# CHAPTER 4

# Exponential stabilization of neutral-type neural networks with interval and distributed time-varying delays

In this chapter, we consider the problem of exponential stabilization of neutral-type neural networks with interval and distributed time-varying delays. There are various activation functions which are considered in the system and the restriction on differentiability of interval time-varying delays is removed, which means that a fast interval time-varying delay is allowed. Based on the construction of improved Lyapunov-Krasovskii functionals combined with Liebniz-Newton's formula and the integral terms, new delay-dependent sufficient conditions for the exponential stabilization of system are established of LMIs without introducing any free-weighting matrices. The new stability condition is much less conservative and more general than some existing results, see [23, 32, 43–45]. Numerical examples illustrate the effectiveness of the results.

## 4.1 Exponential stabilization for Nominal Interval Time-varying Delay Systems

Consider the following neural networks with mixed time-varying delays and control input:

$$\begin{aligned}
\dot{x}(t) &= -(A+\Delta A(t))x(t) + (W_0+\Delta W_0)f(x(t)) + (W_1+\Delta W_1)g(x(t-h(t))) \\
&\quad +(W_2+\Delta W_2)\int_{t-k(t)}^{t} h(x(s))ds + B_0\dot{x}(t-\eta(t)) + Bu(t) \\
x(t) &= \phi(t), t\in[-d,0], d=\max\{h_2,k,\eta\}, \qquad (4.1)
\end{aligned}$$

where $x(t)\in\mathcal{R}^n$ is the state of the neural networks, $u(.)\in L_2([0,t],\mathcal{R}^m)$ is the control; $n$ is the number of neurals, and

$$\begin{aligned}
f(x(t)) &= [f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))]^T, \\
g(x(t)) &= [g_1(x_1(t)), g_2(x_2(t)), ..., g_n(x_n(t))]^T, \\
h(x(t)) &= [h_1(x_1(t)), h_2(x_2(t)), ...h_n(x_n(t))]^T,
\end{aligned}$$

are the activation functions, $A = \text{diag } (\bar{a_1}, \bar{a_2}, ..., \bar{a_n}), \bar{a_i} > 0$ represents the self-feedback term and $W_0, W_1, W_2$ denote the connection weights, the discretely delayed connection weights and the distributively delayed connection weight, respectively. In this thesis we consider various activation functions and assume that the activation functions $f(.), g(.), h(.)$ are Lipschitzian with the Lipschitz constants $a_i, b_i, c_i > 0$ :

$$\begin{aligned} |f_i(\xi_1) - f_i(\xi_2)| &\le a_i|\xi_1 - \xi_2|, \qquad i = 1, 2, ..., n, \forall \xi_1, \xi_2 \in \mathcal{R}, \\ |g_i(\xi_1) - g_i(\xi_2)| &\le b_i|\xi_1 - \xi_2|, \qquad i = 1, 2, ..., n, \forall \xi_1, \xi_2 \in \mathcal{R}, \\ |h_i(\xi_1) - h_i(\xi_2)| &\le c_i|\xi_1 - \xi_2|, \qquad i = 1, 2, ..., n, \forall \xi_1, \xi_2 \in \mathcal{R}. \end{aligned} \tag{4.2}$$

The time-varying delay functions $h(t), k(t), \eta(t)$ satisfy the condition

$$0 \le h_1 \le h(t) \le h_2, \tag{4.3}$$

$$0 \le k(t) \le k, \tag{4.4}$$

$$0 \le \eta(t) \le \eta, \dot{\eta}(t) \le \delta < 1. \tag{4.5}$$

It is worth noting that the state time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, and the lower bound not restricted to being zero. The initial functions $\phi(t) \in C^1([-d, 0], \mathcal{R}^n)$, with the norm

$$\| \phi \| = \sup_{t \in [-d,0]} \sqrt{\| \phi(t) \|^2 + \| \dot{\phi}(t) \|^2}.$$

The uncertainties satisfy the following condition:

$$\begin{aligned} \Delta A(t) = E_a F_a(t) H_a, \;\; \Delta W_0(t) = E_0 F_0(t) H_0, \\ \Delta W_1(t) = E_1 F_1(t) H_1, \;\; \Delta W_2(t) = E_2 F_2(t) H_2, \end{aligned} \tag{4.6}$$

where $E_i,\ H_i,\ i = a, 0, 1, 2$ are given constant matrices with appropriate dimensions, $F_i(t),\ i = a, 0, 1, 2$ are unknown, real matrices with Lebesgue measurable elements satisfying

$$F_i^T(t) F_i(t) \le I, \quad i = a, 0, 1, 2 \;\; \forall t \ge 0. \tag{4.7}$$

**Definition 4.1.1** *The zero solution of system (4.8) is exponentially stabilizable if there exists a feedback control $u(t) = Kx(t), K \in \mathcal{R}^{m \times n}$ such that the resulting closed-loop system*

$$\begin{aligned} \dot{x}(t) \;\;=\;\; & -[A - BK]x(t) + W_0 f(x(t)) + W_1 g(x(t - h(t))) + W_2 \int_{t-k(t)}^{t} h(x(s)) ds \\ & + B_0 \dot{x}(t - \eta(t)) \end{aligned}$$

*is exponentially stable.*

The nominal system (4.1) is given by

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2 \int_{t-k(t)}^{t} h(x(s))ds \\ &\quad + B_0 \dot{x}(t-\eta(t)) + Bu(t) \\ x(t) &= \phi(t), t \in [-d, 0]. \end{aligned} \tag{4.8}$$

