CHAPTER 2
Preliminaries

In this chapter, the underlying concepts of stability, important definitions, lemmas, propositions and results, are provided which will be used in later chapters.

2.1 Types of Matrix

Let $M \in \mathcal{R}^{n \times n}$, then we have the following definitions.
Definition 2.1.1 Matrix M is semi-positive definite if $x^T M x \geq 0$ for all $x \in \mathcal{R}^n$.
Definition 2.1.2 Matrix M is positive definite $(M > 0)$ if for all $x \in \mathcal{R}^n$, $x \neq 0$.
Definition 2.1.3 Matrix M is semi-negative definite $(M \leq 0)$ if for all $x \in \mathcal{R}^n$.
Definition 2.1.4 Matrix M is negative definite $(M < 0)$ if for all $x \in \mathcal{R}^n$, $x \neq 0$.

2.2 Notations

We give some important notations will be used throughout this thesis:
$\mathcal{R}^+$ denotes the set of all non-negative real numbers;
$\mathcal{R}^n$ denotes the n-dimensional Euclidean space;
$M > 0$ $(M \geq 0)$ denotes the square symmetric, M is positive (semi-) definite matrix;
$M < 0$ $(M \leq 0)$ denotes the square symmetric, M is negative (semi-) definite matrix;
$M > N$ $(M \geq N)$ denotes the $M - N$ matrix is square symmetric positive (semi-) definite matrix;
$M < N$ $(M \leq N)$ denotes the $M - N$ matrix is square symmetric negative (semi-) definite matrix;
$\mathcal{R}^{n \times m}$ denotes the space of all $(n \times m)$ real matrices;
A^T denotes the transpose of the vector/matrix A;
A^{-1} denotes the inverse of a non-singular matrix A;
I denotes the identity matrix;
$\lambda(A)$ denotes the set of all eigenvalues of A;

$\lambda_{\max}(A) = \max\ \{\mathrm{Re}\ \lambda : \lambda \in \lambda(A)\}$,
$\lambda_{\min}(A) = \min\ \{\mathrm{Re}\ \lambda : \lambda \in \lambda(A)\}$;
$\langle x, y\rangle$ or $x^\top y-$ the scalar product of two vector x, y ;
$\|x\|$ denotes the Euclidean vector norm of x;
$L_2([0,t], \mathcal{R}^m)$ denotes the set of all the $\mathcal{R}^m$-valued square integrable functions on $[0,t]$;
$C([0,t], \mathcal{R}^n)$ denotes the set of all $\mathcal{R}^n$-valued continuous functions on $[0,t]$;
$C_h = C([-h,0], \mathcal{R}^n), h > 0$ denotes the Banach space of continuous functions;
mapping the interval $[-h,0]$ into $\mathcal{R}^n$, with the topology of uniform convergence;
$x_t = \{x(t+s) : s \in [-h,0]\}, \|x_t\| = \sup\limits_{s\in[-h,0]} \|x(t+s)\|$;
$\|x_t\| \in C_h$ defined $x_t = x(t+\theta), -h \leq \theta \leq 0$ and $\|x_t\|_{C_h} = \sup_{-h\leq\theta\leq 0} \|x(t+\theta)\|$.

2.3 Stability of Ordinary Differential Equation

Consider a dynamical system described by

$$\dot{x}(t) = f(t, x(t)) \tag{2.1}$$

where $x \in \mathcal{R}^n$ and f is a vector having components $f_i(t, x_1, ..., x_n)$, $i = 1, 2, ..., n$. We shall assume that the f_i are continuous and satisfy standard conditions, such as having continuous first partial derivatives so that the solution of (2.1) exists and is unique for the given initial conditions. If f_i do not depend explicitly on t, (2.1) is called autonomous (otherwise, nonautonomous). If $f(t, c) = 0$ for all t, where c is some constant vector, then it follows at once from (2.1) that if $x(t_0) = c$ then $x(t) = c$ for all $t \geq t_0$. Thus solutions starting at c remain there, and c is said to be an *equilibrium* or *critical point*. Clearly, by introducing new variables $\acute{x}_i = x_i - c_i$ we can arrange for the equilibrium point to be transferred to the origin; we shall assume that this has been done for any equilibrium point under consideration (there may well be several for a given system (2.1) so that we then have $f(t, 0) = 0$, $t \geq t_0$.

