CHAPTER 7
Novel delay-dependent exponential stability criteria for neutral-type neural networks with time-varying discrete and neutral delays

In this chapter, we investigate the exponential stability problem for uncertain neutral-type neural networks with both interval time-varying delays and generalized activation functions. We note that discrete and neutral delays are both time-varying where the discrete delay is not necessarily differentiable and the information on derivative of neutral delay is not required. To the best of our knowledge, this is the first study under this conditions on discrete and neutral delays. Meanwhile, this restriction is required in some existing result, see [37, 46, 48, 69]. Furthermore, a new activation function which has not been considered yet in other literature is proposed and utilized to reduce the conservatism of stability criterion. Based on the Lyapunov-Krasovskii functional, we derive new delay-dependent stability conditions in LMIs which can be solved by various available algorithms. Finally, numerical examples are given to illustrate the effectiveness of theoretical results and to show less conservativeness than some existing results in the literature.

7.1 Exponential stability criteria for neutral-type neural networks

Consider the following neural networks with interval time-varying discrete and neutral delays:

$$\begin{aligned} \dot{u}(t) - D_1\dot{u}(t-\eta(t)) &= -Au(t) + W_0 g(u(t)) + W_1 g(u(t-h(t))) \\ &\quad + W_2 g(\dot{u}(t-\eta(t))) + J, \end{aligned} \tag{7.1}$$

where $u(t) = [u_1(t)\ u_2(t)\ .\ .\ .\ u_n(t)]^T \in \mathcal{R}^n$ is the neural state vector. $A = \mathrm{diag}\{a_1, a_2,\ .\ .\ .\ , a_n\} > 0$ is the state feedback coefficient matrix; $g(u(t)) =$

$[g_1(u_1(t))\ g_2(u_2(t))\ .\ .\ .\ g_n(u_n(t))]^T$ is the activation of neurons. W_0 is the connection weight matrix and W_1, W_2 are the delayed connection weight matrices; $J = [J_1\ J_2\ .\ .\ .\ J_n]^T$ represents the external inputs.

Assumption 7.1.1 *$h(t), \eta(t)$ are time-varying discrete and neutral delays respectively satisfying:*

$$0 \le h_1 \le h(t) \le h_2; \quad 0 \le \eta(t) \le \eta_2,$$

where h_1, h_2 and η_2 are known constants, where $h(t)$ is not necessarily differentiable.

Remark 7.1.2 *In this work, $h(t)$ and $\eta(t)$ are continuous functions satisfying $0 \le h_1 \le h(t) \le h_2; \quad 0 \le \eta(t) \le \eta_2, \forall t > 0$. The discrete delay, $h(t)$ is not necessarily differentiable and information on derivative of neutral delay is not required in this work. However, If derivative of discrete or neutral delays is assumed to be bounded, such as $0 \le h_1 \le h(t) \le h_2, \dot{h}(t) \le a_1; \quad 0 \le \eta(t) \le \eta_2, \dot{\eta}(t) \le a_2, \forall t > 0$, where a_1, a_2 are constants. We can add some more terms such as $\int_{t-h(t)}^{t} e^{2\alpha s} x^T(s) Q x(s)\, ds, \int_{t-\eta(t)}^{t} e^{2\alpha s} x^T(s) S x(s)\, ds$ to derive stability conditions.*

It should be noted that the activation functions $g_i(.)(i = 1, ..., n)$ satisfy the following assumption.

Assumption 7.1.3 *For any $j = \{1, 2, ..., n\}$, $g_j(0) = 0$ and there exist constants F_j^- and F_j^+ such that*

$$F_j^- \le \frac{g_j(\alpha_1) - g_j(\alpha_2)}{\alpha_1 - \alpha_2} \le F_j^+$$

for any $\alpha_1, \alpha_2 \in \mathcal{R}, \alpha_1 \ne \alpha_2$. Moreover, we assume that the initial condition of system (7.1) has the form $u(t) = \phi(t), \dot{u}(t) = \varphi(t), t \in [-m, 0], m = max\{h_2, \eta_2\}$ where function $\phi(t), \varphi(t) \in C[-m, 0]$.