First, we present a delay-dependent exponential stabilizability conditions for the nominal system (4.8). Let us set for simplicity. we denote

$$\begin{aligned}\lambda_1 &= \lambda_{min}(P^{-1}), \\ \lambda_2 &= \lambda_{max}(P^{-1}) + 2h_2\lambda_{max}(P^{-1}QP^{-1}) + 2h_2^2\lambda_{max}(P^{-1}RP^{-1}) \\ &\quad + \eta\lambda_{max}(P^{-1}Q_1P^{-1}) + 2\lambda_{max}(HD_2^{-1}H) + (h_2-h_1)^2\lambda_{max}(P^{-1}UP^{-1}).\end{aligned}$$

**Assumption 4.1.2** *All the eigenvalues of matrix $B_0$ are inside the unit circle.*

**Theorem 4.1.3** *Given $\alpha > 0$. The system (4.8) is exponentially stabilizable if there exist symmetric positive definite matrices $P, Q, R, U, Q1$, diagonal matrices $D_i, i = 0, 1, 2$ such that the following LMIs hold:*

$$\begin{aligned}\mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix} < 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned}\mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix} < 0, \end{aligned} \tag{4.10}$$

$$\mathcal{M}_3 = \begin{bmatrix} -0.1BB^T - 0.1(e^{-2\alpha h_1} + e^{-2\alpha h_2})R & 2kPH & 2PF \\ * & -2kD_2 & 0 \\ * & * & -2D_0 \end{bmatrix} < 0, \tag{4.11}$$

$$\mathcal{M}_4 = \begin{bmatrix} -0.1e^{-2\alpha h_2}U & 2PG \\ * & -2D_1 \end{bmatrix} < 0, \tag{4.12}$$

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & 0 \\ * & M_{22} & 0 & 0 & M_{25} & 0 \\ * & * & M_{33} & 0 & 0 & M_{36} \\ * & * & * & M_{44} & 0 & M_{46} \\ * & * & * & * & M_{55} & 0 \\ * & * & * & * & * & M_{66} \end{bmatrix},$$

*where*

$$
\begin{aligned}
M_{11} &= [-A+\alpha I]P + P[-A+\alpha I]^T - 0.9BB^T + 2Q + W_0D_0W_0^T + W_1D_1W_1^T \\
&\quad + ke^{2\alpha k}W_2D_2W_2^T - 0.9e^{-2\alpha h_1}R - 0.9e^{-2\alpha h_2}R], \\
M_{12} &= B_0P, \quad M_{13} = e^{-2\alpha h_1}R, \quad M_{14} = e^{-2\alpha h_2}R, \quad M_{15} = -PA^T - 0.5BB^T, \\
M_{22} &= -(1-\delta)e^{-2\alpha\eta}Q_1, M_{25} = PB_0^T, M_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \\
M_{36} &= e^{-2\alpha h_2}U, \quad M_{44} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \quad M_{46} = e^{-2\alpha h_2}U, \\
M_{55} &= h_1^2R + h_2^2R + (h_2-h_1)^2U + Q_1 - 2P + W_0D_0W_0^T + W_1D_1W_1^T \\
&\quad + ke^{2\alpha k}W_2D_2W_2^T], \\
M_{66} &= -1.9e^{-2\alpha h_2}U.
\end{aligned}
$$

The memoryless feedback control is

$$u(t) = -0.5B^TP^{-1}x(t), \qquad t \geq 0, \tag{4.13}$$

and the solution $x(t,\phi)$ of the system satisfies

$$\| x(t,\phi) \| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \| \phi \|, \quad \forall t \geq 0.$$

Let $Y = P^{-1}$ and $y(t) = Yx(t)$.
Proof. Using the feedback control (4.13) we consider the following Lyapunov-Krasovskii functional

$$V(t,x_t) = \sum_{i=1}^{8} V_i, \tag{4.14}$$

where

$$
\begin{aligned}
V_1 &= x^T(t)Yx(t), \\
V_2 &= \int_{t-h_1}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)\,ds, \\
V_3 &= \int_{t-h_2}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)\,ds, \\
V_4 &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)\,d\tau\,ds, \\
V_5 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)\,d\tau\,ds, \\
V_6 &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YUY\dot{x}(\tau)\,d\tau\,ds,
\end{aligned}
$$

$$
\begin{aligned}
V_7 &= \int_{t-\eta(t)}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)YQ_1Y\dot{x}(s)\,ds,\\
V_8 &= 2\int_{-k}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}h^T(x(\tau))D_2^{-1}h(x(\tau))d\tau\,ds.
\end{aligned}
$$

It easy to check that

$$
\lambda_1 \parallel x(t) \parallel^2 \leq V(t,x_t) \leq \lambda_2 \parallel x_t \parallel^2, \quad \forall t \geq 0. \tag{4.15}
$$

