$$

So $x \equiv \pm mur (\text{mod } n)$. Similarly,

$$\begin{aligned} y &\equiv jm(cu + dn) \pmod n \\ &\equiv jmcu \pmod n \\ &\equiv ijmus \pmod n. \end{aligned}$$

That is $y \equiv \pm mus (\text{mod } n)$. Also, $ry - sx = (ij)[am(cu + dn) - c(mau + bn)] = (ij)(mad - bc)n = \pm n$.

Now, suppose that $m \mid y, ry - sx = kn$ and $x \equiv kmur (\text{mod } n), y \equiv kmus (\text{mod } n)$ where $k = \pm 1$. Since $m \mid y$ and $(m, n) = 1$, there are integers b, d such that $kx = mur + bn, ky = mus + mdn$. Taking $a = r, c = s$ we have that $mad - bc = (kry - mrus)/n - s(kx - mur)/n = k(ry - sx)/n = k^2 = 1$. We may take

$$T(z) = \frac{a\sqrt{m}z + b}{cz + d\sqrt{m}}$$

so that $T(\infty) = (r/s)\sqrt{m}$ and $T((u/n)\sqrt{m}) = (mau + bn)/(cu + dn)\sqrt{m} = (x/y)\sqrt{m}$. So, $((r/s)\sqrt{m}, (x/y)\sqrt{m}) \in O(\infty, (u/n)\sqrt{m})$. That is, there exists an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, (u/n)\sqrt{m})$.

If $m \mid s, ry - sx = kn$ and $x \equiv kur (\text{mod } n), y \equiv kus (\text{mod } n)$ where $k = \pm 1$. Then there are integers b, d such that $kx = ur + bn, ky = us + dn$. Taking $a = r$ and $c = s/m$, we have $ad - bcm = 1$. So, with

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d}$$

we can reach the same conclusion as the earlier case. □

We can also prove the following theorem in the same way.

Theorem 3.2.2. *Suppose $(m, n) = m$, then there exists an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, (u/n)\sqrt{m})$ if and only if either*

(i) $m \mid s, ry - sx = \pm n$ *and* $x \equiv \pm ur (\text{mod } n)$, $y \equiv \pm us (\text{mod } n)$ *or*

(ii) $ry - sx = \pm n/m$ *and* $x \equiv \pm ur (\text{mod } n)$, $y \equiv \pm us (\text{mod } n)$.

From now on we only consider the case where $(m, n) = 1$.

Consider the suborbital graph $\mathcal{G}(\infty, \sqrt{m})$. With hyperbolic geodesics as its edges, we have

Lemma 3.2.3. *No edges of $\mathcal{G}(\infty, \sqrt{m})$ cross in $\mathbb{H}^2$.*

Proof. Let $(r_1/s_1)\sqrt{m} \to (r_2/s_2)\sqrt{m}$ be an edge in $\mathcal{G}(\infty, \sqrt{m})$.Let $T(z) = z + \sqrt{m}$, then $T \in \mathcal{H}(\sqrt{m})$ and $T(\infty) = \infty, T(0) = \sqrt{m}$. So $O(\infty, 0) = O(\infty, \sqrt{m})$ $= O((r_1/s_1)\sqrt{m}, (r_2/s_2)\sqrt{m})$. Therefore, there is an element of $\mathcal{H}(\sqrt{m})$ sending the edge $(r_1/s_1)\sqrt{m} \to (r_2/s_2)\sqrt{m}$ to $0 \to \infty$. Since the element preserves the geodesics, we may assume that an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ cross with $0 \to \infty$ instead of assuming that two random edges cross in $\mathbb{H}^2$. But it is impossible since $ry - sx = \pm 1$ which contradicts to the fact that either $r/s < 0$ or $x/y < 0$. □

