CHAPTER 6
A new absolute stability criteria for Lur'e systems of neutral-type with time-varying delays

In this chapter, we deal with the problem of absolute stability of neutral type Lur'e systems with time-varying delays. By using a set of improved Lyapunov-Krasovskii functional includes some integral terms, a matrix-based on quadratic convex [68], combined with Wirtinger inequalities [58] and some useful integral inequality, so some new cross terms will be introduced which enhance the feasible stability criterion. By introducing new augmented Lyapunov-Krasovskii functional which have not been consider yet in stability analysis of Lur'e systems, see [28, 36, 55, 56, 65]. The new stability condition is much less conservative and more general than some existing results. Numerical examples are given to illustrate the effectiveness of our theoretical results.

6.1 Absolute stability criteria for Lur'e systems of neutral type

Consider the following Lur'e system of neutral type with interval time-varying delay:

$$\dot{x}(t) = A_1\dot{x}(t-\tau(t)) + Ax(t) + Bx(t-h(t)) + Cf(\omega(t)) + Dh(\sigma(t)), \tag{6.1}$$

$$\omega(t) = Ex(t) = [E_1 \;\; E_2 \;\; \dots \;\; E_{k_1}]^T x(t), \;\; \forall t \geq 0, \tag{6.2}$$

$$\sigma(t) = Fx(t-h(t)) = [F_1 \;\; F_2 \;\; \dots \;\; F_{k_2}]^T x(t-h(t)), \;\; \forall t \geq 0, \tag{6.3}$$

$$x(t+s) = \phi(t+s), \; \dot{x}(t+s) = \varphi(t+s), s \in [-m, 0], \; m = \max\{h_2, \tau_2\},$$

where $x(t) \in \mathcal{R}^n$, $\omega(t) \in \mathcal{R}^{k_1}$ and $\sigma(t) \in \mathcal{R}^{k_2}$ denote the state vector and output ones of the system, respectively; $A, A_1, B \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{n\times k_1}$, $D \in \mathcal{R}^{n\times k_2}$ are constant matrices; $f(Ex(\cdot)) = [f_1(E_1^T x(\cdot)), ..., f_{k_1}(E_{k_1}^T x(\cdot))]^T$, $h(Fx(\cdot)) = [h_1(F_1^T x(\cdot)), ..., h_{k_2}(F_{k_2}^T x(\cdot))]^T$ are the nonlinear elements.

Lemma 6.1.1 *The Lur'e system of neutral type described by (6.1) is said to be robustly absolutely stable in the sector if the trivial solution* $x(t) = 0$ *is globally uniformly asymptotically stable for any nonlinear function* $\psi(t, z(t))$ *satisfying (6.2)and (6.3), where* $\psi(t, z(t)) : [0, \infty) \times \mathcal{R}^m \longrightarrow \mathcal{R}^m$ *is a memoryless, possibly time-varying, nonlinear function which is piecewise continuous in* t *and globally Lipschitz in* $z(t)$ *and satisfies the sector condition (6.2), (6.3).*

Assumption 6.1.2 *The delays* $\tau(t)$ *and* $h(t)$ *are time-varying continuous functions satisfying*

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \mu_1 \leq \dot{h}(t) \leq \mu_2, \tag{6.4}$$

$$0 \leq \tau(t) \leq \tau_2, \qquad \dot{\tau}(t) \leq \delta < 1, \tag{6.5}$$

in which $h_1, h_2, \tau_2, \mu_1, \mu_2$ *and* δ *are known real numbers.*

Assumption 6.1.3 *For any* $\epsilon_1, \epsilon_2 \in \mathcal{R}$*, the nonlinear functions* $f_i(\cdot)$ *and* $h_j(\cdot)$ *satisfy* $f_i(0) = h_j(0) = 0$*, and*

$$\sigma_i^- \leq \frac{f_i(\epsilon_1) - f_i(\epsilon_2)}{\epsilon_1 - \epsilon_2} \leq \sigma_i^+, \delta_j^- \leq \frac{h_j(\epsilon_1) - h_j(\epsilon_2)}{\epsilon_1 - \epsilon_2} \leq \delta_j^+, \epsilon_1 \neq \epsilon_2, i = 1, ..., k_1;$$
$$j = 1, ..., k_2,$$

where $\sigma_i^+, \sigma_i^-, \delta_j^+$*, and* δ_j^- *are given constants. This work, we denote*

$$\begin{aligned}
\Upsilon_1 &= diag(\sigma_1^+\sigma_1^-, ..., \sigma_{k_1}^+\sigma_{k_1}^-), \Upsilon_2 = diag(\frac{\sigma_1^+ + \sigma_1^-}{2}, ..., \frac{\sigma_{k_1}^+ + \sigma_{k_1}^-}{2}),\\
\Upsilon_3 &= diag(\delta_1^+\delta_1^-, ..., \delta_{k_2}^+\delta_{k_2}^-), \Upsilon_4 = diag(\frac{\delta_1^+ + \delta_1^-}{2}, ..., \frac{\delta_{k_2}^+ + \delta_{k_2}^-}{2}),\\
\overline{\Upsilon}_1 &= diag(\sigma_1^+, ..., \sigma_{k_1}^+), \overline{\Upsilon}_2 = diag(\sigma_1^-, ..., \sigma_{k_1}^-),\\
\overline{\Upsilon}_3 &= diag(\delta_1^+, ..., \delta_{k_2}^+), \overline{\Upsilon}_4 = diag(\delta_1^-, ..., \delta_{k_2}^-).
\end{aligned} \tag{6.6}$$