2.3.1 Autonomous systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{2.2}$$

where $f : D \to \mathcal{R}^n$ is locally Lipschitz map from a domain $D \subset \mathcal{R}^n$ into $\mathcal{R}^n$. We shall always assume that $f(x)$ satisfies $f(0) = 0$, and study stability of the origin $x = 0$.

Definition 2.3.1 *[20] The equilibrium point $x = 0$ of (2.2) is*

(i) **stable** *if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x\| < \epsilon, \quad \forall t \geq 0,$$

(ii) **unstable** *if it is not stable, that is, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exist an $x(0)$ with $\|x(0)\| < \delta$ so that $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > 0$. If this holds for every $x(0)$ in $\|x(0)\| < \delta$ the equilibrium is completely unstable.*

(iii) **asymptotically stable** *if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

Definition 2.3.2 *[20] A function $V(\cdot) : \mathcal{R}^n \to \mathcal{R}$ is said to be Lyapunov-Krasovskii functional if it satisfies the following:*

1. $V(x)$ and all its partial derivatives $\dfrac{\partial V}{\partial x_i}$ are continuous.

2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighborhood $\| x \| \leq k$ of the origin.

3. The derivative of V with respect to (2.2), namely

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + ... + \frac{\partial V}{\partial x_n}f_n \end{aligned} \tag{2.3}$$

is negative semidefinite i.e. $\dot{V}(0) = 0$, and for all x satisfy $\|x\| \leq k$, $\dot{V}(x) \leq 0$.

Theorem 2.3.3 *[20] Let $x = 0$ be an equilibrium point for (2.2) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V(x) : D \to \mathcal{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \quad and \quad V(x) > 0 \quad in \quad D - \{0\},$$
$$\dot{V}(x) \leq 0 \quad in \quad D.$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \quad in \quad D - \{0\},$$

then $x = 0$ is asymptotically stable.

Theorem 2.3.4 *[20] Let $x = 0$ be an equilibrium point for (2.2). Let $V(x) : \mathcal{R}^n \to \mathcal{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \quad \textit{and} \quad V(x) > 0, \quad \forall x \neq 0,$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty,$$
$$\dot{V}(x) < 0, \quad \forall x \neq 0,$$

then $x = 0$ is globally asymptotically stable.

Theorem 2.3.5 *[20] Let $x = 0$ be an equilibrium point for (2.2) and $f : D \to \mathcal{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let*

$$A = \frac{\partial f}{\partial x}(x) \mid_{x=0} .$$

Then,

1. The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A.

2. The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A.

2.3.2 Non-autonomous systems

Consider the non-autonomous system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x(t) \in \mathcal{R}^n, \quad t \in \mathcal{R}^+, \tag{2.4}$$

where $f : \mathcal{R}^+ \times D \to \mathcal{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $\mathcal{R}^+ \times \mathcal{R}^n$ and $D \subset \mathcal{R}^n$ is domain that contains the origin $x = 0$. The origin is an equilibrium point for (2.4) if

$$f(t, 0) = 0, \quad \forall t \geq t_0.$$

Definition 2.3.6 *[20] The equilibrium point $x = 0$ of the system (2.4) is*

(i) **stable** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0, \tag{2.5}$$

(ii) **uniformly stable** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (2.5) is satisfied,*

(iii) **unstable** *if not stable,*

(iv) **asymptotically stable** *if it is stable and there is $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$,*

(v) **uniformly asymptotically stable** *if it is uniformly stable and there is $c > 0$, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0, for each $\epsilon > 0$, there is $T = T(\epsilon) > 0$ such that*

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon), \quad \forall \|x(t_0)| < c,$$

(vi) **globally uniformly asymptotically stable** *if it is uniformly stable and, for each pair of positive numbers ϵ and c, there is $T = T(\epsilon, c) > 0$ such that*

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon, c), \quad \forall \|x(t_0)| < c.$$

