CHAPTER 5
Robust stability of a class of uncertain Lur'e systems of neutral-type with time-varying delay

In this chapter, we make the problem of robust stability for a class of uncertain neutral and Lur'e dynamical systems with sector-bounded nonlinear functions. The time delay is a continuous function belonging to a given interval, which means that the lower and upper bounds for the time varying delay are available, but the delay function is not necessary to be differentiable. To the best of the authors knowledge, there were no global stability results for uncertain neutral and Lur'e dynamical systems with some sector bounded conditions [10, 13, 55, 65]. Based on the construction of improved Lyapunov-Krasovskii functionals combined with Liebniz-Newton's formula and the integral terms, new delay-dependent sufficient conditions for the uncertain neutral and Lur'e dynamical of system are established of LMIs. Numerical examples are given to illustrate the effectiveness of the results.

Consider the following uncertain Lur'e system of neutral type with interval time-varying delays and sector-bounded nonlinear functions:

$$\begin{aligned} \dot{x}(t) - C\dot{x}(t-\eta(t)) &= (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-h(t)) \\ &\quad + (B + \Delta B(t))f(\sigma(t)), \qquad (5.1) \\ \sigma(t) &= H^T x(t) = [\bar{h}_1 \ \ \bar{h}_2 \ \ \ldots \ \ \bar{h}_m]^T x(t), \quad \forall t \geq 0, \\ x(t+s) &= \phi(t+s), \ \dot{x}(t+s) = \varphi(t+s), s \in [-m, 0], \\ m &= \max\{h_2, \eta\}, \end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the state vector; $\sigma(t) \in \mathcal{R}^n$ is the output vector; $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{n\times n}$, $A_1 \in \mathcal{R}^{n\times n}$, $H \in \mathcal{R}^{n\times m}$ are known constant matrices; $f(\sigma(t)) \in \mathcal{R}^m$ is the nonlinear function in the feedback path, which is denoted by f for simplicity in the sequel. The nonlinear function $f(\sigma(t))$ satisfies the following:

$$\begin{aligned} f(\sigma(t)) &= [f_1(\sigma_1(t)) \ \ f_2(\sigma_2(t)) \ \ \ldots \ \ f_m(\sigma_m(t))]^T, \\ \sigma(t) &= [\sigma_1(t) \ \ \sigma_2(t) \ \ \ldots \ \ \sigma_m(t)]^T = [h_1^T x(t) \ \ h_2^T x(t) \ \ \ldots \ \ h_m^T x(t)], \quad (5.2) \end{aligned}$$

where, each $f_i(\sigma_i(t))$, $i = 1, 2, ..., m$ term satisfies one of the following sector bounded conditions:

$$\begin{aligned} &f_i(\sigma_i(t)) \in K_{[0,k_i]} = \{f_i(\sigma_i(t)) \mid f_i(0) = 0, 0 < \sigma_i(t)f_i(\sigma_i(t)) \leq k_i\sigma_i(t)^2, \\ &\sigma_i(t) \neq 0\} \end{aligned} \tag{5.3}$$

or

$$f_i(\sigma_i(t)) \in K_{[0,\infty]} = \{f_i(\sigma_i(t)) \mid f_i(0) = 0, \sigma_i(t)f_i(\sigma_i(t)) > 0, \sigma_i(t) \neq 0\}. \tag{5.4}$$

$\Delta A(t), \Delta B(t), \Delta A_1(t)$ are time-varying uncertainties of appropriate dimensions, which are assumed to be of the following form:

$$[\Delta A(t) \;\; \Delta B(t) \;\; \Delta A_1(t)] = DF(t)[E_1 \;\; E_2 \;\; E_3], \tag{5.5}$$

where D, E_1, E_2 and E_3 are know matrices of appropriate dimensions, and the time-varying matrix $F(t)$ satisfies

$$F^T(t)F(t) \leq I, \quad \forall t \geq 0. \tag{5.6}$$

The delays $h(t)$ and $\tau(t)$ are time-varying continuous functions that satisfy

$$0 \leq h_1 \leq h(t) \leq h_2, \quad 0 \leq \eta(t) \leq \eta, \quad \dot{\eta}(t) \leq \eta_d < 1. \tag{5.7}$$

5.1 Stability of uncertain Lur'e systems

Now we present a new delay-dependent condition for the uncertain system (5.1) satisfying the sector bounded condition (5.3).