Assumption 6.1.4 *All the eigenvalues of matrix* A_1 *are inside the unit circle.*

Remark 6.1.5 *Clearly, Lemma 2.5.7 contains a tighter lower bound for* $\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du$ *than Jensen's inequality. This inequality encompasses the Jensen one and also given to tractable LMI criteria to further reduce the conservatism over the existing results [28, 36, 56, 63, 66].*

For stability analysis of the Lur'e system (6.1) satisfying the conditions (6.2), (6.3) with interval time-varying delays, we consider a Lyapunov-Krasovskii functional as following:

$$V(t, x_t, \dot{x}_t) = \sum_{i=1}^{4} V_i(t), \tag{6.7}$$

where

$$
\begin{aligned}
V_1(t) =& \eta^T(t)P\eta(t) + \int_{t-h_1}^{t} \dot{x}^T(s)Q_0\dot{x}(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^T(s)J\dot{x}(s)ds \\
V_2(t) =& \int_{t-h_1}^{t} [x^T(t)\ x^T(s)]Q_1[x^T(t)\ x^T(s)]^T ds \\
&+ \int_{t-h(t)}^{t-h_1} [x^T(t)\ \ x^T(s)]Q_2[x^T(t)\ \ x^T(s)]^T ds \\
&+ \int_{t-h_2}^{t-h(t)} [x^T(t)\ \ x^T(s)]Q_3[x^T(t)\ \ x^T(s)]^T ds \\
V_3(t) =& \int_{t-h_1}^{t} \left\{ h_1(h_1-t+s)\dot{x}^T(s)W_1\dot{x}(s) + (h_1-t+s)^2\dot{x}^T(s)W_2\dot{x}(s) \right\} ds \\
&+ \int_{t-h_2}^{t-h_1} \left\{ h_{21}(h_2-t+s)\dot{x}^T(s)R_1\dot{x}(s) + (h_2-t+s)^2\dot{x}^T(s)R_2\dot{x}(s) \right\} ds \\
V_4(t) =& 2\sum_{i=1}^{n} \int_0^{E_i^T x} [k_i[f_i(s) - \sigma_i^-(s)] + l_i[\sigma_i^+(s) - f_i(s)]]ds \\
&+ 2\sum_{i=1}^{n} \int_0^{F_i^T x} [g_i[h_i(s) - \delta_i^-(s)] + t_i[\delta_i^+(s) - h_i(s)]]ds
\end{aligned}
\tag{6.8}
$$

where P is real matrices, $Q_0 > 0, Q_j > 0, W_q > 0, R_q > 0, J > 0 (j = 1, 2, 3; q = 1, 2)$, $K = \text{diag}(k_1, ..., k_n) > 0, L = \text{diag}(l_1, ..., l_n) > 0, G = \text{diag}(g_1, ..., g_n) > 0, T = \text{diag}(t_1, ..., t_n) > 0$ and $h_{21} = h_2 - h_1$, $\eta(t) = \text{col}\{x(t),\ x(t-h_1),\ \int_{t-h_2}^{t-h(t)} x(s)ds,\ \int_{t-h(t)}^{t-h_1} x(s)ds,\ \int_{t-h_1}^{t} x(s)ds\ \}$.

Remark 6.1.6 $(I.)$ *In [28], the augmented vector* $\eta(t) = [x(t), \int_{t-\tau_0}^{t} x(s)ds]$ *has been used to derive stability criteria. However in our work, we consider the augmented vector as* $[x(t), \int_{t-h_2}^{t-h(t)} x(s)ds, x(t-h_1), \int_{t-h(t)}^{t-h_1} x(s)ds]$. *One can see that the adoption of this new augmented variables. There are more cross terms of variables and more multiple integral terms in stability analysis which may reduce the conservatism.*
$(II.)$ *In [28, 62], the augmented Lyapunov matrix P must satisfy $P > 0$, but this condition is not necessary in our work, which can be seen from Lemma 6.1.7. below.*

For simplicity of presentation, we let ω_1, ω_2 are defined as in Lemma 2.5.8 and $\omega_3 = \frac{1}{h_1}\int_{t-h_1}^{t} x(s)ds$. Denoted by $\tilde{e}_i \in \mathcal{R}^{n\times 5n} (i = 1, \dots, 5)$ the block-row vectors of the $5n \times 5n$ identify matrix such that $x(t) = \tilde{e}_1\eta(t), x(t-h_1) = \tilde{e}_2\eta(t)$ and so on. Then we have the following result.

Lemma 6.1.7 *[68] For the Lyapunov-Krasovskii functional (6.8), there exist scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that*

$$\epsilon_1 \|x\|^2 \leq V(t, x_t, \dot{x}_t) \leq \epsilon_2 \|x_t\|_W^2 \tag{6.9}$$

if the following LMIs are satisfied

$$\begin{aligned} &\tilde{e}_1 P \tilde{e}_1^T > 0, \quad P_0 \geq 0, \quad \Lambda_1(h_1) + \Lambda_2(h_1) \geq 0, \\ &\Lambda_1(h_2) + \Lambda_2(h_2) \geq 0, \end{aligned} \tag{6.10}$$