Definition 2.3.7 *[20] The equilibrium point $x = 0$ of the system (2.4) is exponentially stable if there exist three positive real constants ϵ, K and λ such that*

$$\|x(t)\| \leq K\|x_0\|e^{-\lambda(t-t_0)}, \quad \forall \|x_0\| < \epsilon, \quad t \geq t_0;$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

Definition 2.3.8 *[20] The function $W(x)$ is said to be positive (negative) definite if $W(x) > 0(-W(x) > 0)$ and $W(x) = 0$ if and only if $x = 0$. The function $W(x)$ is said to be positive (negative) semi-definite if $W(x) \geq 0(-W(x) \geq 0)$.*

Definition 2.3.9 *[20] The function $W(x)$ is said to be radially unbounded, positive definite if $W(x)$ is positive definite and $W(x) \to \infty$ as $\|x\| \to \infty$.*

Let B_ϵ be a ball of size ϵ around the origin,

$$B_\epsilon = \{x \in \mathcal{R}^n : \|x\| < \epsilon\}.$$

Definition 2.3.10 *[38] A function $V(\cdot) : \mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}$ is said to be Lyapuno-Krasovskii functional if it satisfies the following:*

(i) *$V(t, x)$ and all its partial derivatives $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial x_i}$ are continuous for all $i = 1, 2, 3, ..., n$.*

(ii) *$V(t,x)$ is positive definite function, i.e., $V(0) = 0$ and $V(t,x) > 0$, $x \neq 0$, $\forall x \in B_\epsilon$.*

(iii) *The derivative of $V(t,x)$ with respect to system (2.4), namely*

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + ... + \frac{\partial V}{\partial x_n}f_n. \qquad (2.6)\end{aligned}$$

$\dot{V}(t,x)$ is negative semi-definite i.e., $\dot{V}(t,0) = 0$ and $\forall x \in B_\epsilon$, $\dot{V}(t,x) \leq 0$.

Theorem 2.3.11 *[20] Let $x = 0$ be an equilibrium point for (2.4) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{R}^+ \times D \to \mathcal{R}$ be a continuously differentiable function, such that*

$$W_1(x) \leq V(t,x) \leq W_2(x), \qquad (2.7)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \leq -W_3(x) \qquad (2.8)$$

$\forall t \geq t_0 \geq 0$, $\forall x \in D$ where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable.

Corollary 2.3.12 *[20] Suppose that all the assumptions of Theorem 2.3.11 are satisfied globally (for all $x \in \mathcal{R}^n$) and $W_1(x)$ is radially unbounded. Then, $x = 0$ is globally uniformly asymptotically stable.*

Corollary 2.3.13 *[20] Suppose all the assumptions of Theorem 2.3.11 are satisfied with*

$$W_1(x) \geq k_1\|x\|^c, \quad W_2(x) \leq k_2\|x\|^c, \quad W_3(x) \geq k_3\|x\|^c$$

for some positive constants k_1, k_2, k_3 and c. Then, $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then, $x = 0$ is globally exponentially stable.

Theorem 2.3.14 *[20] Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x}(t) = f(t,x)$$

where $f : [0,\infty) \times D \to \mathcal{R}^n$ is continuously differentiable, $D = \{x \in \mathcal{R}^n | \|x\|_2 < r\}$, and Jacobian matrix $[\frac{\partial f}{\partial x}]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t,x)|_{x=0}.$$

Then, the origin is an exponentially stable equilibrium point for nonlinear system if and only if it is an exponentially stable equilibrium point for linear system

$$\dot{x}(t) = A(t).$$

2.4 Time Delays System

We consider the system with time-delay of the form [11]

$$\begin{aligned} \dot{x}(t) &= f(t, x(t-h)), \qquad \forall t \geq 0, \\ x(t_0+\theta) &= \phi(\theta), \qquad\qquad \forall \theta \in [-h, 0], \end{aligned} \tag{2.9}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, $h \in \mathcal{R}^+$ is the delay and $f : \mathcal{R}^+ \times C[-h, 0] \to \mathcal{R}^n$. $\phi(t)$ is a continuous vector-valued initial condition. We assume $f(t, 0) = 0$ so that system (2.9) admits the trivial solution. The following Theorem guarantee for an existence and uniqueness solution of (2.9) through (t_0, ϕ).