Taking the derivative of $V(x_t)$ along the solution of system (4.8) we have

$$
\begin{aligned}
\dot{V}_1 &= 2x^T(t)Y\dot{x}(t),\\
&= 2y^T(t)[-Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2\int_{t-k(t)}^{t} h(x(s))ds\\
&\quad +B_0\dot{x}(t-\eta(t)) - 0.5BB^TP^{-1}x(t)]\\
&= 2y^T(t)[-APy(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2\int_{t-k(t)}^{t} h(x(s))ds\\
&\quad +B_0P\dot{y}(t-\eta(t)) - 0.5BB^Ty(t)]\\
&= y^T(t)[-AP - PA^T]y(t) + 2y^T(t)W_0 f(x(t)) + 2y^T(t)W_1 g(x(t-h(t)))\\
&\quad +2y^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + 2y^T(t)B_0P\dot{y}(t-\eta(t)) - y^T(t)BB^Ty(t)\\
&\quad +2\alpha y^T(t)Py(t) - 2\alpha V_1,\\
\dot{V}_2 &= y^T(t)Qy(t) - e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1) - 2\alpha V_2,\\
\dot{V}_3 &= y^T(t)Qy(t) - e^{-2\alpha h_2}y^T(t-h_2)Qy(t-h_2) - 2\alpha V_3,\\
\dot{V}_4 &\leq h_1^2\dot{y}^T(t)R\dot{y}(t) - h_1e^{-2\alpha h_1}\int_{t-h_1}^{t}\dot{y}^T(s)R\dot{y}(s)ds - 2\alpha V_4,\\
\dot{V}_5 &\leq h_2^2\dot{y}^T(t)R\dot{y}(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{y}^T(s)R\dot{y}(s)ds - 2\alpha V_5,\\
\dot{V}_6 &\leq (h_2-h_1)^2\dot{y}^T(t)U\dot{y}(t) - (h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds - 2\alpha V_6,\\
\dot{V}_7 &\leq \dot{y}^T(t)Q_1\dot{y}(t) - (1-\delta)e^{-2\alpha\eta}\dot{y}^T(t-\eta(t))Q_1\dot{y}(t-\eta(t)) - 2\alpha V_7,\\
\dot{V}_8 &\leq 2kh^T(x(t))D_2^{-1}h(x(t)) - 2e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds - 2\alpha V_8.
\end{aligned} \tag{4.16}
$$

Using (4.2) and since matrices $D_i^{-1}$ , $i = 0, 1, 2$ are positive diagonal matrices, we have

$$
\begin{aligned}
kh^T(x(t))D_2^{-1}h(x(t)) &\le kx^T(t)HD_2^{-1}Hx(t) = ky^T(t)PHD_2^{-1}HPy(t),\\
f^T(x(t))D_0^{-1}f(x(t)) &\le x^T(t)FD_0^{-1}Fx(t) = y^T(t)PFD_0^{-1}FPy(t),\\
g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t))) &\le x^T(t-h(t))GD_1^{-1}Gx(t-h(t)),\\
&= y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)), \qquad (4.17)
\end{aligned}
$$

From (4.2) and the Lemma 2.5.5, we have the following estimations:

$$
\begin{aligned}
2y^T(t)W_0f(x(t)) &\le y^T(t)W_0D_0W_0^Ty(t) + f^T(x(t))D_0^{-1}f(x(t))\\
&\le y^T(t)W_0D_0W_0^Ty(t) + x^T(t)FD_0^{-1}Fx(t) \qquad (4.18)\\
&\le y^T(t)W_0D_0W_0^Ty(t) + y^T(t)PFD_0^{-1}FPy(t),\\
2y^T(t)W_1g(x(t-h(t))) &\le y^T(t)W_1D_1W_1^Ty(t)\\
&\quad +g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t)))\\
&\le y^T(t)W_1D_1W_1^Ty(t)\\
&\quad +x^T(t-h(t))GD_1^{-1}Gx(t-h(t))\\
&\le y^T(t)W_1D_1W_1^Ty(t)\\
&\quad +y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)), \qquad (4.19)\\
2y^T(t)W_2\int_{t-k(t)}^t h(x(s))ds &\le ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t)\\
&\quad +k^{-1}e^{-2\alpha k}\Big(\int_{t-k(t)}^t h(x(s)ds)\Big)^TD_2^{-1}\\
&\quad \times\Big(\int_{t-k(t)}^t h(x(s))ds\Big)\\
&\le ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t)\\
&\quad +e^{-2\alpha k}\int_{t-k}^t h^T(x(s))D_2^{-1}h(x(s))ds. \qquad (4.20)
\end{aligned}
$$

Applying Lemma 2.5.6 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_1\int_{t-h_1}^t \dot{y}^T(s)R\dot{y}(s)ds &\le -\Big[\int_{t-h_1}^t \dot{y}(s)\Big]^TR\Big[\int_{t-h_1}^t \dot{y}(s)\Big]\\
&\le -[y(t)-y(t-h_1)]^TR[y(t)-y(t-h_1)]\\
&= -y^T(t)Ry(t) + 2y^T(t)Ry(t-h_1)\\
&\quad -y^T(t-h_1)Ry(t-h_1), \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
-h_2 \int_{t-h_2}^{t} \dot{y}^T(s) R \dot{y}(s) ds &\leq -\Big[ \int_{t-h_2}^{t} \dot{y}(s) \Big]^T R \Big[ \int_{t-h_2}^{t} \dot{y}(s) \Big] \\
&\leq -[y(t) - y(t-h_2)]^T R [y(t) - y(t-h_2)] \\
&= -y^T(t) R y(t) + 2 y^T(t) R y(t-h_2) \\
&\quad -y^T(t-h_2) R y(t-h_2). \qquad (4.22)
\end{aligned}
$$

Note that

$$
\begin{aligned}
-(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds &= -(h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\quad -(h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds \\
&= -(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\quad -(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\quad -(h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\quad -(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds.
\end{aligned}
$$