Then, by using the well-known Brouwers fixed-point theorem, one can easily prove that there exists at least one equilibrium point, say u^*, for system (7.1). For the sake of simplicity in our study, we use the transformation $x(\cdot) = u(\cdot) - u^*$, then (7.1) becomes

$$\begin{aligned} \dot{x}(t) - D_1\dot{x}(t - \eta(t)) &= -Ax(t) + W_0 f(x(t)) + W_1 f(x(t - h(t))) \\ &\quad + W_2 f(\dot{x}(t - \eta(t))) \end{aligned} \tag{7.2}$$

where $x(t) = [x_1(t), ..., x_n(t)] \in \mathcal{R}^n$ is the state vector of the transformed system and u^* is an equilibrium point of system (7.1). Note that $f_j(x_j(t)) = g_j(x_j(t) + u_j^*) - g_j(u_j^*)$ with $f_j(0) = 0, (j = 1, 2, ..., n)$. From Assumption 7.1.3, $f_j(\cdot)$ satisfies the following condition

Assumption 7.1.4 *For any $j = \{1, 2, ..., n\}$, $f_j(0) = 0$ and there exist constants F_j^- and F_j^+ such that*

$$F_j^- \leq \frac{f_j(s)}{s} \leq F_j^+ \quad \forall s \in \mathcal{R}, \quad f_j(0) = 0, \quad (j = 1, 2, ..., n).$$

It is obvious that the equilibrium point of system (7.1) is exponentially stable if and only if the zero solution of system (7.2) is exponentially stable.

Rewrite system (7.2) in the following descriptor system:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ y(t) &= -Ax(t) + W_0 f(x(t)) + W_1 f(x(t - h(t))) + W_2 f(\dot{x}(t - \eta(t))) + D_1 \dot{x}(t - \eta(t)). \end{aligned} \tag{7.3}$$

For presentation convenience, in the following, we denote
$F_1 = \mathrm{diag}(F_1^-, F_2^-, ..., F_n^-)$, $F_2 = \mathrm{diag}(F_1^+, F_2^+, ..., F_n^+)$,
$F_3 = \mathrm{diag}(F_1^- F_1^+, F_2^- F_2^+, ..., F_n^- F_n^+)$, $F_4 = \mathrm{diag}(\frac{F_1^- + F_1^+}{2}, \frac{F_2^- + F_2^+}{2}, ..., \frac{F_n^- + F_n^+}{2})$.

Theorem 7.1.5 *Given $\alpha > 0$. The system (7.3) is $\alpha-$exponentially stable if there exist symmetric positive definite matrices P_1, Q, R, U, S positive diagonal matrices $G_1 = diag(\lambda_1, \lambda_2, ..., \lambda_m)$, $G_2 = diag(\delta_1, \delta_2, ..., \delta_m)$, $L_j (j = 1, 2, 3)$, and any matrices $P_k (k = 2, ..., 11), H$ such that the following LMIs hold:*

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \tag{7.5}$$

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & M_{18} & M_{19} & M_{1a} & 0 \\ * & M_{22} & 0 & M_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & M_{33} & M_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & M_{44} & M_{45} & M_{46} & M_{47} & M_{48} & M_{49} & M_{4a} & 0 \\ * & * & * & * & M_{55} & M_{56} & M_{57} & M_{58} & M_{59} & M_{5a} & 0 \\ * & * & * & * & * & M_{66} & M_{67} & M_{68} & M_{69} & M_{6a} & 0 \\ * & * & * & * & * & * & M_{77} & 0 & 0 & M_{7a} & 0 \\ * & * & * & * & * & * & * & M_{88} & 0 & M_{8a} & 0 \\ * & * & * & * & * & * & * & * & M_{99} & M_{9a} & 0 \\ * & * & * & * & * & * & * & * & * & M_{10a} & 0 \\ * & * & * & * & * & * & * & * & * & * & M_{11b} \end{bmatrix},$$

where
$M_{11} = -P_7A - A^TP_7 - 4\alpha F_1G_1 + 4\alpha G_2F_2 + S + Q + Q^T - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R - F_3L_1 + 2\alpha P_1$, $M_{12} = e^{-2\alpha h_1}R$, $M_{13} = e^{-2\alpha h_2}R$, $M_{14} = P_2^T - A^TP_8^T$, $M_{15} = P_2 + F_2G_2 - F_1G_1 - A^TP_9^T$, $M_{16} = -A^TP_{10}^T + P_7D_1 + P_7W_2$, $M_{17} = 2\alpha G_1^T - 2\alpha F_2^T + F_4L_1 + P_7W_0$, $M_{18} = P_7W_1$, $M_{19} = P_7W_2$, $M_{1a} = P_1 - P_2 - P_7 - A^TP_{11}^T - P_7 - A^TP_{11}^T$, $M_{22} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U$, $M_{24} = e^{-2\alpha h_2}U$, $M_{33} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U$, $M_{34} = e^{-2\alpha h_2}U$, $M_{44} = -2e^{-2\alpha h_2}U - F_3L_2$, $M_{45} = P_3$, $M_{46} = P_8W_2 + P_8D_1$, $M_{47} = P_8W_0$, $M_{48} = P_8W_1 + F_4L_2$, $M_{49} = P_8W_2$, $M_{4a} = -P_3 - P_8$, $M_{55} = h_1^2R + h_2^2R + (h_2 - h_1)^2U + P_4 + P_4^T - H - H^T$, $M_{56} = P_5^T + P_9W_2 + P_9D_1$, $M_{57} = F_1 - F_2 + P_9W_0$, $M_{58} = P_9W_1$, $M_{59} = P_9W_2$, $M_{5a} = -P_4 - P_9 + H + P_6^T$, $M_{66} = -F_3L_3 + P_{10}W_2 + P_{10}D_1 + W_2^TP_{10}^T + D_1^TP_{10}^T - L_3$, $M_{67} = P_{10}W_0$, $M_{68} = P_{10}W_1$, $M_{69} = P_{10}W_2 + F_4L_3$, $M_{6a} = -P_5 - P_{10} + D_1^TP_{11}^T + W_2^TP_{11}^T$, $M_{77} = -L_1, M_{7a} = W_0^TP_{11}^T, M_{88} = -L_2$, $M_{8a} = W_1^TP_{11}^T$, $M_{99} = -L_3$, $M_{9a} = W_2^TP_{11}^T$, $M_{10a} = -P_6 - P_6^T - P_{11} - P_{11}^T$, $M_{11b} = -e^{-2\alpha\eta_2}S$.
The solution $x(t)$ of the system satisfies,