Theorem 2.4.1 *[11] (uniqueness) Suppose that $\Omega \subseteq \mathcal{R} \times C$ is an open set, $f : \Omega \to \mathcal{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω. That is, for each given compact set $\Omega_0 \subset \Omega$, there exists a constant L, such that*

$$\| f(t, \phi_1) - f(t, \phi_2) \| \leq L \| \phi_1 - \phi_2 \| \tag{2.10}$$

for any $(t, \phi_1) \in \Omega_0$ and $(t, \phi_2) \in \Omega_0$. If $(t_0, \phi) \in \Omega$, then there exists an existence and uniqueness solution of (2.9) through (t_0, ϕ).

Definition 2.4.2 *For the system described by (2.9), the trivial solution $x(t) = 0$ is said to be*

(i) **stable** *if for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$\|x_{t_0}\|_c < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0,$$

(ii) **uniformly stable** *if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of t_0,*

(iii) **asymptotically stable** *if it is stable, and for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$ there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that*

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \lim_{t \to \infty} = 0,$$

(iv) **uniformly asymptotically stable** *if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that*

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \|x(t)\| < \eta, \quad \forall t \geq t_0 + T,$$

(v) **exponentially stable** *if there exists constants* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\| x(t) \| \leq \beta \sup_{-h \leq \theta \leq 0} \| x(\theta) \| e^{-\alpha t}, \tag{2.11}$$

then the trivial solution of (2.9) is globally exponentially stable, and α *is called the exponential convergence rate.*

Definition 2.4.3 *[20] A functional* $V : \mathcal{R}^+ \times C \to \mathcal{R}^+$ *is called a Lyapunov-Krasovskii functional for the system* (2.9) *if it has the following properties. There exist* $\lambda_1, \lambda_2, \lambda_3 > 0$ *such that*

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2$.

Lemma 2.4.4 *[11] Consider the non-autonomous time-delay system* (2.9). *If there exist a Lyapunov-Krasovskii functional* $V(t, x_t)$ *and* $\lambda_1, \lambda_2 > 0$ *such that for every solution* $x(t)$ *of the system, the following conditions hold,*

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq 0$,

then the solution of the system is bounded, i.e., there exists $N > 0$ *such that* $\|x(t, \phi)\| \leq N \|\phi\|, \forall t \geq 0$.

Theorem 2.4.5 *[11] Suppose that* $u, v, w : \mathcal{R}^+ \to \mathcal{R}^+$ *are continuous nondecreasing functions, where additionally* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$, *and* $u(0) = v(0) = 0$. *If there exist a continuous differentiable functional* $V : \mathcal{R}^+ \times C \to \mathcal{R}$ *such that*

$$u(\|\phi(0)\|) \leq V(t, x(t)) \leq v(\|\phi\|),$$

the equilibrium point $x^* = 0$ *of system* (2.9) *is*

(i) *uniformly stable if*

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

(ii) *asymptotically stable if*

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

where $w(0) = 0$,

(iii) uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

where $w(s) > 0$ *for* $s > 0$*,*

(iv) globally uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

and $u(s)$ *is radially unbounded.*

Lemma 2.4.6 *[11] Consider the autonomous time-delay system* (2.9)*. If there exist a Lyapunov-Krasovskii functional* $V(x_t)$ *and* $\lambda_1, \lambda_2, \lambda_3 > 0$ *such that for every solution* $x(t)$ *of the system, the following conditions hold,*

(i) $\lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(x_t) \leq -\lambda_3 \|x(t)\|^2$,

then the solution of the system (2.9) *is exponentially stable.*

Definition 2.4.7 *suppose* $f : \mathcal{R}^+ \times C_h \to \mathcal{R}^n, D : \mathcal{R}^+ \times C_h \to \mathcal{R}^n$ *are given continuous functions. The relation*

$$\frac{d}{dt} D(t, x_t) = f(t, x_t),$$

is called the neutral differential equation. The function D *will be called the operator for the neutral differential equation.*

Definition 2.4.8 *The operator* D *is said to be stable if solution* $\bar{x} = 0$ *of the homogeneous difference equation* $D(t, x_t) = 0, \quad t \geq 0$ *is stable where*
$D : \mathcal{R}^n \times C_h \to \mathcal{R}^n$.