Assumption 5.1.1 *All the eigenvalues of matrix C are inside the unit circle.*

Theorem 5.1.2 *Under assumption 5.1.1, given $\alpha > 0$. The system (5.1) satisfying the sector condition (5.3) is exponentially stable if there exist symmetric positive definite matrices P, Q, R, U, F, L; symmetric positive semi-definite matrices $Z = diag(z_1, z_2, ..., z_m)$, and $J = diag(j_1, j_2, ..., j_m)$; scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$; matrices N_i; $i = 1, 2, 3$ of appropriate dimensions such that the following LMIs hold:*

$$\begin{aligned} \mathcal{M}_1 \;\; = \;\; & \mathcal{M} - \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ & \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{5.8}$$

$$\begin{aligned}\mathcal{M}_2 \;&=\; \mathcal{M} - \begin{bmatrix} 0 & -I & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,\end{aligned} \tag{5.9}$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} & \phi_{17} & \phi_{18} & \phi_{19} \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{28} & \phi_{29} \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \phi_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \phi_{55} & \phi_{56} & \phi_{57} & \phi_{58} & \phi_{59} \\ * & * & * & * & * & \phi_{66} & \phi_{67} & \phi_{68} & 0 \\ * & * & * & * & * & * & \phi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \phi_{88} & 0 \\ * & * & * & * & * & * & * & * & \phi_{99} \end{bmatrix},$$

where

$$\begin{aligned}
\phi_{11} &= PA + A^TP + Q + Q^T - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R + 2\alpha x^T(t)Px(t), \\
\phi_{13} &= e^{-2\alpha h_1}R, \phi_{14} = e^{-2\alpha h_2}R, \phi_{17} = PC, \phi_{18} = PD, \phi_{19} = \epsilon_i E_1^T, \\
\phi_{16} &= A^TL + A^TN_3^T, \phi_{15} = A^THZ^T + PB + HKJ + A^TN_2^T + 2\alpha HZ, \\
\phi_{22} &= -2e^{-2\alpha h_2}U + N_1A_1 + A_1^TN_1^T, \phi_{23} = e^{-2\alpha h_2}U, \phi_{24} = e^{-2\alpha h_2}U, \\
\phi_{25} &= A_1^THZ^T + N_1B + A_1^TN_2^T, \phi_{26} = A_1^TL - N_1 + A_1^TN_3^T, \phi_{27} = N_1C, \\
\phi_{28} &= N_1D, \phi_{29} = \epsilon_i E_3^T, \phi_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \\
\phi_{44} &= -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \phi_{57} = ZH^TC + N_2C, \\
\phi_{55} &= ZH^TB + B^THZ^T - J - J^T + N_2B + B^TN_2^T, \phi_{56} = B^TL - N_2 + B^TN_3^T, \\
\phi_{58} &= ZH^TD + N_2D, \phi_{59} = \epsilon_i E_2^T, \phi_{88} = -\epsilon_i I, \phi_{99} = -\epsilon_i I, \\
\phi_{66} &= h_1^2R + h_2^2R + F + (h_2 - h_1)^2U - L - L^T - N_3 - N_3^T, \phi_{12} = PA_1 + A^TN_1^T, \\
\phi_{67} &= LC + N_3C, \phi_{68} = LD + N_3D, \phi_{77} = -e^{-2\alpha\eta}(1 - \eta_D)F.
\end{aligned}$$

The solution $x(t)$ of the system satisfies,

$$\|x(t)\| \le \sqrt{\frac{a\|\phi\|^2 + b\|M_1\|^2 + c\|M_2\|^2}{\lambda_m(P)}}, \tag{5.10}$$

where

$a = \lambda_M(P) + 2h_2\lambda_M(R) + h_2\lambda_M(U) + 2\lambda_M(HZKH^T)$; $K = diag(k_1, k_2, ..., k_m)$,
$b = 2\lambda_M(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha} + 2h_2\lambda_M(R)\frac{1-e^{-2\alpha h_2}}{2\alpha} + h_2\lambda_M(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}$,
$c = \lambda_M(F)\frac{1-e^{-2\alpha\eta}}{2\alpha}$, $\|M_1\| = \sup_{-m\le s\le 0}\|x(s)\|, \|M_2\| = \sup_{-m\le s\le 0}\|\dot{x}(s)\|$.

Proof. Using (5.5), the uncertain system (5.1) can be represented as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + A_1x(t-h(t)) + Bf(\sigma(t)) + C\dot{x}(t-\eta(t)) + Dp(t),\\
p(t) &= F(t)(E_1x(t) + E_2f(\sigma(t)) + E_3x(t-h(t))),\\
\sigma(t) &= H^Tx(t) = [\bar{h}_1 \ \ \bar{h}_2 \ \ \ldots \ \ \bar{h}_m]^T x(t), \quad \forall t \geq 0,\\
x(s) &= \phi(s), \quad s \in [-\max(h_2, \eta_2), 0].
\end{aligned}
\tag{5.11}
$$