$$\|x(t)\| \leq \sqrt{\frac{a\|\phi\|^2 + b\|M_1\|^2}{\lambda_{min}(P_1)}}.e^{-\alpha t}, \tag{7.6}$$

where $a = \lambda_{max}(P_1) + 2h_2\lambda_{max}(R) + h_2\lambda_{max}(U) + 2\lambda_{max}(F_2 - F_1)(\lambda_{max}(G_1) + \lambda_{max}(G_2))$,
$b = 2\lambda_{max}(Q)\frac{1-e^{-2\alpha h_2}}{2\alpha} + \lambda_{max}(S)\frac{1-e^{-2\alpha\eta_2}}{2\alpha} + 2h_2\lambda_{max}(R)\frac{1-e^{-2\alpha h_2}}{2\alpha} + h_2\lambda_{max}(U)\frac{1-e^{-2\alpha h_2}}{2\alpha}$,
$\|M_1\| = \sup_{-m\leq s\leq 0}\|x(s)\|$.

Proof. Rewrite (7.2) to descriptor system (7.3), we consider the following Lyapunov-Krasovskii functional

$$V(t,x_t)=\sum_{i=1}^{8}V_i, \tag{7.7}$$

where

$$\begin{aligned}
V_1 &= e^{2\alpha t}\xi^T(t)E^TP\xi(t)=e^{2\alpha t}x^T(t)P_1x(t),\\
V_2 &= 2e^{2\alpha t}\Sigma_{i=1}^{n}\{\int_0^{x_i(t)}\lambda_i(f_i(s)-r_i^-(s))ds+\int_0^{x_i(t)}\delta_i(r_i^+(s)+f_i(s))ds\},\\
V_3 &= \int_{t-h_1}^{t}e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_4 &= \int_{t-h_2}^{t}e^{2\alpha s}x^T(s)Qx(s)\,ds,\\
V_5 &= h_1\int_{-h_1}^{0}\int_{t+s}^{t}e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_6 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t}e^{2\alpha\tau}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_7 &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t}e^{2\alpha\tau}\dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds,\\
V_8 &= \int_{t-\eta_2}^{t}e^{2\alpha s}x^T(s)Sx(s)\,ds,
\end{aligned}$$

where $\xi(t)=[x^T(t)\ \ x^T(t-h(t))\ \ \dot{x}^T(t)\ \ \dot{x}^T(t-\eta(t))\ \ y^T(t)]^T$ and

$$E=\begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}, P=\begin{bmatrix} P_1 & 0 & 0 & 0 & 0\\ P_2 & P_3 & P_4 & P_5 & P_6\\ P_7 & P_8 & P_9 & P_{10} & P_{11}\end{bmatrix}.$$

Taking the derivative of $V(x_t)$ along any trajectory of solution of system (7.2), we have