2.5 Preliminary results

Lemma 2.5.1 *[11] (Schur Complement Lemma) Given constant symmetric matrices* $Q,\ S$
and $R \in \mathcal{R}^{n \times n}$ *where* $R(x) < 0,\ Q(x) = Q^T(x)$ *and* $R(x) = R^T(x)$ *we have*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \Leftrightarrow \ Q(x) - S(x)R^{-1}(x)S^T(x) < 0.$$

Lemma 2.5.2 *[11] Suppose $\lambda_{min}(Q)$ is minimum eigenvalue of matrix Q and $\lambda_{max}(Q)$ is maximum eigenvalue of matrix Q. The following inequalities hold:*

$$\lambda_{min}(Q)x^T x \leq x^T Q x \leq \lambda_{max}(Q)x^T x,$$

for symmetric matrix $Q \in \mathcal{R}^{n\times n}$ for all $x \in \mathcal{R}^n$.

Lemma 2.5.3 *[31] Let U, V, W and M be real matrices of appropriate dimensions with M satisfying $M = M^T$, then $M + UVW + W^T V^T U^T < 0$ for all $V^T V \leq I$ if and only if there exists a scalar $\epsilon > 0$ such that $M + \epsilon^{-1} U U^T + \epsilon W^T W < 0$.*

Lemma 2.5.4 *[11] There exists a symmetric matrix X such that*

$$\begin{pmatrix} P_1 - LXL^T & Q_1 \\ Q_1^T & R_1 \end{pmatrix} < 0 \qquad \textit{and}$$

$$\begin{pmatrix} P_2 + X & Q_2 \\ Q_2^T & R_2 \end{pmatrix} < 0$$

if and only if

$$\begin{pmatrix} P_1 + LP_2L^T & Q_1 & LQ_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T L^T & 0 & R_2 \end{pmatrix} < 0.$$

Lemma 2.5.5 *(Cauchy inequality) For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathcal{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 2.5.6 *[11] (Jensen's inequality) For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to \mathcal{R}^n$ such that the integrations concerned are well defined, the following inequality holds*

$$\Big(\int_0^\gamma \omega(s)\, ds\Big)^T M \Big(\int_0^\gamma \omega(s)\, ds\Big) \leq \gamma \Big(\int_0^\gamma \omega^T(s) M \omega(s)\, ds\Big).$$

Lemma 2.5.7 *[58] (Wirtinger inequality) For a given matrix $R > 0$, the following inequality holds for all continuously differentiable function ω in $[a, b] \to \mathcal{R}^n$:*

$$\begin{aligned} \int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du \;\; &\geq \;\; \frac{1}{b-a}(\omega(b) - \omega(a))^T R \\ &\quad \times (\omega(b) - \omega(a)) + \frac{3}{b-a} \tilde{\Omega}^T R \tilde{\Omega} \end{aligned} \tag{2.12}$$

where $\tilde{\Omega} = \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(u) du$.

Lemma 2.5.8 *[68] Let $h(t)$ be a continuous function satisfying $0 \leq h_1 \leq h(t) \leq h_2$. For any $n \times n$ real matrix $R_1 > 0$ and a vector $\dot{x} : [-h_2, 0] \to \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n \times 2n$ real matrices S_1 satisfying* $\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0$

$$
\begin{aligned}
&-(h_2 - h_1)\int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_1\dot{x}(s)ds \\
\leq &2\varphi_{11}^T S\varphi_{21} - \varphi_{11}^T\tilde{R}_1\varphi_{11} - \varphi_{21}^T\tilde{R}_1\varphi_{21},
\end{aligned}
\tag{2.13}
$$

where $\tilde{R}_1 \triangleq \mathrm{diag}\{R_1, 3R_1\}$ *and*

$$
\begin{aligned}
\varphi_{11} &\triangleq \begin{bmatrix} x(t-h(t)) - x(t-h_2) \\ x(t-h(t)) + x(t-h_2) - 2\omega_1(t) \end{bmatrix}, \\
\varphi_{21} &\triangleq \begin{bmatrix} x(t-h_1) - x(t-h(t)) \\ x(t-h_1) + x(t-h(t)) - 2\omega_2(t) \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1 &\triangleq \frac{1}{h_2 - h(t)}\int_{t-h_2}^{t-h(t)} x(s)ds, \\
\omega_2 &\triangleq \frac{1}{h(t) - h_1}\int_{t-h(t)}^{t-h_1} x(s)ds.
\end{aligned}
\tag{2.14}
$$