We consider the following Lyapunov-Krasovskii functional

$$
V(x(t)) = \sum_{i=1}^{8} V_i, \tag{5.12}
$$

where

$$
\begin{aligned}
V_1(x(t)) &= e^{2\alpha t}x^T(t)Px(t),\\
V_2(x(t)) &= \int_{t-h_1}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_3(x(t)) &= \int_{t-h_2}^{t} e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_4(x(t)) &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha s}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_5(x(t)) &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha s}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_6(x(t)) &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha s}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds,\\
V_7(x(t)) &= \int_{t-\eta(t)}^{t} e^{2\alpha s}\dot{x}^T(s)F\dot{x}(s)\,ds,\\
V_8(x(t)) &= 2\sum_{i=1}^{m}\lambda_i e^{2\alpha t}\int_{0}^{\sigma_i} f_i(\sigma_i)\,d\sigma_i.
\end{aligned}
$$

Taking the derivative of $V(x_t)$ along the solution of system (5.11), we have

$$
\begin{aligned}
\dot{V}_1(x(t)) &= 2\alpha e^{2\alpha t}x^T(t)Px(t) + 2e^{2\alpha t}x^T(t)P\dot{x}(t),\\
&= e^{2\alpha t}[2\alpha x^T(t)Px(t) + 2x^T(t)P(Ax(t) + A_1x(t-h(t)) + Bf(\sigma(t))\\
&\quad +C\dot{x}(t-\eta(t)) + Dp(t))],\\
\dot{V}_2(x(t)) &= e^{2\alpha t}[x^T(t)Qx(t) - e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1)],\\
\dot{V}_3(x(t)) &= e^{2\alpha t}[x^T(t)Qx(t) - e^{-2\alpha h_2}x^T(t-h_2)Qx(t-h_2)],
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_4(x(t)) &\leq e^{2\alpha t}[h_1^2\dot{x}^T(t)R\dot{x}(t) - h_1e^{-2\alpha h_1}\int_{t-h_1}^{t}\dot{x}^T(s)R\dot{x}(s)ds],\\
\dot{V}_5(x(t)) &\leq e^{2\alpha t}[h_2^2\dot{x}^T(t)R\dot{x}(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{x}^T(s)R\dot{x}(s)ds],\\
\dot{V}_6(x(t)) &\leq e^{2\alpha t}[(h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t) - (h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)ds],\\
\dot{V}_7(x(t)) &\leq e^{2\alpha t}[\dot{x}^T(t)F\dot{x}(t) - e^{-2\alpha\eta}(1-\eta_d)\dot{x}^T(t-\eta(t))F\dot{x}(t-\eta(t))],\\
\dot{V}_8(x(t)) &= 2\sum_{i=1}^{m}\lambda_i e^{2\alpha t}(2\alpha\int_0^{\sigma_i(t)} f_i(\sigma_i)d\sigma_i + f_i(\sigma_i(t))\dot{\sigma}_i(t)),\\
&\leq e^{2\alpha t}[4\alpha f^T(\sigma(t))Z\sigma(t) + 2f^T(\sigma(t))Z\dot{\sigma}(t)]\\
&\leq e^{2\alpha t}[4\alpha f^T(\sigma(t))ZHx(t) + 2f^T(\sigma(t))ZH^T\dot{x}(t)],\\
&= e^{2\alpha t}[4\alpha f^T(\sigma(t))ZHx(t) + 2f^T(\sigma(t))ZH^T(Ax(t) + A_1x(t-h(t))\\
&\quad +Bf(\sigma(t)) + C\dot{x}(t-\eta(t)) + Dp(t))]. \qquad (5.13)
\end{aligned}
$$

Applying Lemma 2.5.6 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_1\int_{t-h_1}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\leq -\Big[\int_{t-h_1}^{t}\dot{x}(s)\Big]^T R\Big[\int_{t-h_1}^{t}\dot{x}(s)\Big]\\
&\leq -[x(t)-x(t-h_1)]^T R[x(t)-x(t-h_1)]\\
&= -x^T(t)Rx(t) + 2x^T(t)Rx(t-h_1)\\
&\quad -x^T(t-h_1)Rx(t-h_1), \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
-h_2\int_{t-h_2}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\leq -\Big[\int_{t-h_2}^{t}\dot{x}(s)\Big]^T R\Big[\int_{t-h_2}^{t}\dot{x}(s)\Big]\\
&\leq -[x(t)-x(t-h_2)]^T R[x(t)-x(t-h_2)]\\
&= -x^T(t)Rx(t) + 2x^T(t)Rx(t-h_2)\\
&\quad -x^T(t-h_2)Rx(t-h_2). \qquad (5.15)
\end{aligned}
$$