$$\begin{aligned}
\dot{V}_1 &= 2\alpha e^{2\alpha t}x^T(t)P_1x(t)+2e^{2\alpha t}x^T(t)P_1\dot{x}(t)=2\alpha e^{2\alpha t}\xi^T(t)E^TP\xi(t)\\
&\quad +2e^{2\alpha t}\xi^T(t)P^T\begin{bmatrix} y(t)\\ 0\\ 0\end{bmatrix}\\
&= 2e^{2\alpha t}\xi^T(t)P^T\begin{bmatrix} y(t)\\ \dot{x}(t)-y(t)\\ -y(t)-Ax(t)+W_0f(x(t))+D_1\dot{x}(t-\eta(t))\\ +W_2f(\dot{x}(t-\eta(t)))+W_1f(x(t-h(t)))\end{bmatrix}\\
&\quad +2\alpha e^{2\alpha t}x^T(t)P_1x(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq e^{2\alpha t}\{2\alpha x^T(t)P_1x(t) + x^T(t)P_1y(t) + x^T(t)P_2\dot{x}(t) - x^T(t)P_2y(t) \\
&\quad -x^T(t)P_7y(t) + x^T(t)P_7W_1f(x(t-h(t))) + x^T(t)P_7W_2f(\dot{x}(t-\eta(t))) \\
&\quad +x^T(t)P_7D_1\dot{x}(t-\eta(t)) + x^T(t)P_7W_0f(x(t)) + x^T(t-h(t))P_3\dot{x}(t) \\
&\quad -x^T(t-h(t))P_3y(t) - x^T(t-h(t))P_8y(t) - x^T(t-h(t))P_8Ax(t) \\
&\quad +x^T(t-h(t))P_8W_0f(x(t)) + x^T(t-h(t))P_8W_1f(x(t-h(t))) \\
&\quad +x^T(t-h(t))P_8W_2f(\dot{x}(t-\eta(t))) + x^T(t-h(t))P_8D_1\dot{x}(t-\eta(t)) \\
&\quad -\dot{x}^T(t)P_9Ax(t) + \dot{x}^T(t)P_9W_0f(x(t)) + \dot{x}^T(t)P_9W_1f(x(t-h(t))) \\
&\quad +\dot{x}^T(t)P_9W_2f(\dot{x}(t-\eta(t))) + \dot{x}^T(t)P_9D_1\dot{x}(t-\eta(t)) - \dot{x}^T(t)P_9y(t) \\
&\quad -\dot{x}^T(t)P_4y(t) - x^T(t)Ax(t) + \dot{x}^T(t-\eta(t))P_{10}W_1f(x(t-h(t))) \\
&\quad +\dot{x}^T(t-\eta(t))P_5\dot{x}(t) - \dot{x}^T(t-\eta(t))P_5y(t) - \dot{x}^T(t-\eta(t))P_{10}y(t) \\
&\quad -\dot{x}^T(t-\eta(t))P_{10}Ax(t) + \dot{x}^T(t-\eta(t))P_{10}W_0f(x(t)) + \dot{x}^T(t)P_4\dot{x}(t) \\
&\quad +\dot{x}^T(t-\eta(t))P_{10}W_2f(\dot{x}(t-\eta(t))) + \dot{x}^T(t-\eta(t))P_{10}D_1\dot{x}(t-\eta(t)) \\
&\quad +y^T(t)P_6\dot{x}(t) - y^T(t)P_{11}Ax(t) + y^T(t)P_{11}W_0f(x(t)) \\
&\quad +y^T(t)P_{11}W_1f(x(t-h(t))) + y^T(t)P_{11}W_2f(\dot{x}(t-\eta(t))) \\
&\quad +y^T(t)P_{11}D_1\dot{x}(t-\eta(t)) - y^T(t)P_6y(t) - y^T(t)P_{11}y(t)\} \\
\dot{V}_2 &\leq 4\alpha e^{2\alpha t}[(f^T(x(t)) - x^T(t)F_1)G_1x(t) + (x^T(t)G_2 - f^T(x(t)))F_2x(t)] \\
&\quad +2e^{2\alpha t}[f^T(x(t))(F_1-F_2) + x^T(t)(F_2G_2 - F_1G_1)]\dot{x}(t), \\
\dot{V}_3 &= e^{2\alpha t}x^T(t)Qx(t) - e^{2\alpha(t-h_1)}x^T(t-h_1)Qx(t-h_1), \\
\dot{V}_4 &= e^{2\alpha t}x^T(t)Qx(t) - e^{2\alpha(t-h_2)}x^T(t-h_2)Qx(t-h_2), \\
\dot{V}_5 &\leq e^{2\alpha t}[h_1^2\dot{x}^T(t)R\dot{x}(t) - h_1\int_{t-h_1}^{t} e^{2\alpha s}\dot{x}^T(s)R\dot{x}(s)ds], \\
\dot{V}_6 &\leq e^{2\alpha t}[h_2^2\dot{x}^T(t)R\dot{x}(t) - h_2\int_{t-h_2}^{t} e^{2\alpha s}\dot{x}^T(s)R\dot{x}(s)ds], \\
\dot{V}_7 &\leq e^{2\alpha t}[(h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t) - (h_2-h_1)\int_{t-h_2}^{t-h_1} e^{2\alpha s}\dot{x}^T(s)U\dot{x}(s)ds], \\
\dot{V}_8 &= e^{2\alpha t}[x^T(t)Sx(t) - e^{2\alpha(t-\eta_2)}x^T(t-\eta_2)Sx(t-\eta_2)].
\end{aligned}
$$