Lemma 2.5.9 *[11] There exists a symmetric matrix X such that*

$$
\begin{bmatrix} P_1 - LXL^T & Q_1 \\ Q_1^T & R_1 \end{bmatrix} < 0 \ \ \textit{and} \ \ \begin{bmatrix} P_2 + X & Q_2 \\ Q_2^T & R_2 \end{bmatrix} < 0
$$

if and only if

$$
\begin{bmatrix} P_1 + LP_2L^T & Q_1 & LQ_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T L^T & 0 & R_2 \end{bmatrix} < 0.
$$

Lemma 2.5.10 *[67] For any constant symmetric matrix $M \in \mathcal{R}^{n\times n}$, $M = M^T > 0$, $0 \leq h_m \leq h(t) \leq h_M$, $t \geq 0$, and any differentiable vector function $x(t) \in \mathcal{R}^n$, we have*

$$
\begin{aligned}
(a) \quad \Big[\int_{t-h_m}^{t} \dot{x}(s)ds\Big]^T M\Big[\int_{t-h_m}^{t} \dot{x}(s)ds\Big] &\leq h_m\int_{t-h_m}^{t} \dot{x}^T(s)M\dot{x}(s)ds, \\
(b) \Big[\int_{t-h(t)}^{t-h_m} \dot{x}(s)ds\Big]^T M\Big[\int_{t-h(t)}^{t-h_m} \dot{x}(s)ds\Big] &\leq (h(t) - h_m)\int_{t-h(t)}^{t-h_m} \dot{x}^T(s)M\dot{x}(s)ds \\
&\leq (h_M - h_m)\int_{t-h(t)}^{t-h_m} \dot{x}^T(s)M\dot{x}(s)ds.
\end{aligned}
$$

Lemma 2.5.11 *[67] Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ with appropriate dimensions. Then*

$$Q + HFE + E^T F^T H^T < 0$$

for all F satisfying $F^T F \leq R$, if and only if there exists an $\epsilon > 0$ such that

$$Q + \epsilon H H^T + \epsilon^{-1} E^T R E < 0.$$

Lemma 2.5.12 *[72] The following inequalities are true:*

$$\begin{aligned} 0 &\leq \int_0^{x_i(t)} (f_i(s) - r_i^-(s))ds \leq (f_i(x_i(t)) - r_i^- x_i(t))x_i(t) \\ 0 &\leq \int_0^{x_i(t)} (r_i^+(s) - f_i(s))ds \leq (r_i^+ x_i(t) - f_i(x_i(t)))x_i(t). \end{aligned}$$

Lemma 2.5.13 *[68] Let ξ_0, ξ_1 and ξ_2 be $m \times m$ real symmetric matrices and a continuous function h satisfy $h_1 \leq h \leq h_2$, where h_1 and h_2 are constants satisfying $0 \leq h_1 \leq h_2$. If $\xi_0 \geq 0$, then*

$$\begin{aligned} & h^2\xi_0 + h\xi_1 + \xi_2 < 0(\leq 0),\ \forall h \in [h_1, h_2], \\ \Longleftrightarrow\quad & h_i^2\xi_0 + h_i\xi_1 + \xi_2 < 0(\leq 0),\ (i = 1, 2), \end{aligned} \tag{2.15}$$

or

$$\begin{aligned} & h^2\xi_0 + h\xi_1 + \xi_2 > 0(\geq 0),\ \forall h \in [h_1, h_2], \\ \Longleftrightarrow\quad & h_i^2\xi_0 + h_i\xi_1 + \xi_2 > 0(\geq 0),\ (i = 1, 2). \end{aligned} \tag{2.16}$$

Lemma 2.5.14 *[11] A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are non-negative (positive).*

Lemma 2.5.15 *[11] A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are non-positive (negative).*