For each integer n we have an $\mathcal{H}(\sqrt{m})$-invariant relation R defined earlier. Recall that$((r/s)\sqrt{m}, (x/y)\sqrt{m}) \in R$ if and only if $ry - sx \equiv 0 (\text{mod } n)$. If there is an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, (u/n)\sqrt{m})$, then $ry - sx = \pm n$. That is $((r/s)\sqrt{m}, (x/y)\sqrt{m}) \in R$. Thus, each connected component of $\mathcal{G}(\infty, (u/n)\sqrt{m})$ is in the same block for R.
Let $\mathcal{F}(\infty, (u/n)\sqrt{m})$ be a subgraph of $\mathcal{G}(\infty, (u/n)\sqrt{m})$ with the set of vertices $[\infty] = \{(x/y)\sqrt{m} : y \equiv 0 (\text{mod } n)\}$. Each block is permuted transitively by $\mathcal{H}(\sqrt{m})$ on $\sqrt{m}\widehat{\mathbb{Q}}$ and the subgraph corresponding to each block is all isomorphic. We can apply the same techniques used on $\mathcal{G}(\infty, (u/n)\sqrt{m})$ to $\mathcal{F}(\infty, (u/n)\sqrt{m})$ and give this theorem:

Theorem 3.2.4. *There is an edge* $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ *in* $\mathcal{F}(\infty, (u/n)\sqrt{m})$ *if and only if* $ry - sx = \pm n$ *and either*

(i) $m \mid s$ *and* $x \equiv \pm ur (\text{mod } n)$ *or*

(ii) $m \mid y$ *and* $x \equiv \pm mur (\text{mod } n)$.

Lemma 3.2.5. $T : \mathcal{F}(\infty, (u/n)\sqrt{m}) \to \mathcal{F}(\infty, ((n-u)/n)\sqrt{m})$ *given by* $T(v) = \sqrt{m} - v$ *is an isomorphism.*

Proof. We see that T is bijective. Suppose that there is an edge $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{F}(\infty, (u/n)\sqrt{m})$. We will show that the edge $T((r/s)\sqrt{m}) \to T((x/y)\sqrt{m}) = ((s-r)/s)\sqrt{m} \to ((y-x)/y)\sqrt{m}$ is in $\mathcal{F}(\infty, ((n-u)/n\sqrt{m})$. Since $m \mid s, ry - sx = \pm n$, and $x \equiv \pm ur (\text{mod } n)$ or $m \mid y, ry - sx = \pm n$, and $x \equiv \pm mur (\text{mod } n)$, we have $y(s-r) - s(y-x) = -ry + sx = \pm n$. Since $(r/s)\sqrt{m}, (x/y)\sqrt{m} \in \mathcal{F}(\infty, (u/n)\sqrt{m})$, then if $m \mid s$, $y - x \equiv \pm(n-u)(s-r) (\text{mod } n)$, and if $m \mid y$, then $y - x \equiv \pm m(n-u)(s-r) (\text{mod } n)$. By Theorem 3.2.4, there is an edge $((s-r)/s)\sqrt{m} \to ((y-x)/y)\sqrt{m}$ in $\mathcal{F}(\infty, ((n-u)/n)\sqrt{m})$. □

Again, we represent the edges of $\mathcal{F}(\infty, (u/n)\sqrt{m})$ as hyperbolic geodesics in $\mathbb{H}^2$. We have

Lemma 3.2.6. *No edges of* $\mathcal{F}(\infty, (u/n)\sqrt{m})$ *cross in* $\mathbb{H}^2$.

Proof. Suppose that the edges $(r/sn)\sqrt{m} \to (x/yn)\sqrt{m}$ and $(r'/s'n)\sqrt{m} \to (x'/y'n)\sqrt{m}$ cross in $\mathbb{H}^2$. Then $ry - sx = \pm 1$ and $m \mid yn$ or $m \mid y'$. Also, $r'y' - s'x' = \pm 1$, and $m \mid y'n$ or $m \mid s'n$. Since $(m, n) = 1$ and m is a prime, $m \mid s$ or $m \mid y$ and $m \mid s'$ or $m \mid y'$. Therefore, the edges $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ and $(r'/s')\sqrt{m} \to (x'/y')\sqrt{m}$ in $\mathcal{G}(\infty, \sqrt{m})$ cross in $\mathbb{H}^2$. A contradiction. □

Lemma 3.2.7. *There is no element of* $\sqrt{m}\mathbb{Z} = \{k\sqrt{m} : k \in \mathbb{Z}\}$ *between two adjacent vertices in* $\mathcal{F}(\infty, (u/n)\sqrt{m})$ *except when one of the two vertices is* ∞.