where

$$\Lambda_1(h(t)) = \begin{cases} \Delta, & h_1 = 0 \\ \Delta + \frac{1}{h_1}\Gamma_2^T diag\{Q_0, 3Q_0\}\Gamma_2, & h_1 \neq 0 \end{cases} \tag{6.11}$$

$$\begin{aligned} \Lambda_2(h(t)) \;=\;& h_1[\tilde{e}_1^T \; \tilde{e}_5^T]Q_1[\tilde{e}_1^T \; \tilde{e}_5^T]^T + (h(t) - h_1)[\tilde{e}_1^T \; \tilde{e}_4^T]Q_2[\tilde{e}_1^T \; \tilde{e}_4^T]^T \\ &+ (h_2 - h(t))[\tilde{e}_1^T \; \tilde{e}_3^T]Q_3[\tilde{e}_1^T \; \tilde{e}_3^T]^T \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} \Gamma_1 &= col\{\tilde{e}_1, \tilde{e}_2, (h_2 - h(t))\tilde{e}_3, (h(t) - h_1)\tilde{e}_4, h_1\tilde{e}_5\}, \\ \Gamma_2 &= col\{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_1 + \tilde{e}_2 - 2\tilde{e}_5\}, \\ P_0 &= (\tilde{e}_4^T\tilde{e}_4 - \tilde{e}_3^T\tilde{e}_3)P(\tilde{e}_4^T\tilde{e}_4 - \tilde{e}_3^T\tilde{e}_3), \\ \Delta &= \Gamma_1^T P \Gamma_1 - \tilde{e}_1^T\tilde{e}_1 P \tilde{e}_1^T\tilde{e}_1. \end{aligned}$$

Based on Lemma 6.1.1, we derive absolutely stable of the system (6.1) satisfying the sector conditions (6.2) and (6.3) as in the following theorem 6.1.8.

Theorem 6.1.8 *The system (6.1) satisfying the sector condition (6.2), (6.3), for given scalars h_1, h_2, μ_1, μ_2 and δ is absolutely stable if there exist real matrices P, symmetric positive definite matrices $Q_0 > 0, Q_j > 0, W_q > 0, R_q > 0, J > 0 (j = 1, 2, 3; q = 1, 2)$, and $n \times n$ diagonal matrices $K > 0, L > 0, G > 0, T > 0, U > 0, V > 0$ such that (6.10) and the following LMIs hold:*

$$\begin{aligned} &\sum_{i=1, h(t)=h_1, \dot{h}(t)=\mu_1}^{4} \Sigma_i < 0, \qquad \sum_{i=1, h(t)=h_1, \dot{h}(t)=\mu_2}^{4} \Sigma_i < 0, \\ &\sum_{i=1, h(t)=h_2, \dot{h}(t)=\mu_1}^{4} \Sigma_i < 0, \qquad \sum_{i=1, h(t)=h_2, \dot{h}(t)=\mu_2}^{4} \Sigma_i < 0, \end{aligned} \tag{6.13}$$

$$\begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2) \begin{bmatrix} Z_3 & N_2 \\ N_3^T & W_2 \end{bmatrix} \geq 0, \tag{6.14}$$

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0, Z_1 \geq Z_2, \tag{6.15}$$

where $\tilde{R}_1 = diag\{R_1, 3R_1\}$;*and*

$$\begin{aligned}
\Sigma_1(h(t), \dot{h}(t)) &= \Delta_1^T P \Delta_2 + \Delta_2^T P \Delta_1 + \Delta_0^T Q_0 \Delta_0 - e_8^T Q_0 e_8 + \Delta_0^T J \Delta_0 \\
&\quad -(1 - \dot{\tau}(t)) e_{11}^T J e_{11} \\
\Sigma_2(h(t), \dot{h}(t)) &= \Psi_{20} + [h(t) - h_1]\Psi_{21} + [h_2 - h(t)]\Psi_{22} \\
\Sigma_3(h(t)) &= \tilde{\varphi}_1^T S_1 \tilde{\varphi}_2 + \tilde{\varphi}_2^T S_1^T \tilde{\varphi}_1 - \tilde{\varphi}_1^T \tilde{R}_1 \tilde{\varphi}_1 + (h_2 - h(t))^2 (Z_1 - Z_2) \\
&\quad +(h_2 - h(t))\Psi_{31} + (h(t) - h_1)\Psi_{32} + h_{21}^2 Z_2 - \tilde{\varphi}_2^T \tilde{R}_1 \tilde{\varphi}_2 \\
\Sigma_4 &= -\tilde{\varphi}_3^T \tilde{W}_1 \tilde{\varphi}_3 + \Delta_0^T (h_1^2 W_1 + h_1^2 W_2)\Delta_0 + 2h_1 N_3 (e_1 - e_7) \\
&\quad +e_8^T (h_{21}^2 R_1 + h_{21}^2 R_2) e_8 + 2h_1 (e_1 - e_7)^T N_3^T + h_1^2 Z_3 \\
&\quad +e_{10}^T [K - L] E \Delta_0 + \Delta_0^T E^T [K - L]^T e_{10} \\
&\quad +e_1^T E^T [\bar{\Upsilon}_1 L - \bar{\Upsilon}_2 K] E \Delta_0 + \Delta_0^T E^T [\bar{\Upsilon}_1 L - \bar{\Upsilon}_2 K]^T E e_1 \\
&\quad +e_9^T [G - T] F \Delta_0 + \Delta_0^T F^T [G - T]^T e_9 + e_1^T F^T [\bar{\Upsilon}_3 G - \bar{\Upsilon}_4 T] F \Delta_0 \\
&\quad +\Delta_0^T F^T [\bar{\Upsilon}_3 G - \bar{\Upsilon}_4 T]^T F e_1 - [e_1^T E^T U \Upsilon_1 E e_1 - 2 e_1^T E^T U \Upsilon_2 e_{10} \\
&\quad +e_{10}^T U e_{10}] - [e_1^T F^T V \Upsilon_3 F e_1 - 2 e_1^T F^T V \Upsilon_4 e_9 + e_9^T V e_9]
\end{aligned} \tag{6.16}$$