Using Lemma 2.5.6 gives

$$
\begin{aligned}
-(h_2 - h(t)) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s)\, ds &\leq -\Big[ \int_{t-h_2}^{t-h(t)} \dot{y}(s) ds \Big]^T U \Big[ \int_{t-h_2}^{t-h(t)} \dot{y}(s) ds \Big] \\
&\leq -[y(t-h(t)) - y(t-h_2)]^T U [y(t-h(t)) \\
&\quad -y(t-h_2)], \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t) - h_1) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds &\leq -\Big[ \int_{t-h(t)}^{t-h_1} \dot{y}(s) ds \Big]^T U \Big[ \int_{t-h(t)}^{t-h_1} \dot{y}(s) ds \Big] \\
&\leq -[y(t-h_1) - y(t-h(t))]^T U [y(t-h_1) \\
&\quad -y(t-h(t))]. \qquad (4.24)
\end{aligned}
$$

Let $\beta = \frac{h_2 - h(t)}{h_2 - h_1} \leq 1$. Then, we obtain

$$
\begin{aligned}
-(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds &= -\beta \int_{t-h(t)}^{t-h_1} (h_2 - h_1) \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\leq -\beta \int_{t-h(t)}^{t-h_1} (h(t) - h_1) \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\leq -\beta [y(t-h_1) - y(t-h(t))]^T U [y(t-h_1) \\
&\quad -y(t-h(t))], \qquad (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s)\, ds &= -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h_1) \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\leq -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h(t)) \dot{y}^T(s) U \dot{y}(s)\, ds \\
&\leq -(1-\beta)[y(t-h(t)) - y(t-h_2)]^T \qquad (4.26) \\
&\quad \times U[y(t-h(t)) - y(t-h_2)].
\end{aligned}
$$

Therefore, from (4.23)-(4.26), we obtain

$$
\begin{aligned}
-(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{y}^T(s) U \dot{y}(s)\, ds \leq\ & -[y(t-h(t)) - y(t-h_2)]^T U [y(t-h(t)) \\
& -y(t-h_2)] - [y(t-h_1) - y(t-h(t))]^T \\
& \times U[y(t-h_1) - y(t-h(t))] \qquad (4.27) \\
& -\beta [y(t-h_1) - y(t-h(t))]^T U [y(t-h_1) \\
& -y(t-h(t))] - (1-\beta)[y(t-h(t)) \\
& -y(t-h_2)]^T U [y(t-h(t)) - y(t-h_2)].
\end{aligned}
$$

By using the following identity relation

$$
\begin{aligned}
& -P\dot{y}(t) - APy(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_0 \int_{t-k(t)}^{t} h(x(s)) ds \\
& +B_0 P \dot{y}(t - \eta(t)) - 0.5 BB^T y(t) = 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
& -2\dot{y}^T(t) P \dot{y}(t) - 2\dot{y}^T(t) APy(t) + 2\dot{y}^T(t) W_0 f(x(t)) + 2\dot{y}^T(t) W_1 g(x(t-h(t))) \\
& +2\dot{y}^T(t) W_0 \int_{t-k(t)}^{t} h(x(s)) ds + 2\dot{y}^T(t) B_0 P \dot{y}(t-\eta(t)) \\
& -\dot{y}^T(t) BB^T y(t) = 0. \qquad (4.28)
\end{aligned}
$$

By using Lemma 2.5.5, we have

$$\begin{aligned}
2\dot{y}^T(t)W_0f(x(t)) &\le \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t) + f^T(x(t))D_0^{-1}f(x(t)) \\
&\le \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t) \\
&\quad + y^T(t)PFD_0^{-1}FPy(t), \qquad (4.29) \\
2\dot{y}^T(t)W_1g(x(t-h(t))) &\le \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t) \\
&\quad + g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t))) \\
&\le \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t) \\
&\quad + y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)), \qquad (4.30) \\
2\dot{y}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds &\le ke^{2\alpha k}\dot{y}^T(t)W_2D_2W_2^T\dot{y}(t) \\
&\quad + e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds. \qquad (4.31)
\end{aligned}$$

Then, from (4.16) - (4.31), we obtain

$$\begin{aligned}
\dot{V}(t,x_t) + 2\alpha V(t,x_t) \le & y^T(t)[-AP - PA^T + 2\alpha P - BB^T + 2Q + 2kPHD_2^{-1}HP \\
& + W_0D_0W_0^T + W_1D_1W_1^T + ke^{2\alpha k}W_2D_2W_2^T \\
& - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R + PFD_1^{-1}FP]y(t) \\
& + 2y^T(t)B_0P\dot{y}(t-\eta(t)) + y^T(t-h_1)[-e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R \\
& - e^{-2\alpha h_2}U]y(t-h_1) + y^T(t-h_2)[-e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R \\
& - e^{-2\alpha h_2}U]y(t-h_2) + \dot{y}^T(t)[h_1^2R + h_2^2R + (h_2-h_1)^2U + Q_1 \\
& + W_1D_1W_1^T + ke^{2\alpha k}W_2D_2W_2^T]\dot{y}(t) - (1-\delta)e^{-2\alpha\eta} \\
& \times \dot{y}^T(t-\eta(t))Q_1\dot{y}(t-\eta(t)) + y^T(t-h(t))[2PGD_1^{-1}GP \\
& - 2e^{-2\alpha h_2}U]y(t-h(t)) + 2e^{-2\alpha h_1}y^T(t)Ry(t-h_1) \\
& + 2e^{-2\alpha h_2}y^T(t-h(t))Uy(t-h_2) - 2P + W_0D_0W_0^T \\
& + 2e^{-2\alpha h_2}y^T(t-h(t))Uy(t-h_1) - 2\dot{y}^T(t)APy(t) \\
& + 2\dot{y}^T(t)B_0P\dot{y}(t-\eta(t)) - \dot{y}^T(t)BB^Ty(t) + 2e^{-2\alpha h_2}y^T(t) \\
& \times Ry(t-h_2) - e^{-2\alpha h_2}\beta[y(t-h_1) - y(t-h(t))]^TU[y(t-h_1) \\
& - y(t-h(t))] - e^{-2\alpha h_2}(1-\beta)[y(t-h(t)) - y(t-h_2)]^T \\
& \times U[y(t-h(t)) - y(t-h_2)]
\end{aligned}$$