Proof. Suppose that there exists an edge $(r/sn)\sqrt{m} \to (x/yn)\sqrt{m}$ in $\mathcal{F}(\infty, (u/n)\sqrt{m})$ and assume that $(r/sn)\sqrt{m} < k\sqrt{m} < (x/yn)\sqrt{m}$. Then $(r/s)\sqrt{m} < kn\sqrt{m} < (x/y)\sqrt{m}$. Since there are edges $kn\sqrt{m} \to \infty$ and $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ in $\mathcal{G}(\infty, \sqrt{m})$, they cross in $\mathcal{G}(\infty, \sqrt{m})$ which is impossible by Lemma 3.2.3. □

3.3 Circuits in $\mathcal{G}(\infty,(u/n)\sqrt{m})$

In $\mathcal{G}(\infty,(u/n)\sqrt{m})$, every edge is a directed edge. For $v, w \in \sqrt{m}\widehat{\mathbb{Q}}$, we say that $v \leftrightarrows w$ if there is the edge $v \to w$ or $w \to v$ in $\mathcal{G}(\infty,(u/n)\sqrt{m})$. We call a sequence of n different vertices $v_1, v_2, ..., v_n$ with $v_1 \to v_2 \leftrightarrows \cdots \leftrightarrows v_n \leftrightarrows v_1$ where $n \geq 3$ a circuit of length n. A forest is a graph containing no circuit.

Lemma 3.3.1. *If $n > 1$, then $\mathcal{G}(\infty,(u/n)\sqrt{m})$ contains a circuit if and only if $n \mid mu^2 \pm mu + 1$.*

Proof. It's sufficient to assume that $\mathcal{F}(\infty,(u/n)\sqrt{m})$ contains a circuit $v_1 \to v_2 \leftrightarrows \cdots \leftrightarrows v_n \leftrightarrows v_1$ where every v_j is different from one another. Since $(v_1, v_2) \in O(\infty,(u/n)\sqrt{m})$, there exist some $T \in \mathcal{H}(\sqrt{m})$ such that $T(\infty,(u/n)\sqrt{m}) = (v_1, v_2)$. We have $T \in H_0^m(n), T^{-1} \in H_0^m(n)$. Also, if $v \in [\infty]$, then $T^{-1}(v) \in [\infty]$. So, we may assume that $\infty \to \frac{u}{n}\sqrt{m} \leftrightarrows w_3 \leftrightarrows \cdots \leftrightarrows w_{k-1} \leftrightarrows w_k \leftrightarrows \infty = T^{-1}(v_1) \to T^{-1}(v_2) \leftrightarrows \cdots \leftrightarrows T^{-1}(v_n) \leftrightarrows T^{-1}(v_1)$ is a circuit in $\mathcal{F}(\infty,(u/n)\sqrt{m})$.

Since no edges of $\mathcal{F}(\infty,(u/n)\sqrt{m})$ cross in $\mathbb{H}^2$, we have

$$\frac{u}{n}\sqrt{m} < w_3 < \cdots < w_{k-1} < w_k$$

or

$$\frac{u}{n}\sqrt{m} > w_3 > \cdots > w_{k-1} > w_k.$$

If $(u/n)\sqrt{m} < w_3 < \cdots < w_{k-1} < w_k$, then we will show that $w_k \to \infty$. Suppose that $\infty \to w_k = (r/sn)\sqrt{m}$. Then, we have $sn1 - 0r = n$, so $s = 1$. Since $m \mid 0$, we have from Theorem 3.2.4 that $r \equiv u (\text{mod } n)$. Since $n > 1, w_k = (r/n)\sqrt{m}$ and $r \neq u$, there is an element of $\sqrt{m}\mathbb{Z}$ between $(u/n)\sqrt{m}$ and $(r/n)\sqrt{m}$. Because all the vertices in the circuit lie in $\mathcal{F}(\infty,(u/n)\sqrt{m})$, they don't belong to $\sqrt{m}\mathbb{Z}$. That means there is an element of $\sqrt{m}\mathbb{Z}$ between two adjacent vertices in $\mathcal{F}(\infty,(u/n)\sqrt{m})$ which is impossible by Lemma 3.2.6. So $w_k \to \infty$. In a similar way, we can prove that $w_k = (c/n)\sqrt{m}$ and $1 + muc \equiv 0 (\text{mod } n)$.