Note that

$$
\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds &= -(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&\quad -(h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&= -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-(h(t)-h_1)\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds \\
&-(h(t)-h_1)\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \qquad (5.16)\\
&-(h_2-h(t))\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds.
\end{aligned}
$$

Using Lemma 2.5.6 gives

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds &\le -\Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big]^T U\Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big] \\
&\le -[x(t-h(t))-x(t-h_2)]^T U[x(t-h(t)) \\
&\quad -x(t-h_2)], \qquad (5.17)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds &\le -\Big[\int_{t-h(t)}^{t-h_1} \dot{x}(s)ds\Big]^T U\Big[\int_{t-h(t)}^{t-h_1} \dot{x}(s)ds\Big] \\
&\le -[x(t-h_1)-x(t-h(t))]^T U[x(t-h_1) \\
&\quad -x(t-h(t))]. \qquad (5.18)
\end{aligned}
$$

Let $\beta = \frac{h_2-h(t)}{h_2-h_1} \le 1$, so we have

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds &= -\beta\int_{t-h(t)}^{t-h_1} (h_2-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
&\le -\beta\int_{t-h(t)}^{t-h_1} (h(t)-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
&\le -\beta[x(t-h_1)-x(t-h(t))]^T U[x(t-h_1) \\
&\quad -x(t-h(t))], \qquad (5.19)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)\,ds &= -(1-\beta)\int_{t-h_2}^{t-h(t)} (h_2-h_1)\dot{x}^T(s)U\dot{x}(s)\,ds \\
&\le -(1-\beta)\int_{t-h_2}^{t-h(t)} (h_2-h(t))\dot{x}^T(s)U\dot{x}(s)\,ds \\
&\le -(1-\beta)[x(t-h(t))-x(t-h_2)]^T \qquad (5.20)\\
&\quad \times U[x(t-h(t))-x(t-h_2)].
\end{aligned}
$$

Therefore, from (5.16)-(5.20), we obtain

$$\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \;\; \leq \;\; & -[x(t-h(t))-x(t-h_2)]^T U[x(t-h(t)) \\
& -x(t-h_2)] - [x(t-h_1)-x(t-h(t))]^T \\
& \times U[x(t-h_1)-x(t-h(t))] - \beta[x(t-h_1) \\
& -x(t-h(t))]^T U[x(t-h_1)-x(t-h(t))] \\
& -(1-\beta)[x(t-h(t))-x(t-h_2)]^T \\
& \times U[x(t-h(t))-x(t-h_2)]. \qquad (5.21)
\end{aligned}$$

We add the following zero equation:

$$\begin{aligned}
& 2\xi^T(t)\bar{N}[Ax(t)+A_1x(t-h(t))+Bf(\sigma(t))+C\dot{x}(t-\eta(t))+Dp(t) \\
& -\dot{x}(t)] = 0, \qquad (5.22)
\end{aligned}$$

where $\bar{N} = [N_1^T \;\; N_2^T \;\; N_3^T]^T$, $\xi(t) = [x^T(t-h(t)) \;\; f^T(\sigma(t)) \;\; \dot{x}^T(t)]^T$ and by using the identity relation

$$-\dot{x}(t)+Ax(t)+A_1x(t-h(t))+Bf(\sigma(t))+C\dot{x}(t-\eta(t))+Dp(t) = 0,$$

we have

$$\begin{aligned}
& -2\dot{x}^T(t)L\dot{x}(t)+2\dot{x}^T(t)LAx(t)+2\dot{x}^T(t)LA_1x(t-h(t))+2\dot{x}^T(t)LBf(\sigma(t)) \\
& +2\dot{x}^T(t)LC\dot{x}(t-\eta(t))+2\dot{x}^T(t)LDp(t) = 0. \qquad (5.23)
\end{aligned}$$

From sector bounded condition (5.3) and $J = \mathrm{diag}(j_1, j_2, ..., j_m)$, we have

$$j_if_i(\sigma_i)[k_ih_i^Tx(t)-f_i(\sigma_i)] \geq 0, i = 1, 2, ..., m, \qquad (5.24)$$

which is equivalent to

$$x^T(t)HKJf(\sigma(t)) - f^T(\sigma(t))Jf(\sigma(t)) \geq 0. \qquad (5.25)$$

Similarly, for any $\epsilon > 0$, from (5.11), we have

$$\begin{aligned}
& -\epsilon p^T(t)p(t)+\epsilon(E_1x(t)+E_2f(\sigma(t))+E_3x(t-h(t)))^T(E_1x(t)+E_2f(\sigma(t)) \\
& +E_3x(t-h(t))) \geq 0. \qquad (5.26)
\end{aligned}$$