$$
\begin{aligned}
=&\xi^T(t)\Omega\xi(t) - e^{-2\alpha h_2}\beta[y(t-h_1) - y(t-h(t))]^T U \\
&\times [y(t-h_1) - y(t-h(t))] - e^{-2\alpha h_2}(1-\beta)[y(t-h(t)) \\
&- y(t-h_2)]^T U[y(t-h(t)) - y(t-h_2)] \\
=&\xi^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2]\xi(t) + y^T(t)M_3 y(t) \\
&+ y^T(t-h(t))M_4 y(t-h(t))
\end{aligned} \tag{4.32}
$$

where

$$
\begin{aligned}
M_3 &= -0.1BB^T - 0.1(e^{-2\alpha h_1} + e^{-2\alpha h_2})R + 2kPHD_2^{-1}HP + 2PFD_0^{-1}FP, \\
M_4 &= -0.1e^{-2\alpha h_2}U + 2PGD_1^{-1}GP,
\end{aligned}
$$

and

$$
\zeta(t) = [y(t), \dot{y}(t-\eta(t)), y(t-h_1), y(t-h_2), \dot{y}(t), y(t-h(t))].
$$

Since $0 \leq \beta \leq 1$, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2$ is a convex combination of $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ is equivalent to $\mathcal{M}_1 < 0$ and $\mathcal{M}_2 < 0$. Applying Lemma 2.5.1, the inequalities $M_3 < 0$ and $M_4 < 0$ are equivalent to $\mathcal{M}_3 < 0$ and $\mathcal{M}_4 < 0$, respectively. Thus, From (4.10) - (4.12) and (4.32), we get

$$
\dot{V}(t, x_t) \leq -2\alpha V(t, x_t), \quad \forall t \geq 0. \tag{4.33}
$$

Integrating both sides of (4.33) from 0 to $t$, we obtain

$$
V(t, x_t) \leq V(\phi)e^{-2\alpha t}, \quad \forall t \geq 0.
$$

Furthermore, taking condition (4.15) into account, we have

$$
\lambda_1 \parallel x(t,\phi) \parallel^2 \leq V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \lambda_2 e^{-2\alpha t} \parallel \phi \parallel^2,
$$

which imply that

$$
\parallel x(t,\phi) \parallel \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel, \quad t \geq 0.
$$

Therefore, nominal system (4.8) is exponentially stabilizable. The proof is completed.□

## 4.2 Exponential stabilization for Interval Time-varying Delay Systems

Based on Theorem 4.1.2, we derive stabilizability conditions of uncertain linear control systems with interval time-varying delay (4.1) in terms of LMIs.

**Theorem 4.2.1** *Given $\alpha > 0$. The system (4.1) is exponentially stabilizable if there exist symmetric positive definite matrices $P, Q, R, U, Q_1$, diagonal matrices $D_i, i = 0, 1, 2$ and $\epsilon_i > 0, i = 1, 2, ..., 6$ such that the following LMIs hold:*

$$\begin{aligned} \mathcal{W}_1 &= \mathcal{W} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix} < 0, \end{aligned} \tag{4.34}$$

$$\begin{aligned} \mathcal{W}_2 &= \mathcal{W} - \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix}^T \\ &\quad \times \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix} < 0, \end{aligned} \tag{4.35}$$

$$\mathcal{W}_3 = \begin{bmatrix} \Delta & 4kPH & 2PF & PH_a^T & PH_a^T & PFH_0^T & PFH_0^T \\ * & -4kD_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -2D_0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_4 I & 0 & 0 \\ * & * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & * & -\epsilon_5 I \end{bmatrix} < 0, \tag{4.36}$$

$$\mathcal{W}_4 = \begin{bmatrix} -0.1e^{-2\alpha h_2} U & 2PG & PGH_1^T & PGH_1^T \\ * & -2D_1 & 0 & 0 \\ * & * & -\epsilon_3 I & 0 \\ * & * & * & -\epsilon_6 I \end{bmatrix} < 0, \tag{4.37}$$

$$\mathcal{W} = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & 0 \\ * & W_{22} & 0 & 0 & W_{25} & 0 \\ * & * & W_{33} & 0 & 0 & W_{36} \\ * & * & * & W_{44} & 0 & W_{46} \\ * & * & * & * & W_{55} & 0 \\ * & * & * & * & * & W_{66} \end{bmatrix},$$