with $\tilde{e}_i \in \mathcal{R}^{n \times 11n} (i = 1, \ldots, 11)$ *the i-th row-block vector of the* $11n \times 11n$ *identity matrix.* $\tilde{W}_1 = diag\{W_1, 3W_1\}$;*and*

$$\begin{aligned}
\Psi_{20} &= [e_1^T\ e_3^T](Q_2 - Q_1)[e_1^T\ e_3^T]^T + h_1[\Delta_0^T\ 0]Q_1[e_1^T\ e_7^T]^T + h_1[e_1^T\ e_7^T]Q_1[\Delta_0^T\ 0]^T \\
&\quad - (1 - \dot{h}(t))[e_1^T\ e_2^T](Q_2 - Q_3)[e_1^T\ e_2^T]^T - [e_1^T\ e_4^T]Q_3[e_1^T\ e_4^T]^T + [e_1^T\ e_1^T]Q_1[e_1^T\ e_1^T]^T \\
\Psi_{21} &= [e_1^T\ e_6^T]Q_2[\Delta_0^T\ 0]^T + [\Delta_0^T\ 0]Q_2[e_1^T\ e_6^T]^T \\
\Psi_{22} &= [e_1^T\ e_5^T]Q_3[\Delta_0^T\ 0]^T + [\Delta_0^T\ 0]Q_3[e_1^T\ e_5^T]^T \\
\Psi_{31} &= 2N_1(e_2 - e_5) + 2N_2(e_3 - e_2) + 2(e_3 - e_2)^T N_2^T + 2(e_2 - e_5)^T N_1^T \\
\Psi_{32} &= 2N_2(e_3 - e_6) + 2(e_3 - e_6)^T N_2^T \\
\tilde{\varphi}_1 &= col\{e_2 - e_4, e_2 + e_4 - 2e_5\} \\
\tilde{\varphi}_2 &= col\{e_3 - e_2, e_3 + e_2 - 2e_6\} \\
\tilde{\varphi}_3 &= col\{e_1 - e_3, e_1 + e_3 - 2e_7\} \\
\Delta_1 &= col\{e_1, e_3, (h_2 - h(t))e_5, (h(t) - h_1)e_6, h_1 e_7\} \\
\Delta_2 &= col\{\Delta_0, e_8, (1 - \dot{h}(t))e_2 - e_4, e_3 - (1 - \dot{h}(t))e_2, e_1 - e_3\}.
\end{aligned}$$

For simplicity of presentation, we denote

$$\Theta = col\{x(t), x(t-h(t)), x(t-h_1), x(t-h_2), \omega_1(t), \omega_2(t), \omega_3(t), \dot{x}(t-h_1), h(\sigma(t)), f(\omega(t)), \dot{x}(t-\tau(t))\}.$$

Then system 6.1 may be rewritten as $\dot{x}(t) = \Delta_0\Theta(t)$,
$\Delta_0 = A_1e_{11} + Ae_1 + Be_2 + Ce_{10} + De_9$. Taking the derivative of $V(t)$ as in (6.8) along the solution of system (6.1), we obtain

$$\begin{aligned}
\dot{V}_1(t) &= 2\eta^T(t)P\dot{\eta}(t) + \dot{x}^T(t)Q_0\dot{x}(t) - \dot{x}^T(t-h_1)Q_0\dot{x}(t-h_1) + \dot{x}^T(t)J\dot{x}(t) \\
&\quad -(1-\dot{\tau}(t))\dot{x}^T(t-\tau(t))J\dot{x}(t-\tau(t)), \\
\dot{V}_2(t) &= [x^T(t)\ x^T(t)]Q_1[x^T(t)\ x^T(t)]^T - [x^T(t)\ x^T(t-h_1)]Q_1 \\
&\quad \times[x^T(t)\ x^T(t-h_1)]^T + 2\int_{t-h_1}^{t}[x^T(t)\ x^T(s)]Q_1[\dot{x}(t)^T\ 0]^T ds \\
&\quad +[x^T(t)\ x^T(t-h_1)]Q_2[x^T(t)\ x^T(t-h_1)]^T \\
&\quad -(1-\dot{h}(t))[x^T(t)\ x^T(t-h(t))]Q_2[x^T(t)\ x^T(t-h(t))]^T \qquad (6.17)\\
&\quad +2\int_{t-h(t)}^{t-h_1}[x^T(t)\ x^T(s)]Q_2[\dot{x}^T(t)\ 0]^T ds + [x^T(t)\ x^T(t-h_2)]Q_3 \\
&\quad \times[x^T(t)\ x^T(t-h_2)]^T - (1-\dot{h}(t))[x^T(t)\ x^T(t-h(t))]Q_3 \\
&\quad \times[x^T(t)\ x^T(t-h(t))]^T + 2\int_{t-h_2}^{t-h(t)}[x^T(t)\ x^T(s)]Q_3[\dot{x}^T(t)\ 0]^T ds, \\
\dot{V}_3(t) &= \Theta^T(t)(h_1^2\Delta_0^TW_1\Delta_0 + h_1^2\Delta_0^TW_2\Delta_0)\Theta(t) - h_1\int_{t-h_1}^{t}\dot{x}^T(s)W_1\dot{x}(s)ds \\
&\quad -2\int_{t-h_1}^{t}(h_1-t+s)\dot{x}^T(s)W_2\dot{x}(s)ds + h_{21}^2\dot{x}^T(t-h_1)R_1\dot{x}(t-h_1) \\
&\quad +h_{21}^2\dot{x}^T(t-h_1)R_2\dot{x}(t-h_1) - h_{21}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad -2\int_{t-h_2}^{t-h_1}(h_2-t+s)\dot{x}^T(s)R_2\dot{x}(s)ds, \\
\dot{V}_4(t) &= 2f^T(Ex(t))[K-L]E\dot{x}(t) + 2x^T(t)E^T[\bar{\Upsilon}_1L - \bar{\Upsilon}_2K]E\dot{x} \\
&\quad +2h^T(Fx(t))[G-T]F\dot{x}(t) + 2x^T(t)F^T[\bar{\Upsilon}_3G - \bar{\Upsilon}_4T]F\dot{x}(t).
\end{aligned}$$