Applying Lemma 2.5.6 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_1\int_{t-h_1}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\le -\Big[\int_{t-h_1}^{t}\dot{x}(s)\Big]^T R\Big[\int_{t-h_1}^{t}\dot{x}(s)\Big]\\
&\le -[x(t)-x(t-h_1)]^T R[x(t)-x(t-h_1)]\\
&= -x^T(t)Rx(t)+2x^T(t)Rx(t-h_1)\\
&\quad -x^T(t-h_1)Rx(t-h_1), \qquad (7.8)\\
-h_2\int_{t-h_2}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\le -\Big[\int_{t-h_2}^{t}\dot{x}(s)\Big]^T R\Big[\int_{t-h_2}^{t}\dot{x}(s)\Big]\\
&\le -[x(t)-x(t-h_2)]^T R[x(t)-x(t-h_2)]\\
&= -x^T(t)Rx(t)+2x^T(t)Rx(t-h_2)\\
&\quad -x^T(t-h_2)Rx(t-h_2), \qquad (7.9)
\end{aligned}
$$

note that

$$
\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds &= -(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&\quad -(h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&= -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&\quad -(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&\quad -(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds\\
&\quad -(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds \qquad (7.10)
\end{aligned}
$$

Using Lemma 2.5.6 gives

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U\dot{x}(s)\,ds &\le -\Big[\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds\Big]^T U\Big[\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds\Big]\\
&\le -[x(t-h(t))-x(t-h_2)]^T U\\
&\quad \times[x(t-h(t))-x(t-h_2)], \qquad (7.11)
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)\,ds &\le -\Big[\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds\Big]^T U\Big[\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds\Big]\\
&\le -[x(t-h_1)-x(t-h(t))]^T U\\
&\quad \times[x(t-h_1)-x(t-h(t))]. \qquad (7.12)
\end{aligned}
$$

Let $\beta = \frac{h_2 - h(t)}{h_2 - h_1} \leq 1$. Then, we get

$$\begin{aligned}
-(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds &= -\beta \int_{t-h(t)}^{t-h_1} (h_2 - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
&\leq -\beta \int_{t-h(t)}^{t-h_1} (h(t) - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
&\leq -\beta [x(t-h_1) - x(t-h(t))]^T U \\
&\quad \times [x(t-h_1) - x(t-h(t))], \qquad (7.13)
\end{aligned}$$

and

$$\begin{aligned}
-(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) U \dot{x}(s)\, ds &= -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h_1) \dot{x}^T(s) U \dot{x}(s)\, ds \\
&\leq -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h(t)) \dot{x}^T(s) U \dot{x}(s)\, ds \\
&\leq -(1-\beta) [x(t-h(t)) - x(t-h_2)]^T \qquad (7.14) \\
&\quad \times U [x(t-h(t)) - x(t-h_2)].
\end{aligned}$$

Therefore, from (7.8)-(7.14), we obtain

$$\begin{aligned}
-(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{x}^T(s) U \dot{x}(s)\, ds \leq\; & -[x(t-h(t)) - x(t-h_2)]^T U \\
& \times [x(t-h(t)) - x(t-h_2)] \\
& -[x(t-h_1) - x(t-h(t))]^T U \\
& \times [x(t-h_1) - x(t-h(t))] \\
& -\beta [x(t-h_1) - x(t-h(t))]^T U \\
& \times [x(t-h_1) - x(t-h(t))] \\
& -(1-\beta) [x(t-h(t)) - x(t-h_2)]^T \\
& \times U [x(t-h(t)) - x(t-h_2)], \qquad (7.15)
\end{aligned}$$

For any positive diagonal matrices $L_1 > 0, L_2 > 0, L_3 > 0$, it follows from Assumption 7.1.4 that

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} -F_3 L_1 & F_4 L_1 \\ F_4 L_1 & -L_1 \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \geq 0, \qquad (7.16)$$

$$\begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix}^T \begin{bmatrix} -F_3 L_2 & F_4 L_2 \\ F_4 L_2 & -L_2 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix} \geq 0, \qquad (7.17)$$

$$\begin{bmatrix} \dot{x}(t-\eta(t)) \\ f(\dot{x}(t-\eta(t))) \end{bmatrix}^T \begin{bmatrix} -F_3L_3 & F_4L_3 \\ F_4L_3 & -L_3 \end{bmatrix} \begin{bmatrix} \dot{x}(t-\eta(t)) \\ f(\dot{x}(t-\eta(t))) \end{bmatrix} \geq 0. \tag{7.18}$$