Hence, according to (5.14)-(5.15), (5.21), and by adding the zero term (5.23) and (5.25)-(5.26), we get

$$\begin{aligned}
\dot{V}(x(t)) \leq\ & e^{2\alpha t}\{2\alpha x^T(t)Px(t) + 2x^T(t)P(Ax(t) + A_1x(t-h(t)) + Bf(\sigma(t)) \\
& +Dp(t)) + x^T(t)Qx(t) - e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1) + C\dot{x}(t-\eta(t)) \\
& -e^{-2\alpha h_2}x^T(t-h_2)Qx(t-h_2) + h_1^2\dot{x}^T(t)R\dot{x}(t) + h_2^2\dot{x}^T(t)R\dot{x}(t) \\
& -e^{-2\alpha h_1}x^T(t)Rx(t) + 2e^{-2\alpha h_1}x^T(t)Rx(t-h_1) + x^T(t)Qx(t) \\
& -e^{-2\alpha h_1}x^T(t-h_1)Rx(t-h_1) - e^{-2\alpha h_2}x^T(t-h_2)Rx(t-h_2) \\
& -e^{-2\alpha h_2}x^T(t)Rx(t) + 2e^{-2\alpha h_2}x^T(t)Rx(t-h_2) + Bf(\sigma(t)) \\
& \dot{x}^T(t)F\dot{x}(t) - e^{-2\alpha\eta}(1-\eta_D)\dot{x}^T(t-\eta(t))F\dot{x}(t-\eta(t)) \\
& 2\alpha f^T(\sigma(t))ZHx(t) + 2f^T(\sigma(t))ZH^T(Ax(t) + A_1x(t-h(t)) \\
& +C\dot{x}(t-\eta(t)) + Dp(t)) - e^{-2\alpha h_2}x^T(t)Rx(t) \\
& (h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t) + 2x^T(t)HKJf(\sigma(t)) - 2f^T(\sigma(t))Jf(\sigma(t)) \\
& -2\dot{x}^T(t)L\dot{x}(t) + 2\dot{x}^T(t)LAx(t) + 2\dot{x}^T(t)LA_1x(t-h(t)) \\
& +2\dot{x}^T(t)LBf(\sigma(t)) + 2\dot{x}^T(t)LC\dot{x}(t-\eta(t)) + 2\dot{x}^T(t)LDp(t) \\
& -\epsilon p^T(t)p(t) + \epsilon(E_1x(t) + E_2f(\sigma(t)) + E_3x(t-h(t)))^T \\
& \times(E_1x(t) + E_2f(\sigma(t)) + E_3x(t-h(t))) + 2e^{-2\alpha h_2}x^T(t)Rx(t-h_2) \\
& -e^{-2\alpha h_2}[x(t-h(t)) - x(t-h_2)]^TU[x(t-h(t)) - x(t-h_2)] \\
& -e^{-2\alpha h_2}[x(t-h_1) - x(t-h(t))]^TU[x(t-h_1) - x(t-h(t))]\} \\
& -\beta[x(t-h_1) - x(t-h(t))]^TU[x(t-h_1) - x(t-h(t))] \\
& -(1-\beta)[x(t-h(t)) - x(t-h_2)]^TU[x(t-h(t)) - x(t-h_2)] \\
=\ & e^{2\alpha t}\{\zeta^T(t)\mathcal{M}\zeta(t) - \beta[x(t-h_1) - x(t-h(t))]^Te^{-2\alpha h_2} \\
& \times U[x(t-h_1) - x(t-h(t))] - (1-\beta)[x(t-h(t)) - x(t-h_2)]^T \\
& \times e^{-2\alpha h_2}U[x(t-h(t)) - x(t-h_2)]\} \\
=\ & e^{2\alpha t}\{\zeta^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2]\zeta(t)\}, \qquad (5.27)
\end{aligned}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as in (5.8) and (5.9), respectively, and $\zeta(t) = [x(t) \quad x(t-h(t)) \quad x(t-h_1) \quad x(t-h_2) \quad f(\sigma(t)) \quad \dot{x}(t) \quad \dot{x}(t-\eta(t)) \quad p(t)]$. By $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ holds if and only if $\mathcal{M}_1 < 0$ and $\mathcal{M}_2 < 0$. For showing the convergence rate, we have $\dot{V}(x(t)) \leq 0$, and then $V(x(t)) \leq V(x(0))$. However,