Form (6.6), we get

$$\begin{aligned}
0 \ \leq\ & -[x^T(t)E^TU\Upsilon_1Ex(t) - 2x^T(t)E^TU\Upsilon_2f(Ex(t)) \\
& +f^T(Ex(t))Uf(Ex(t))] - [x^T(t)F^TV\Upsilon_3Fx(t) \qquad (6.18)\\
& -2x^T(t)F^TV\Upsilon_4h(Fx(t)) + h^T(Fx(t))Vh(Fx(t))].
\end{aligned}$$

With the consideration of some terms of $\dot{V}_2(t), \dot{V}_3(t)$, we obtain the following equalities:

$$\int_{t-h_1}^{t} [x^T(t)\ x^T(s)]Q_1[\dot{x}(t)^T\ 0]^T ds = [\int_{t-h_1}^{t} x^T(t)ds \quad \int_{t-h_1}^{t} x^T(s)ds]Q_1[\dot{x}^T(t)\ \ 0]^T$$
$$= h_1[x^T(t)\ \ \omega_3^T]Q_1[\dot{x}^T(t)\ \ 0]^T, \tag{6.19}$$

$$\int_{t-h(t)}^{t-h_1} [x^T(t)\ x^T(s)]Q_2[\dot{x}(t)^T\ 0]^T ds = [\int_{t-h(t)}^{t-h_1} x^T(t)ds \quad \int_{t-h(t)}^{t-h_1} x^T(s)ds]Q_2[\dot{x}^T(t)\ \ 0]^T$$
$$= (h(t)-h_1)[x^T(t)\ \ \omega_2^T]Q_2[\dot{x}^T(t)\ \ 0]^T, \tag{6.20}$$

and

$$\int_{t-h_2}^{t-h(t)} [x^T(t)\ x^T(s)]Q_3[\dot{x}(t)^T\ 0]^T ds = [\int_{t-h_2}^{t-h(t)} x^T(t)ds \quad \int_{t-h_2}^{t-h(t)} x^T(s)ds]Q_3[\dot{x}^T(t)\ \ 0]^T$$
$$= (h_2-h(t))[x^T(t)\ \ \omega_1^T]Q_3[\dot{x}^T(t)\ \ 0]^T. \tag{6.21}$$

Utilizing Lemma 2.5.7, we obtain

$$-\int_{t-h_1}^{t} \dot{x}^T(s)h_1W_1\dot{x}(s)ds \le -[x(t)-x(t-h_1)]^T W_1[x(t)-x(t-h_1)] - 3\tilde{\Omega}_1^{\ T} W_1\tilde{\Omega}_1, \tag{6.22}$$

where $\tilde{\Omega}_1 = x(t)+x(t-h_1)-2\omega_3$. Applying [68], we obtain

$$\begin{aligned}&-2\int_{t-h_1}^{t} (h_1-t+s)\dot{x}^T(s)W_2\dot{x}(s)ds \\ &\le h_1^2\Theta^T(t)Z_3\Theta(t) + 2h_1\Theta^T(t)N_3[x(t)-\omega_3] + 2h_1[x(t)-\omega_3]^T N_3^T\Theta(t),\end{aligned} \tag{6.23}$$

$$-\int_{t-h_2}^{t-h_1} \dot{x}^T(s)h_{21}R_1\dot{x}(s)ds \le 2\varphi_{11}^T S_1\varphi_{21} - \varphi_{11}^T\tilde{R}_1\varphi_{11} - \varphi_{21}^T\tilde{R}_1\varphi_{21} \tag{6.24}$$

and

$$\begin{aligned}-2\int_{t-h_2}^{t-h_1} (h_2-t+s)\dot{x}^T(s)R_2\dot{x}(s)ds \ \ \le\ \ & (h_2-h(t))^2\Theta^T(t)Z_1\Theta(t) \\ & +4(h_2-h(t))\Theta^T(t)N_1[x(t-h(t))-\omega_1] \\ & +[(h_2-h_1)^2-(h_2-h(t))^2]\Theta^T(t)Z_2\Theta(t) \\ & +4\Theta^T(t)N_2[(h_2-h(t))[x(t-h_1) \\ & -x(t-h(t))] + (h(t)-h_1)[x(t-h_1) \\ & -\omega_2(t)]].\end{aligned} \tag{6.25}$$