Let $c = u + t, t \geqslant 1$. Then $n \mid (mu(u + t) + 1)$. We will show that $t = 1$. Suppose not, then $c/n < 1$ since otherwise there would be an integer between u/n and c/n. Let

$$\varphi(z) = \frac{-u\sqrt{m}z + (mu(u+t)+1)/n}{-nz + (u+t)\sqrt{m}}.$$

Then, $\varphi \in H_0^m(n)$. Moreover, $\varphi(\infty) = (u/n)\sqrt{m}$, and $\varphi((u+t)/n\sqrt{m}) = \infty$. We can show the vertices adjacent to $(u/n)\sqrt{m}$ are not greater than $\varphi((u/n)\sqrt{m}) = [(u+1/tm)/n]\sqrt{m}$. From then, we can show using mathematical induction that the vertices adjacent to the vertex $\varphi^i((u/n)\sqrt{m})$ are less than or equal to $\varphi^{i+1}((u/n)\sqrt{m})$ for all positive integer i. We can see that $w_j \leqslant \varphi^{j-1}((u/n)\sqrt{m})$ for all $3 \leqslant j \leqslant k$. Again, we can show by using mathematical induction that $\varphi^i((u/n)\sqrt{m}) < \frac{u+\frac{1}{t-1}}{n}\sqrt{m} < \frac{u+1}{n}\sqrt{m}$ for $i \geq 1$. Since $w_k = (c/n)\sqrt{m} = [(u+t)/n]\sqrt{m} \geqslant [(u+2)/n]\sqrt{m}$, we have $[(u+2)/n]\sqrt{m} \leqslant w_k \leqslant \varphi^{k-1}((u/n)\sqrt{m}) < [(u+1)/n]\sqrt{m}$, a contradiction. Thus $t = 1$. That is, $mu^2 + mu + 1 = mu(u+1) + 1 = muc + 1$. Thus, $n \mid mu^2 + mu + 1$.

If $(u/n)\sqrt{m} > w_3 > \cdots > w_{k-1} > w_k$, then

$$\infty \rightarrow \frac{n-u}{n}\sqrt{m} \leftrightarrows \sqrt{m} - w_3 \leftrightarrows \cdots \leftrightarrows \sqrt{m} - w_k \rightarrow \infty$$

with $[(n-u)/n]\sqrt{m} < \sqrt{m} - w_3 < \cdots < \sqrt{m} - w_k$ is a circuit in $\mathcal{F}(\infty, [(n-u)/n]\sqrt{m})$. With the same method we reach the conclution that

$$mu^2 - mu + 1 \equiv m(n-u)^2 + m(n-u) + 1 \equiv 0 (\text{mod } n).$$

Hence, if the exists a circuit in $\mathcal{G}(\infty, (u/n)\sqrt{m})$, then $mu^2 \pm mu + 1 \equiv 0(\text{mod } n)$.

Now let $n \mid mu^2 \pm mu + 1$. Taking

$$T(z) = \frac{-u\sqrt{m}z + (mu^2 \pm mu + 1)/n}{-nz + (u \pm 1)\sqrt{m}},$$

then $T \in H_0^m(n)$ and $T(\infty) = (u/n)\sqrt{m}$. Since T is elliptic, T is of finite order. We can construct the circuit

$$\infty \rightarrow T(\infty) \rightarrow T^2(\infty) \rightarrow \cdots \rightarrow T^{k-1}(\infty) \rightarrow \infty$$

in $\mathcal{G}(\infty, (u/n)\sqrt{m})$ where k is the order of T. □

Theorem 3.3.2. *If $(m, n) = 1$ and $n > 1$, Then $\mathcal{G}(\infty, (u/n)\sqrt{m})$ is a forest if and only if $n \nmid (mu^2 \pm mu + 1)$.*