$$\begin{aligned}
V_1(x(0)) &= e^{2\alpha t}x^T(0)Px(0) \leq \lambda_{\max}(P)\|\phi\|^2 \\
V_2(x(0)) &= \int_{-h_1}^{0} e^{2\alpha s}x^T(s)Qx(s)\,ds, \leq \lambda_{\max}(Q)\int_{-h_1}^{0} e^{2\alpha s}ds\|M_1\|^2 \\
&= \lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|M_1\|^2 \\
V_3(x(0)) &= \int_{-h_2}^{0} e^{2\alpha s}x^T(s)Qx(s)\,ds, \leq \lambda_{\max}(Q)\int_{-h_2}^{0} e^{2\alpha s}ds\|M_1\|^2 \\
&= \lambda_{\max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|M_1\|^2 \\
V_4(x(0)) &= h_1\int_{-h_1}^{0}\int_{s}^{0} e^{2\alpha s}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds \\
&= h_1\int_{-h_1}^{0} e^{2\alpha s}[x^T(0)Rx(0) - x^T(s)Rx(s)]ds \\
&\leq h_2\lambda_M(R)\int_{-h_1}^{0} e^{2\alpha s}ds\|\phi\|^2 - h_2\lambda_M(R)\int_{-h_1}^{0} e^{2\alpha s}ds\|M_1\|^2 \\
&= h_2\lambda_M(R)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi\|^2 - h_2\lambda_M(R)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|M_1\|^2 \\
V_5(x(0)) &= h_2\int_{-h_2}^{0}\int_{s}^{0} e^{2\alpha s}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds, \\
&\leq h_2\lambda_M(R)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi\|^2 - h_2\lambda_M(R)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|M_1\|^2 \\
V_6(x(0)) &= h_2\int_{-h_2}^{-h_1}\int_{s}^{0} e^{2\alpha s}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds, \\
&\leq h_2\lambda_M(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi\|^2 - h_2\lambda_M(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|M_1\|^2 \\
V_7(x(0)) &= \int_{\eta(0)}^{0} e^{2\alpha s}\dot{x}^T(s)F\dot{x}(s)\,ds, \leq \lambda_M(F)\int_{-\eta}^{0} e^{2\alpha s}ds\|M_2\|^2 \\
&\leq \lambda_M(F)\frac{1-e^{-2\alpha\eta}}{2\alpha}\|M_2\|^2 \\
V_8(x(t)) &= 2\sum_{i=1}^{m}\lambda_i e^{2\alpha t}\int_0^{\sigma_i} f_i(\sigma_i)\,d\sigma_i \leq 2\sum_{i=1}^{m}\lambda_i e^{2\alpha t}\int_0^{\sigma_i} k\sigma_i(t)\,d\sigma_i \\
&\leq 2\sum_{i=1}^{m} e^{2\alpha t}\lambda_i k\sigma_i^2(t) \leq 2\sum_{i=1}^{m} e^{2\alpha t}\lambda_i kx^T(t)HH^Tx(t) \\
V_8(x(0)) &\leq 2\lambda_M(HZKH^T)\|\phi\|^2,
\end{aligned}$$

and $V(x(t)) \geq e^{2\alpha t}x^T(t)Px(t) \geq e^{2\alpha t}\lambda_m(P)\|x(t)\|^2$. Then, from Lemma (2.3.18), we conclude that the equilibrium point is globally exponentially stable. This completes the proof. $\square$

Next, we consider system (5.1) with the sector bounded condition (5.4), and if there exists the following
$J = \text{diag}(j_1, j_2, ..., j_m) \geq 0$, it follows that

$$r_i f_i(\sigma_i(t)) h_i^T x(t) \geq 0, \quad i = 1, 2, ..., m, \tag{5.28}$$

which is equivalent to

$$x^T(t) HJf(\sigma(t)) \geq 0. \tag{5.29}$$

Substituting (5.29) condition into (5.25) and letting $\alpha = 0$, this contributes to the following asymptotically stable with the sector bounded condition (5.29):

Corollary 5.1.3 *Under Assumption 5.1.1, the system (5.1) satisfying the sector condition (5.4) is asymptotically stable if there exist symmetric positive definite matrices P, Q, R, U, F, L; symmetric positive semi-definite matrices $Z = diag(z_1, z_2, ..., z_m)$, and $J = diag(j_1, j_2, ..., j_m)$; scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$; matrices N_i; $i = 1, 2, 3$ of appropriate dimension such that the following LMIs hold:*

$$\begin{aligned}\mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,\end{aligned} \tag{5.30}$$

$$\begin{aligned}\mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & -I & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,\end{aligned} \tag{5.31}$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \varpi_{15} & \phi_{16} & \phi_{17} & \phi_{18} & \phi_{19} \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} & \phi_{28} & \phi_{29} \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \phi_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \varpi_{55} & \phi_{56} & \phi_{57} & \phi_{58} & \phi_{59} \\ * & * & * & * & * & \phi_{66} & \phi_{67} & \phi_{68} & 0 \\ * & * & * & * & * & * & \phi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \phi_{88} & 0 \\ * & * & * & * & * & * & * & * & \phi_{99} \end{bmatrix},$$

where

$$\varpi_{15} = A^T HZ^T + PB + HJ, \quad \varpi_{55} = ZH^T B + B^T HZ^T. \tag{5.32}$$

Now we consider system (5.1) with $\Delta A(t), \Delta A_1(t), \Delta B(t) = 0$. We obtain the following Corollary 5.1.4 for exponentially stable with finite sector bounded and Corollary 5.1.5 for asymptotically stable with infinite sector bounded conditions.