Hence, according to (6.17)-(6.25) we get

$$
\begin{aligned}
\dot{V}(t,x_t,\dot{x}_t) \leq\ & 2\eta^T(t)P\dot{\eta}(t) + \dot{x}^T(t)Q_0\dot{x}(t) - \dot{x}^T(t-h_1)Q_0\dot{x}(t-h_1) + \dot{x}^T(t)J\dot{x}(t) \\
& -(1-\delta)\dot{x}^T(t-\tau(t))J\dot{x}(t-\tau(t)) + [x^T(t)\ x^T(t)]Q_1[x^T(t)\ x^T(t)]^T \\
& -[x^T(t)\ x^T(t-h_1)]Q_1[x^T(t)\ x^T(t-h_1)]^T + 2h_1[x^T(t)\ \ \omega_3^T]Q_1 \\
& \times[\dot{x}^T(t)\ \ 0]^T + [x^T(t)\ x^T(t-h_1)]Q_2[x^T(t)\ x^T(t-h_1)]^T \\
& -(1-\dot{h}(t))[x^T(t)\ x^T(t-h(t))]Q_2[x^T(t)\ x^T(t-h(t))]^T \\
& +2(h(t)-h_1)[x^T(t)\ \ \omega_2^T]Q_2[\dot{x}^T(t)\ \ 0]^T + [x^T(t)\ x^T(t-h_2)]Q_3 \\
& \times[x^T(t)\ x^T(t-h_2)]^T - (1-\dot{h}(t))[x^T(t)\ x^T(t-h(t))]Q_3 \\
& \times[x^T(t)\ x^T(t-h(t))]^T + 2(h_2-h(t))[x^T(t)\ \ \omega_1^T]Q_3 \\
& \times[\dot{x}^T(t)\ \ 0]^T\Theta^T(t)(h_1^2\Delta_0^TW_1\Delta_0 + h_1^2\Delta_0^TW_2\Delta_0)\Theta(t) \\
& -[x(t)-x(t-h_1)]^TW_1[x(t)-x(t-h_1)] - 3\tilde{\Omega_1}^TW_1\tilde{\Omega_1} \\
& +h_1^2\Theta^T(t)Z_3\Theta(t) + 2h_1\Theta^T(t)N_3[x(t)-\omega_3] + 2h_1[x(t) \\
& -\omega_3]^TN_3^T\Theta(t) + h_{21}^2\dot{x}^T(t-h_1)R_1\dot{x}(t-h_1) \\
& \\
& +h_{21}^2\dot{x}^T(t-h_1)R_2\dot{x}(t-h_1) + 2\varphi_{11}^TS_1\varphi_{21} - \varphi_{11}^T\tilde{R}_1\varphi_{11} - \varphi_{21}^T\tilde{R}_1\varphi_{21} \\
& +(h_2-h(t))^2\Theta^T(t)Z_1\Theta(t) + 4(h_2-h(t))\Theta^T(t)N_1[x(t-h(t)) \\
& -\omega_1] + [(h_2-h_1)^2 - (h_2-h(t))^2]\Theta^T(t)Z_2\Theta(t) \\
& +4\Theta^T(t)N_2[(h_2-h(t))[x(t-h_1)-x(t-h(t))] \\
& +(h(t)-h_1)[x(t-h_1)-\omega_2(t)]] + 2f^T(Ex(t))[K-L]E\dot{x}(t) \\
& +2x^T(t)E^T[\bar{\Upsilon}_1L - \bar{\Upsilon}_2K]E\dot{x} + 2h^T(Fx(t))[G-T]F\dot{x}(t) \\
& +2x^T(t)F^T[\bar{\Upsilon}_3G - \bar{\Upsilon}_4T]F\dot{x}(t) - [x^T(t)E^TU\Upsilon_1Ex(t) \\
& -2x^T(t)E^TU\Upsilon_2f(Ex(t)) + f^T(Ex(t))Uf(Ex(t))] \\
& -[x^T(t)F^TV\Upsilon_3Fx(t) - 2x^T(t)F^TV\Upsilon_4h(Fx(t)) \\
& +h^T(Fx(t))Vh(Fx(t))] \\
\dot{V}(t,x_t,\dot{x}_t) \leq\ & \Theta^T(t)\Sigma(h(t),\dot{h}(t))\Theta(t) \qquad (6.26)
\end{aligned}
$$

where $\Sigma(h(t),\dot{h}(t)) \triangleq \sum_{i=1}^4 \Sigma_i$. Clearly, $\Sigma(h(t),\dot{h}(t))$ can be rewritten as $\Sigma(h(t),\dot{h}(t)) = h^2(t)\Pi_0 + h(t)\Pi_1 + \Pi_2$ where $\Pi_0 = Z_1 - Z_2$ and Π_1 and Π_2 are $h(t)-$ independent real matrices. From the fact that if $Z_1 - Z_2 \geq 0$ and (6.13) hold, it follow from Lemma 2.5.13 that $\Sigma(h(t),\dot{h}(t)) < 0, \forall h(t) \in [h_1,h_2], \forall\dot{h}(t) \in [\mu_1,\mu_2]$ then $\dot{V}(t,x_t) \leq -\lambda\|x(t)\|$ for some $\lambda > 0, \forall x(t) \neq 0$. We conclude that (6.1) satisfies conditions (6.2) and (6.3) is absolutely stable. □