Using the following identity relation

$$-\dot{x}(t) + y(t) = 0,$$

we have

$$-2\dot{x}^T(t)H\dot{x}(t) + 2\dot{x}^T(t)Hy(t) = 0. \tag{7.19}$$

From (7.9)-(7.15), and by adding the terms (7.16)-(7.19), we get

$$\begin{aligned}
\dot{V}(x(t)) \leq e^{2\alpha t}\{ & 2\alpha e^{2\alpha t}x^T(t)P_1x(t) + x^T(t)P_1y(t) + x^T(t)P_2\dot{x}(t) - x^T(t)P_2y(t) \\
& - x^T(t)P_7y(t) + x^T(t)P_7W_1f(x(t-h(t))) + x^T(t)P_7W_2f(\dot{x}(t-\eta(t))) \\
& + x^T(t)P_7D_1\dot{x}(t-\eta(t)) + x^T(t)P_7W_0f(x(t)) + x^T(t-h(t))P_3\dot{x}(t) \\
& - x^T(t-h(t))P_3y(t) - x^T(t)Ax(t) - x^T(t-h(t))P_8Ax(t) \\
& + x^T(t-h(t))P_8W_0f(x(t)) - x^T(t-h(t))P_8y(t) \\
& + x^T(t-h(t))P_8W_2f(\dot{x}(t-\eta(t))) + x^T(t-h(t))P_8D_1\dot{x}(t-\eta(t)) \\
& + \dot{x}^T(t)P_4\dot{x}(t) - \dot{x}^T(t)P_4y(t) - y^T(t)P_6y(t) - y^T(t)P_{11}y(t) \\
& - \dot{x}^T(t)P_9Ax(t) + \dot{x}^T(t)P_9W_0f(x(t)) + \dot{x}^T(t)P_9W_1f(x(t-h(t))) \\
& + \dot{x}^T(t)P_9W_2f(\dot{x}(t-\eta(t))) + \dot{x}^T(t)P_9D_1\dot{x}(t-\eta(t)) - \dot{x}^T(t)P_9y(t) \\
& + \dot{x}^T(t-\eta(t))P_5\dot{x}(t) - \dot{x}^T(t-\eta(t))P_5y(t) - \dot{x}^T(t-\eta(t))P_{10}y(t) \\
& - \dot{x}^T(t-\eta(t))P_{10}Ax(t) + \dot{x}^T(t-\eta(t))P_{10}W_0f(x(t)) \\
& + y^T(t)P_{11}D_1\dot{x}(t-\eta(t)) + \dot{x}^T(t-\eta(t))P_{10}W_2f(\dot{x}(t-\eta(t))) \\
& + \dot{x}^T(t-\eta(t))P_{10}D_1\dot{x}(t-\eta(t)) + y^T(t)P_6\dot{x}(t) \\
& - y^T(t)P_{11}Ax(t) + y^T(t)P_{11}W_0f(x(t)) + y^T(t)P_{11}W_1f(x(t-h(t))) \\
& + y^T(t)P_{11}W_2f(\dot{x}(t-\eta(t))) + \dot{x}^T(t-\eta(t))P_{10}W_1f(x(t-h(t))) \\
& + x^T(t-h(t))P_8W_1f(x(t-h(t))) + x^T(t)Sx(t) \\
& - e^{-2\alpha\eta_2}x^T(t-\eta)Sx(t-\eta) - e^{-2\alpha h_2}x^T(t-h_2)Qx(t-h_2) \\
& + 4\alpha[(f^T(x(t)) - x^T(t)F_1)G_1x(t) + (x^T(t)G_2 - f^T(x(t)))F_2x(t)] \\
& + 2[f^T(x(t))(F_1-F_2) + x^T(t)(F_2G_2 - F_1G_1)]\dot{x}(t) + x^T(t)Qx(t) \\
& - e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1) + x^T(t)Qx(t) \\
& + h_1^2\dot{x}^T(t)R\dot{x}(t) - e^{-2\alpha h_1}x^T(t)Rx(t) + 2e^{-2\alpha h_1}x^T(t)Rx(t-h_1) \\
& - e^{-2\alpha h_1}x^T(t-h_1)Rx(t-h_1) + (h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t)
\end{aligned}$$