*where*

$$
\begin{aligned}
\Delta &= -0.1BB^T - 0.1(e^{-2\alpha h_1} + e^{-2\alpha h_2})R, \\
W_{11} &= [-A+\alpha I]P + P[-A+\alpha I]^T - 0.9BB^T + 2Q + W_0 D_0 W_0^T + W_1 D_1 W_1^T \\
&\quad + ke^{2\alpha k} W_2 D_2 W_2^T - 0.9e^{-2\alpha h_1}R - 0.9e^{-2\alpha h_2}R + \epsilon_1 E_a^T E_a + \epsilon_3 E_1^T E_1 \\
&\quad + \epsilon_2 E_0^T E_0 + ke^{2\alpha k} E_2^T H_2^T D_2 H_2 E_2, \\
W_{12} &= B_0 P, \quad W_{13} = e^{-2\alpha h_1}R, \quad W_{14} = e^{-2\alpha h_2}R, \quad W_{15} = -PA^T - 0.5BB^T, \\
W_{22} &= -(1-\delta)e^{-2\alpha\eta}Q_1, W_{25} = PB_0^T, W_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \\
W_{36} &= e^{-2\alpha h_2}U, \quad W_{44} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \quad W_{46} = e^{-2\alpha h_2}U, \\
W_{55} &= h_1^2 R + h_2^2 R + (h_2-h_1)^2 U + Q_1 - 2P + W_0 D_0 W_0^T + W_1 D_1 W_1^T \\
&\quad + ke^{2\alpha k} W_2 D_2 W_2^T + \epsilon_4 E_a^T E_a + \epsilon_5 E_0^T E_0 + \epsilon_6 E_1^T E_1 + ke^{2\alpha k} E_2^T H_2^T D_2 H_2 E_2, \\
W_{66} &= -1.9e^{-2\alpha h_2}U.
\end{aligned}
$$

Proof. Choose Lyapunov-Krasovskii functional as in (4.14) by changing $V_8$ into the following form
$V_8 = 4\int_{-k}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)} h^T(x(\tau)) D_2^{-1} h(x(\tau)) d\tau\, ds$,
we may proof the Theorem by using a similar argument as in the proof of Theorem 4.1.3. By replacing $A, W_0, W_1$ and $W_2$ with $A+E_aF_a(t)H_a, W_0+E_0F_0(t)H_0, W_1+E_1F_1(t)H_1$ and $W_2+E_2F_2(t)H_2$, respectively. We have the following

$$
\begin{aligned}
\dot{V}(t,x_t) + 2\alpha V(t,x_t) \le\, & y^T(t)[(-A+E_aF_a(t)H_a)P + P(-A+E_aF_a(t)H_a)^T \\
& + 2Q - 2R]y(t) + 2y^T(t)(W_0 + E_0F_0(t)H_0)f(x(t)) \\
& + 2y^T(t)(W_1 + E_1F_1(t)H_1)g(x(t-h(t))) + 2y^T(t)(W_2 \\
& + E_2F_2(t)H_2)\int_{t-k(t)}^{t} h(x(s))ds + 2y^T(t)B_0P\dot{y}(t-\eta(t)) \\
& - e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1) - e^{-2\alpha h_2}y^T(t-h_2)Qy(t-h_2) \\
& + h_1^2\dot{y}^T(t)R\dot{y}(t) + h_2^2\dot{y}^T(t)R\dot{y}(t) + (h_2-h_1)^2\dot{y}^T(t)U\dot{y}(t) \\
& + 2y^T(t)Ry(t-h_1) - y^T(t-h_1)Ry((t-h_1) \\
& - y^T(t-h_2)Ry(t-h_2) - \dot{y}^T(t)BB^Ty(t) - BB^T + 2\alpha P \\
& - [y(t-h(t) - y(t-h_2)]^T U[y(t-h(t) - y(t-h_2)] \\
& - [y(t-h_1) - y(t-h(t))]^T U[y(t-h_1) - y(t-h(t))] \\
& - \beta[y(t-h_1) - y(t-h(t))]^T U[y(t-h_1) - y(t-h(t)) \\
& - [y(t-h_1) - y(t-h_(t))]^T U[y(t-h_1) - y(t-h(t))] \\
& - \beta[y(t-h_1) - y(t-h(t))]^T U[y(t-h_1) - y(t-h(t))
\end{aligned}
$$

$$
\begin{aligned}
&- (1-\beta)[y(t-h(t)) - y(t-h_2)]^T U [y(t-h(t)) - y(t-h_2)] \\
&+ \dot{y}^T(t) Q_1 \dot{y}(t) - (1-\delta) e^{-2\alpha\eta} \dot{y}^T(t-\eta(t)) Q_1 \dot{y}(t-\eta(t)) \\
&+ 4k y^T(t) P H D_2^{-1} H P y(t) - 4e^{-2\alpha k} \int_{t-k}^{t} h^T(x(s)) D_2^{-1} h(x(s)) ds \\
&- 2\dot{y}^T(t) P \dot{y}(t) - 2\dot{y}^T(t)(A + E_a F_a(t) H_a) P y(t) \\
&+ 2\dot{y}^T(t)(W_0 + E_0 F_0(t) H_0) f(x(t)) + 2y^T(t) R y(t-h_2) \\
&+ 2\dot{y}^T(t)(W_1 + E_1 F_1(t) H_1) g(x(t-h(t))) \\
&+ 2\dot{y}^T(t)(W_2 + E_2 F_2(t) H_2) \int_{t-k(t)}^{t} h(x(s)) ds + 2\dot{y}^T(t) B_0 P \dot{y}(t-\eta(t)) \\
=& \xi^T(t)[(1-\beta)\mathcal{M}_1 + \beta \mathcal{M}_2]\xi(t) + y^T(t) M_3 y(t) \\
&+ y^T(t-h(t)) M_4 y(t-h(t))
\end{aligned} \tag{4.38}
$$