To show effectiveness for our result, we consider the following. When $D = 0$, time-varying delay system in (6.1) reduces to

$$\dot{x}(t) = A_1\dot{x}(t-\tau(t)) + Ax(t) + Bx(t-h(t)) + Cf(\omega(t)). \tag{6.27}$$

In the following, we present a stability criterion for $h_1 = 0$. We consider the following Lyapunov-Krasovskii funcional candidate

$$\begin{aligned}
\hat{V}(t, x_t, \dot{x}_t) \;=\; & \bar{\eta}^T(t)P\bar{\eta}(t) + \int_{t-\tau(t)}^{t} \dot{x}^T(s)J\dot{x}(s)ds + \int_{t-h(t)}^{t} [x^T(t) \;\; x^T(s)]Q_2 \\
& \times [x^T(t) \;\; x^T(s)]^T ds + \int_{t-h_2}^{t-h(t)} [x^T(t) \;\; x^T(s)]Q_3[x^T(t) \;\; x^T(s)]^T ds \\
& + \int_{t-h_2}^{t} \Big\{ h_2(h_2 - t + s)\dot{x}^T(s)R_1\dot{x}(s) + (h_2 - t + s)^2\dot{x}^T(s)R_2\dot{x}(s) \Big\}\, ds \\
& + 2\sum_{i=1}^{n} \int_0^{E_i^T x} [k_i[f_i(s) - \sigma_i^-(s)] + l_i[\sigma_i^+(s) - f_i(s)]]ds,
\end{aligned} \tag{6.28}$$

where $\bar{\eta} = [x(t) \;\; \int_{t-h_2}^{t-h(t)} x(s)ds \;\; \int_{t-h(t)}^{t} x(s)ds]$ and by using the following inequality
$-[x^T(t)E^TU\Upsilon_1Ex(t) - 2x^T(t)E^TU\Upsilon_2f(Ex(t)) + f^T(Ex(t))Uf(Ex(t))] \geq 0$ instead of (6.18).

Corollary 6.1.9 *The system (6.27) satisfying the sector condition (6.2), for given scalars h_2, μ_1, μ_2 and δ is absolutely stable if there exist $P = P^T$ with $P_{11} > 0$ symmetric positive definite matrices $R_q > 0, J > 0 (q = 1, 2)$, and $n \times n$ diagonal matrices $K > 0, L > 0, U > 0$ such that (6.10) and the following LMIs hold:*

$$\begin{cases} F(h(t), \dot{h}(t)) + h_2\Omega_1 + (1 - \dot{h}(t))\Omega_3 + \Omega_4 < 0, \\ \qquad h(t) = 0, \dot{h}(t) = \mu_1, \mu_2 \end{cases} \tag{6.29}$$

$$\begin{cases} F(h(t), \dot{h}(t)) + h_2\Omega_2 + (1 - \dot{h}(t))\Omega_3 + \Omega_4 < 0, \\ \qquad h(t) = h_2, \dot{h}(t) = \mu_1, \mu_2 \end{cases} \tag{6.30}$$

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0, \begin{bmatrix} P_{22} & P_{23} \\ * & P_{33} \end{bmatrix} \geq 0, Z_1 \geq Z_2, \tag{6.31}$$

$$\begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2), \tag{6.32}$$

$$\begin{bmatrix} h_2 Q_{21} & h_2(P_{13}+Q_{22}) \\ * & h_2^2 P_{33} + h_2 Q_{23} \end{bmatrix} \geq 0, \tag{6.33}$$

$$Q_j = \begin{bmatrix} Q_{j1} & Q_{j2} \\ * & Q_{j3} \end{bmatrix} \geq 0, (i = 2, 3), \tag{6.34}$$

$$\begin{bmatrix} h_2 Q_{11} & h_2(P_{12}+Q_{12}) \\ * & h_2^2 P_{22} + h_2 Q_{13} \end{bmatrix} \geq 0, \tag{6.35}$$