$$
\begin{aligned}
& -e^{-2\alpha h_2}x^T(t)Rx(t) + 2e^{-2\alpha h_2}x^T(t)Rx(t-h_2) - e^{-2\alpha h_2}x^T(t-h_2)Rx(t-h_2) \\
& -e^{-2\alpha h_2}[x(t-h(t)) - x(t-h_2)]^T U[x(t-h(t)) - x(t-h_2)] + h_2^2\dot{x}^T(t)R\dot{x}(t) \\
& -e^{-2\alpha h_2}[x(t-h_1) - x(t-h(t))]^T U[x(t-h_1) - x(t-h(t))] \\
& -\beta[x(t-h_1) - x(t-h(t))]^T U[x(t-h_1) - x(t-h(t))] \\
& -(1-\beta)[x(t-h(t)) - x(t-h_2)]^T U[x(t-h(t)) - x(t-h_2)]\} \\
= \;& e^{2\alpha t}\{\zeta^T(t)\mathcal{M}\zeta(t) - \beta[x(t-h_1) - x(t-h(t))]^T e^{-2\alpha h_2} \\
& \times U[x(t-h_1) - x(t-h(t))] - (1-\beta)[x(t-h(t)) - x(t-h_2)]^T \\
& \times e^{-2\alpha h_2}U[x(t-h(t)) - x(t-h_2)]\} \\
= \;& e^{2\alpha t}\{\zeta^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2]\zeta(t)\}, \qquad (7.20)
\end{aligned}
$$

and $V(x(t)) \geq e^{2\alpha t}x^T(t)P_1x(t) \geq e^{2\alpha t}\lambda_{\min}(P_1)\|x(t)\|^2$. Then, from Lemma 2.4.6, we conclude that the equilibrium point is exponentially-stable. This completes the proof. $\square$

When $D_1 = 0, W_2 = 0$, time-varying delay system (7.2) becomes

$$
\dot{x}(t) = -Ax(t) + W_0f(x(t)) + W_1f(x(t-h(t))). \qquad (7.21)
$$

Rewrite system (7.21) in the following descriptor system:

$$
\begin{aligned}
\dot{x}(t) &= y(t) \\
y(t) &= -Ax(t) + W_0f(x(t)) + W_1f(x(t-h(t))), \qquad (7.22)
\end{aligned}
$$

we obtain the following Corollary 7.1.6 for exponentially stable of system (7.21).

Corollary 7.1.6 *Given $\alpha > 0$. The system (7.21) is exponentially stable if there exist symmetric positive definite matrices P_1, Q, R, U positive diagonal matrices $G_1 = diag(\lambda_1, \lambda_2, ..., \lambda_m)$, $G_2 = diag(\delta_1, \delta_2, ..., \delta_m)$, $L_j (j = 1, 2)$, and matrices $P_k(k = 2, ..., 11, k \neq 5, 10), H$, with appropriate dimensions such that the following LMIs hold:*

$$
\begin{aligned}
\mathcal{M}_1 = \mathcal{M} - & \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha h_2}U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \qquad (7.23)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_2 = \mathcal{M} - & \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
& \times e^{-2\alpha h_2}U \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \qquad (7.24)
\end{aligned}
$$

$$
\mathcal{M} = \begin{bmatrix}
M_{11}^* & M_{12} & M_{13} & M_{14} & M_{15} & M_{17} & M_{18} & M_{1a} \\
* & M_{22} & M_{23} & M_{24} & M_{25} & M_{27} & M_{28} & M_{2a} \\
* & * & M_{33} & M_{34} & M_{35} & M_{37} & M_{38} & M_{3a} \\
* & * & * & M_{44} & M_{45} & M_{47} & M_{48} & M_{4a} \\
* & * & * & * & M_{55} & M_{57} & M_{58} & M_{5a} \\
* & * & * & * & * & M_{77} & M_{78} & M_{7a} \\
* & * & * & * & * & * & M_{88} & M_{8a} \\
* & * & * & * & * & * & * & M_{9a} \\
* & * & * & * & * & * & * & M_{10a}
\end{bmatrix},
$$

where $M_{11}^* = -P_7A - A^TP_7 - 4\alpha F_1G_1 + 4\alpha G_2F_2 + Q + Q^T - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R - F_3L_1$.

Remark 7.1.7 *It is worth pointing out that we can extend this method to more complex dynamical network models, such as BAM neutral-type neural networks [35, 69] with a new activation function.*

Remark 7.1.8 *In this work, We note that discrete and neutral delays are both time-varying where the discrete delay is not necessarily differentiable and the information on derivative of neutral delay is not required. To the best of our knowledge, this is the first study under this conditions on discrete and neutral delays. Meanwhile, this restriction is required in some existing result, see [37, 46, 48, 69]. Moreover, a new activation function which has not been considered yet in other literature is proposed and utilized to reduce the conservatism of stability criterion.*

Based on Theorem 7.1.5, we perform the robust stability analysis for systems (7.2) with uncertainties as the following system;

$$
\begin{aligned}
\dot{x}(t) - (D_1 + \Delta D_1(t))\dot{x}(t-\eta(t) &= -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0(t))f(x(t)) \\
&\quad +(W_1 + \Delta W_1(t))f(x(t-h(t))) \\
&\quad +(W_2 + \Delta W_2(t))f(\dot{x}(t-\eta(t))) \qquad (7.25)
\end{aligned}
$$