Corollary 5.1.4 *Assumption 5.1.1, given $\alpha > 0$. The system (5.1) without uncertainties satisfying the sector condition (5.3) is exponentially stable if there exist symmetric positive definite matrices P, Q, R, U, F, L; symmetric positive semi-definite matrices $Z = diag(z_1, z_2, ..., z_m)$, and $J = diag(j_1, j_2, ..., j_m)$; matrices N_i; $i = 1, 2, 3$ of appropriate dimensions such that the following LMIs hold:*

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{5.33}$$

$$\begin{aligned} \mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & -I & 0 & I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{5.34}$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} & \phi_{17} \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \phi_{44} & 0 & 0 & 0 \\ * & * & * & * & \phi_{55} & \phi_{56} & \phi_{57} \\ * & * & * & * & * & \phi_{66} & \phi_{67} \\ * & * & * & * & * & * & \phi_{77} \end{bmatrix}.$$

Corollary 5.1.5 *Under Assumption 5.1.1, the system (5.1) without uncertainties satisfying the sector condition (5.4) is asymptotically stable if there exist symmetric positive definite matrices P, Q, R, U, F, L; symmetric positive semi-definite matrices $Z = diag(z_1, z_2, ..., z_m)$, and $J = diag(j_1, j_2, ..., j_m)$; matrices N_i; $i = 1, 2, 3$ of appropriate dimensions such that the following LMIs hold:*

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{5.35}$$

$$\begin{aligned} \mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & -I & 0 & I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{5.36}$$

$$\mathcal{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \varpi_{15} & \phi_{16} & \phi_{17} \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} & \phi_{27} \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \phi_{44} & 0 & 0 & 0 \\ * & * & * & * & \varpi_{55} & \phi_{56} & \phi_{57} \\ * & * & * & * & * & \phi_{66} & \phi_{67} \\ * & * & * & * & * & * & \phi_{77} \end{bmatrix}.$$

Remark 5.1.6 In most of studies on stability of Lur'e system, the state delay is assumed to be bounded and differentiable, see [10, 13, 55, 65]. However in our study, the restriction on derivative of the state delay is not required which allows the state delay to be fast time-varying.

Remark 5.1.7 It is worth pointing out that using the method, we can develop to more complex dynamical network models, such as neutral-type neural networks [57, 64], or BAM neutral-type neural networks [35, 69].

5.2 Numerical examples

In this section, we provide numerical examples to show the effectiveness of our theoretical results.

Example 5.2.1 Consider the following nominal Lur'e system with time-varying delays which is studied in [10, 55, 65] :

$$\begin{aligned} \dot{x}(t) - C\dot{x}(t-\eta(t)) &= Ax(t) + A_1x(t-h(t)) + Bf(\sigma(t)), \\ \sigma(t) &= H^T x(t) = [h_1 \;\; h_2]^T x(t), \;\; \forall \geq 0, \end{aligned} \tag{5.37}$$

where

$$A = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix},\; A_1 = \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix},\; B = \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix},\; C = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$

Table 5.1 and Table 5.2 give comparisons of maximum allowable values of h_2 for (5.37) by using Corollary 5.1.4 with nonlinear term satisfying (5.3) for $k = 100$ and by using Corollary 5.1.5 with nonlinear term satisfying (5.4), respectively. It is clear that for $h_1 = 0$ the maximum allowable bounds for h_2 are given in Table 5.1 and Table 5.2 are much larger than those obtained in [10] and [10, 55, 65]

Table 5.1: Maximum allowable upper bound h_2 with $h_1 = 0, k = 100$ for (5.37) obtained in Corollary 5.1.4 with nonlinear term satisfying (5.3).

η_d	Method	Maximum allowable values h_2
0.1	[10]	0.9888 with $\dot{h}(t) \leq 0.6$ 0.7228 with $\dot{h}(t) \leq 0.8$
	Corollary 5.1.4	1.0651 with no restriction on $\dot{h}(t)$
0.5	[10]	0.7793 with $\dot{h}(t) \leq 0.6$ 0.6282 with $\dot{h}(t) \leq 0.8$
	Corollary 5.1.4	0.9210 with no restriction on $\dot{h}(t)$
0.9	[10]	0.0983 with $\dot{h}(t) \leq 0.6$ 0.0967 with $\dot{h}(t) \leq 0.8$
	Corollary 5.1.4	0.1177 with no restriction on $\dot{h}(t)$

respectively. Note that $[10, 55, 65]$ are required the differentiability of time delay $h(t)$ but this restriction is not required in our work.