where $\tilde{R}_1 = diag\{R_1, 3R_1\}$

$$\begin{aligned}
\mathcal{F}(h(t), \dot{h}(t)) &= \mathcal{F}_1^T(h(t))P\mathcal{F}_2(\dot{h}(t)) + \mathcal{F}_2^T(\dot{h}(t))P\mathcal{F}_1(h(t)) \\
\Omega_1 &= [\hat{e}_1^T \ \hat{e}_4^T]Q_3\emptyset_0 + \emptyset_0^T Q_3[\hat{e}_1^T \ \hat{e}_4^T]^T + h_2(Z_1 - Z_2) + 2N_1[\hat{e}_2 - \hat{e}_4] \\
&\quad + 2[\hat{e}_2 - \hat{e}_4]^T N_1^T + 2N_2[\hat{e}_1 - \hat{e}_2] + 2[\hat{e}_1 - \hat{e}_2]^T N_2^T \\
\Omega_2 &= [\hat{e}_1^T \ \hat{e}_5^T]Q_2\emptyset_0 + \emptyset_0^T Q_2[\hat{e}_1^T \ \hat{e}_5^T]^T + 2N_2[\hat{e}_1 - \hat{e}_5] + 2[\hat{e}_1 - \hat{e}_5]^T N_2^T \\
\Omega_3 &= [\hat{e}_1^T \ \hat{e}_2^T](Q_3 - Q_2)[\hat{e}_1^T \ \hat{e}_2^T]^T \\
\Omega_4 &= \emptyset_0^T J\emptyset_0 - (1-\delta)\hat{e}_7^T J\hat{e}_7 + [\hat{e}_1^T \ \hat{e}_1^T]Q_2[\hat{e}_1^T \ \hat{e}_1^T]^T - [\hat{e}_1^T \ \hat{e}_3^T]Q_3[\hat{e}_1^T \ \hat{e}_3^T]^T \\
&\quad + h_2^2\emptyset_0^T(R_1 + R_2)\emptyset_0 + \emptyset_1^T S_1\emptyset_2 + \emptyset_2^T S_1^T\emptyset_1 - \emptyset_1^T\tilde{R}_1\emptyset_1 - \emptyset_2^T\tilde{R}_1\emptyset_2 \\
&\quad + h_2^2 Z_2 + \hat{e}_6^T[K - L]E\emptyset_0 + \emptyset_0^T E^T[K - L]^T\hat{e}_6 + \hat{e}_1 E^T[\bar{\Upsilon}_1 L - \bar{\Upsilon}_2 K]E\emptyset_0 \\
&\quad + \emptyset_0^T E^T[\bar{\Upsilon}_1 L - \bar{\Upsilon}_2 K]^T E\hat{e}_1^T - [\hat{e}_1^T E^T U\Upsilon_1 E\hat{e}_1 - \hat{e}_1^T U\Upsilon_2\hat{e}_6 - \hat{e}_6^T\Upsilon_2^T U^T\hat{e}_1 \\
&\quad + \hat{e}_6^T U\hat{e}_6],
\end{aligned}$$

with $\hat{e}_1 = [I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], ..., \hat{e}_7 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ I]$*; and*

$$\begin{aligned}
\mathcal{F}_1(h(t)) &= col\{\hat{e}_1, (h_2 - h(t))\hat{e}_4, h(t)\hat{e}_5\} \\
\mathcal{F}_2(\dot{h}(t)) &= col\{\emptyset_0, (1 - \dot{h}(t))\hat{e}_2 - \hat{e}_3, \hat{e}_1 - (1 - \dot{h}(t))\hat{e}_2\} \\
\emptyset_0 &= A_1\hat{e}_7 + A\hat{e}_1 + B\hat{e}_2 + C\hat{e}_6 \\
\emptyset_1 &= col\{\hat{e}_2 - \hat{e}_3, \hat{e}_2 + \hat{e}_3 - 2\hat{e}_4\} \\
\emptyset_2 &= col\{\hat{e}_1 - \hat{e}_2, \hat{e}_1 + \hat{e}_2 - 2\hat{e}_5\}.
\end{aligned} \tag{6.36}$$

6.2 Numerical examples

In this section, we provide numerical examples to show the effectiveness of our theoretical results.

Example 6.2.1 Consider the following Lur'e system with time-varying delays which is studied in [66], [63], [56], [28]:

$$\dot{x}(t) \;\; = \;\; A_1\dot{x}(t-\tau(t)) + Ax(t) + Bx(t-h(t)) + Cf(\omega(t)), \qquad (6.37)$$

with the following parameters:

$$A_1 \;\; = \;\; \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix}, \; C = \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix},$$
$$E \;\; = \;\; \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$

From Corollary 6.1.9, the maximum allowable value of h_2 is given in Table 6.1 when $h_1 = 0$, $-\mu_1 = \mu_2 = \mu$ and $\delta = 0.9$. Note that the constraint on the augmented Lyapunov matrix $P > 0$ is not required, so our conditions are generally less conservative than [28].

Table 6.1: Maximum allowable upper bounds h_2 of neutral-type neural networks with $h_1 = 0, \delta = 0.9$ and different values of μ.

δ	Methods	μ			
		0.2	0.4	0.6	0.8
0.5	[66]	1.841	1.315	0.790	0.632
	[63]	2.154	1.704	1.383	1.009
	[56]	2.456	1.801	1.482	1.116
	[28]	2.462	1.810	1.490	1.122
	Corollary 6.1.9	> 10000	> 10000	2.8080	2.1934
0.9	[66]	0.124	0.108	0.099	0.098
	[63]	0.112	0.109	0.109	0.109
	[56]	0.113	0.110	0.110	0.110
	[28]	0.125	0.113	0.113	0.112
	Corollary 6.1.9	0.510	0.451	0.414	0.388

Example 6.2.2 Consider the following neutral system with time-varying delays which is considered in [36]:

$$\dot{x}(t) \;\; = \;\; A_1\dot{x}(t-\tau(t)) + Ax(t) + Bx(t-h(t))$$

with the following parameters:

$$A_1 \;\; = \;\; \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

From Theorem 6.1.8 to the above system, we obtain maximum allowable upper bounds h_2 of neutral-type neural networks with $h_1 = 0$ as listed in Table 6.2. It can be found that the maximum upper bounds $h_2 = 4.2365$, which is larger than in [36]. This means that the proposed ideas in theorem 6.1.8 is effective in reducing the conservatism of stability criterion.

Table 6.2: Maximum allowable upper bounds h_2 of neutral-type neural networks with $h_1 = 0$ and $\tau(t) = h(t)$.

Methods	$h_2[\tau(t) = h(t)]$
[36]	0.985
Theorem 6.1.8	4.2365