Applying Lemma 2.5.11 and (4.6), we have the following estimations

$$
\begin{aligned}
& y^T(t)[(-A + E_a F_a(t) H)_a) P + P(-A^T + H_a^T F_a^T(t) E_a^T)] y(t) && (4.39) \\
&\quad \le y^T(t)[-PA^T - AP] y(t) + \epsilon_1 y^T(t) E_a^T E_a y(t) + \epsilon_1^{-1} y^T(t) P H_a^T H_a P y(t), \\
& 2y^T(t)[W_0 + E_0 F_0(t) H_0] f(x(t)) && (4.40) \\
&\quad = 2y^T(t) W_0 f(x(t)) + 2y^T(t) E_0 F_0(t) H_0 f(x(t)) \le y^T(t) W_0 D_0 W_0^T y(t) \\
&\qquad + y^T(t) P F D_0^{-1} F P y(t) + \epsilon_2 y^T(t) E_0^T E_0 y(t) + \epsilon_2^{-1} y^T(t) P F H_0^T H_0 F P y(t), \\
& 2y^T(t)(W_1 + E_1 F_1(t) H_1) g(x(t-h(t))) \\
&\quad = 2y^T(t) W_1 g(x(t-h(t))) + 2y^T(t) E_1 F_1(t) H_1 g(x(t-h(t))) \\
&\quad \le y^T(t) W_1 D_1 W_1^T y(t) + y^T(t-h(t)) P G D_1^{-1} G P y(t-h(t)) + \epsilon_3 y^T(t) E_1^T E_1 y(t) \\
&\qquad + \epsilon_3^{-1} y^T(t-h(t)) P G H_1^T H_1 G P y^T(t-h(t)), && (4.41) \\
& 2y^T(t)(W_2 + E_2 F_2(t) H_2) \int_{t-k(t)}^{t} h(x(s)) ds \\
&\quad = 2y^T(t) W_2 \int_{t-k(t)}^{t} h(x(s)) ds + 2y^T(t) E_2 F_2(t) H_2 \int_{t-k(t)}^{t} h(x(s)) ds \\
&\quad \le k e^{2\alpha k} y^T(t) W_2 D_2 W_2^T y(t) + e^{-2\alpha k} \int_{t-k}^{t} h^T(x(s)) D_2^{-1} h(x(s)) ds \\
&\qquad + k e^{2\alpha k} y^T(t) E_2^T H_2 D_2 H_2^T E_2 y(t) \\
&\qquad + k^{-1} e^{-2\alpha k} [\int_{t-k(t)}^{t} h(x(s)) ds]^T H_2^T H_2^{-T} D_2^{-1} H_2^{-1} H_2 [\int_{t-k(t)}^{t} h(x(s)) ds]
\end{aligned}
$$

$$\leq ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t) + e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds \quad (4.42)$$

$$+ ke^{2\alpha k}y^T(t)E_2^TH_2D_2H_2^TE_2y(t) + e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds,$$

$$\begin{aligned} &-2\dot{y}^T(t)(A + E_aF_a(t)H_a)Py(t) \\ &\quad \leq -2\dot{y}^T(t)APy(t) - 2\dot{y}^T(t)E_aF_a(t)H_aPy(t) \\ &\quad \leq -2\dot{y}^T(t)APy(t) + \epsilon_4\dot{y}^T(t)E_a^TE_a\dot{y}(t) + \epsilon_4^{-1}y^T(t)PH_a^TH_aPy(t), \quad (4.43) \end{aligned}$$

$$\begin{aligned} 2\dot{y}^T(t)(W_0 + E_0F_0(t)H_0)f(x(t)) &= 2\dot{y}^T(t)[W_0 + E_0F_0(t)H_0]f(x(t)) \\ &\leq \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t) \\ &\quad +y^T(t)PFD_0^{-1}FPy(t) + \epsilon_5\dot{y}^T(t)E_0^TE_0\dot{y}(t) \\ &\quad +\epsilon_5^{-1}y^T(t)PFH_0^TH_0FPy(t), \qquad (4.44) \end{aligned}$$

$$\begin{aligned} 2\dot{y}^T(t)(W_1 + E_1F_1(t)H_1)g(x(t-h(t))) &\leq \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t)\epsilon_6\dot{y}^T(t)E_1^TE_1\dot{y}(t) \\ &\quad +y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)) \\ &\quad +\epsilon_6^{-1}y^T(t-h(t))PGH_1^T \\ &\quad \times H_1GPy^T(t-h(t)), \qquad (4.45) \end{aligned}$$

$$\begin{aligned} 2\dot{y}^T(t)(W_2 + E_2F_2(t)H_2) &\leq ke^{2\alpha k}\dot{y}^T(t)W_2D_2W_2^T\dot{y}(t) \\ &\quad +2e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds \\ &\quad +ke^{2\alpha k}\dot{y}^T(t)E_2^TH_2^TD_2H_2E_2\dot{y}(t). \qquad (4.46) \end{aligned}$$

$\square$

**Remark 4.2.2** *In [23, 32, 51], exponential stability of neutral-type neural networks with time-varying delays were investigated. However, the distributed delays have not been considered so stability conditions in [23, 43–45] are not applicable to our work. Moreover activation function considered in our work are more general than [23, 32, 51]. Therefore, our stability conditions are less conservative than some other existing results.*

**Remark 4.2.3** *In our work, the restriction that the state delay is differentiable is not required which allows the state delay to be fast time-varying. Meanwhile, this restriction is required in some existing results, such as [23, 32, 43–45].*

## 4.3 Numerical examples

In this section, we now provide examples to show the effectiveness of the result in Theorem 4.1.3.