Example 5.2.2 Consider the following uncertain Lur'e system with interval time-varying delays with the following:

$$\begin{aligned} \dot{x}(t) - C\dot{x}(t-\eta(t)) &= (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-h(t)) \\ &\quad + (B + \Delta B(t))f(\sigma(t)), \end{aligned} \tag{5.38}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$D = E_1 = E_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, E_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}.$$

From the conditions (5.8)-(5.9) of Theorem 5.1.2, we let $\alpha = 0.4, h_1 = 0.1, h_2 = 0.4, \eta_d = 0.8, \eta = 0.1$ and $K = 0.5I$. By solving LMIs, we obtain the following

Table 5.2: Maximum allowable upper bound h_2 with $h_1 = 0$ for (5.37) obtained in Corollary 5.1.5 with nonlinearity satisfying (5.4).

η_d	Method	Maximum allowable values h_2
0.5	[10]	0.7793 with $\dot{h}(t) \leq 0.6$
		0.6282 with $\dot{h}(t) \leq 0.8$
	[65]	0.7901 with $\dot{h}(t) \leq 0.6$
		0.6321 with $\dot{h}(t) \leq 0.8$
	Corollary 5.1.5	0.9209 with no restriction on $\dot{h}(t)$
0.9	[10]	0.0983 with $\dot{h}(t) \leq 0.6$
		0.0967 with $\dot{h}(t) \leq 0.8$
	[65]	0.0994 with $\dot{h}(t) \leq 0.6$
		0.0981 with $\dot{h}(t) \leq 0.8$
	[55]	0.1086 with $\dot{h}(t) \leq 0.6$
		0.1086 with $\dot{h}(t) \leq 0.8$
	Corollary 5.1.5	0.1181 with no restriction on $\dot{h}(t)$

solution

$$
\begin{aligned}
P &= \begin{bmatrix} 28.6713 & -0.2237 \\ -0.2237 & 2.4026 \end{bmatrix}, Q = \begin{bmatrix} 25.1176 & 0.7616 \\ 0.7616 & 1.0772 \end{bmatrix}, \\
U &= \begin{bmatrix} 23.6798 & 3.0045 \\ 3.0045 & 3.1148 \end{bmatrix}, F = \begin{bmatrix} 7.0939 & -0.3657 \\ -0.3657 & 0.8018 \end{bmatrix}, \\
N1 &= \begin{bmatrix} -3.1736 & 0.4426 \\ -1.4601 & 1.2008 \end{bmatrix}, L = \begin{bmatrix} 16.5554 & 0 \\ 0 & 16.5554 \end{bmatrix}, \\
N3 &= \begin{bmatrix} -5.5006 & -0.4441 \\ -0.9892 & -14.4101 \end{bmatrix}, N2 = \begin{bmatrix} -1.6410 & 0.2843 \end{bmatrix}, \\
R &= \begin{bmatrix} 12.0906 & 0.2313 \\ 0.2313 & 2.2354 \end{bmatrix}, \quad Z = 0.9717, \; J = 4.7851, \; e_1 = 10.2112, \\
e_2 &= 18.1870.
\end{aligned}
$$

Thus, the system (5.1) is 0.4-exponentially stable. Given $\alpha > 0$, we will give the values of the maximum allowable upper bounds of the uncertain Lur'e system with interval time-varying delay (5.38) for difference η_d of the delay and different decay rates $0.1 \leq \alpha \leq 0.4$ with the same values of parameters $h_1 = 0.1, \eta_d = 0.8, \eta = 0.1$ and $K = 0.5I$. From Theorem 5.1.2, we obtain the maximum allowable upper

Table 5.3: Maximum allowable upper bounds h_2 of the uncertain Lur'e system with interval time-varying delay (5.38) for different values of the η_d and decay rates with $h_1 = 0.1, \eta_d = 0.8, \eta = 0.1, d = 0.2$ and $K = 0.5I$.

	$\eta_d = 0.2$	$\eta_d = 0.4$	$\eta_d = 0.6$	$\eta_d = 0.8$
$\alpha = 0.1$	0.7423	0.7094	0.6541	0.5294
$\alpha = 0.2$	0.6798	0.6513	0.6031	0.4923
$\alpha = 0.3$	0.6291	0.6037	0.5606	0.4604
$\alpha = 0.4$	0.5870	0.5637	0.5245	0.4326

bound of the time-varying delay h_2, as shown in Table 5.3.