**Example 4.3.1** Consider the neural networks with interval time-varying delay and control input with the following:

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t) + W_1 g(x(t-h(t))) + Bu(t) \tag{4.47}$$

where

$$A = \begin{bmatrix} -0.2 & 0 \\ 1 & 2 \end{bmatrix}, W_0 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, W_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix}, F = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix} B = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t) = 0.1 + 0.1|\sin t|$. Therefore, the methods used in [6, 49] are not applicable to this system. We have $h_1 = 0.1, h_2 = 0.2$. Given $\alpha = 0.2$ and any initial function $\phi(t) = C^1([-0.2, 0], \mathcal{R}^2)$. By solving LMIs, we obtain the following solution

$$\begin{aligned} P &= \begin{bmatrix} 0.0370 & 0.0010 \\ 0.0010 & 0.2938 \end{bmatrix}, Q = \begin{bmatrix} 0.0008 & 0.0029 \\ 0.0029 & 0.0250 \end{bmatrix}, \quad U = \begin{bmatrix} 0.0153 & 0.0080 \\ 0.008. & 0.6201 \end{bmatrix} \\ R &= \begin{bmatrix} 0.0377 & 0.0055 \\ 0.0055 & 0.8173 \end{bmatrix}, D_0 = \begin{bmatrix} 0.0353 & 0 \\ 0 & 0.2833 \end{bmatrix}, D_1 = \begin{bmatrix} 0.0215 & 0 \\ 0 & 0.5025 \end{bmatrix}. \end{aligned}$$

Thus, the system (4.47) is 0.2−exponentially stabilizable and the value $\sqrt{\frac{\lambda_2}{\lambda_1}} = 1.6469$, so the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \leq 1.6469 e^{-0.2t} \|\phi\|, \quad \forall t \in \mathcal{R}^+.$$

**Example 4.3.2** Consider the neural networks with mixed interval time-varying delays and control input with the following:

$$\begin{aligned} \dot{x}(t) &= -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0) f(x(t) + (W_1 + \Delta W_1) g(x(t-h(t))) \\ &\quad + (W_2 + \Delta W_2) \int_{t-k(t)}^{t} h(x(s))ds + B_0 \dot{x}(t - \eta(t)) + Bu(t) \end{aligned} \tag{4.48}$$

where

$$A = \begin{bmatrix} 0.15 & 0 \\ 0 & 1 \end{bmatrix}, W_0 = \begin{bmatrix} 0.5 & 0.12 \\ 0.1 & -0.3 \end{bmatrix}, W_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.1 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix}, F = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, H = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad H_a = H_0 = H_1 = H_2 = E_a = E_0 = E_1 = E_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t) = 0.2 + 0.2|\sin t|, k(t) = |\cos t|$ are non-differentiable and $\eta(t) = 0.2\sin^2(t)$. Therefore, the methods used in [23, 32] are not applicable to this system. We have $h_1 = 0.2, h_2 = 0.4, k = 0.1, \delta = 0.1, \eta = 0.2$. Given $\alpha = 0.1$ and any initial function $\phi(t) = C^1([-0.4, 0], \mathcal{R}^2)$. By solving LMIs, we obtain $e_1 = 0.0173, e_2 = 0.0128, e_3 = 0.0111, e_4 = 0.0263, e_5 = 0.0209, e_6 = 0.0192$,

$$\begin{aligned}
P &= \begin{bmatrix} 0.0061 & 0.0002 \\ 0.0002 & 0.0228 \end{bmatrix}, Q = \begin{bmatrix} 0.0003 & 0.0005 \\ 0.0001 & 0.0031 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.0005 & 0.0001 \\ 0.0001 & 0.0024 \end{bmatrix}, \\
U &= \begin{bmatrix} 0.0028 & 0.0004 \\ 0.0004 & 0.00382 \end{bmatrix}, R = \begin{bmatrix} 0.0052 & 0.0008 \\ 0.0008 & 0.0543 \end{bmatrix}, D_0 = \begin{bmatrix} 0.0068 & 0 \\ 0 & 0.0304 \end{bmatrix}, \\
D_1 &= \begin{bmatrix} 0.0038 & 0 \\ 0 & 0.0145 \end{bmatrix}, D_2 = \begin{bmatrix} 0.0433 & 0 \\ 0 & 0.0275 \end{bmatrix}.
\end{aligned}$$

Thus, the system (4.47) is $0.1-$exponentially stabilizable and the value $\sqrt{\frac{\lambda_2}{\lambda_1}} = 2.2939$, so the solution of the closed-loop system satisfies

$$\|x(t, \phi)\| \leq 2.2939 e^{-0.1t} \|\phi\|, \quad \forall t \in \mathcal{R}^+.$$