, where

$$
[\Delta A(t) \;\; \Delta W_0(t) \;\; \Delta W_1(t) \;\; \Delta W_2(t) \;\; \Delta D_1(t)] = DF(t)[E_1 \;\; E_2 \;\; E_3 \;\; E_4 \;\; E_5],
$$

and D, E_1, E_2, E_3, E_4 and E_5 are know matrices of appropriate dimensions, and the time-varying matrix $F(t)$ satisfies

$$
F^T(t)F(t) \leq I, \quad \forall t \geq 0.
$$

Rewrite system (7.25), we get

$$
\begin{aligned}
\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-h(t)) + W_2 f(\dot{x}(t-\eta(t)) \\
&\quad + D_1 \dot{x}(t-\eta(t)) + DEP(t) \\
p(t) &= F(t)(E_1 x(t) + E_2 f(x(t)) + E_3 f(x(t-h(t))) + E_4 f(\dot{x}(t-\eta(t))) \\
&\quad + E_5 \dot{x}(t-\eta(t)))
\end{aligned}
$$

Corollary 7.1.9 *Given $\alpha > 0$. The system (7.25) is $\alpha-$exponentially stable if there exist symmetric positive definite matrices P_1, Q, R, U, S positive diagonal matrices $G_1 = diag(\lambda_1, \lambda_2, ..., \lambda_m)$, $G_2 = diag(\delta_1, \delta_2, ..., \delta_m)$, $L_j (j = 1, 2, 3), \epsilon > 0$, and any matrices $P_k (k = 2, ..., 11), H$ such that the following LMIs hold:*

$$
\begin{aligned}
\mathcal{M}_1^* &= \mathcal{M}^* - \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{7.26}
$$

$$
\begin{aligned}
\mathcal{M}_2^* &= \mathcal{M}^* - \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
&\quad \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & -I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,
\end{aligned} \tag{7.27}
$$

$$
\mathcal{M}^* = \begin{bmatrix}
M_{11}^* & M_{12} & M_{13} & M_{14} & M_{15} & M_{16}^* & M_{17}^* & M_{18}^* & M_{19}^* & M_{1a} & 0 & P^T D \\
* & M_{22} & 0 & M_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & M_{33} & M_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & M_{44} & M_{45} & M_{46} & M_{47} & M_{48} & M_{49} & M_{4a} & 0 & P^T D \\
* & * & * & * & M_{55} & M_{56} & M_{57} & M_{58} & M_{59} & M_{5a} & 0 & P^T D \\
* & * & * & * & * & M_{66}^* & M_{67}^* & M_{68}^* & M_{69}^* & M_{6a} & 0 & P^T D \\
* & * & * & * & * & * & M_{77}^* & M_{78}^* & M_{79}^* & M_{7a} & 0 & 0 \\
* & * & * & * & * & * & * & M_{88}^* & M_{89}^* & M_{8a} & 0 & 0 \\
* & * & * & * & * & * & * & * & M_{99}^* & M_{9a} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & M_{10a} & 0 & P^T D \\
* & * & * & * & * & * & * & * & * & * & M_{11b} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\epsilon I
\end{bmatrix},
$$

where

$$\begin{aligned}
M_{11}^* &= -P_7A - A^TP_7 - 4\alpha F_1G_1 + 4\alpha G_2F_2 + S + Q + Q^T - e^{-2\alpha h_1}R \\
&\quad -e^{-2\alpha h_2}R - F_3L_1 + 2\alpha P_1 + \epsilon E_1^TE_1, \\
M_{16}^* &= -A^TP_{10}^T + P_7D_1 + P_7W_2 + \epsilon E_1^TE_5, \\
M_{17}^* &= 2\alpha G_1^T - 2\alpha F_2^T + F_4L_1 + P_7W_0 + \epsilon E_1^TE_2, \\
M_{18}^* &= P_7W_1 + \epsilon E_1^TE_3, \\
M_{19}^* &= P_7W_2 + \epsilon E_1^TE_4, \\
M_{66}^* &= -F_3L_3 + P_{10}D_1 + D_1^TP_{10}^T - L_3 + \epsilon E_5^TE_5, \\
M_{67}^* &= P_{10}W_0 + \epsilon E_5^TE_2, \\
M_{68}^* &= P_{10}W_1 + \epsilon E_5^TE_3, \\
M_{69}^* &= P_{10}W_2 + F_4L_3 + \epsilon E_5^TE_4, \\
M_{77}^* &= -L_1 + \epsilon E_2^TE_2, \\
M_{78}^* &= \epsilon E_2^TE_3, \\
M_{79}^* &= \epsilon E_2^TE_4, \\
M_{88}^* &= -L_2 + \epsilon E_3^TE_3, \\
M_{89}^* &= \epsilon E_3^TE_4, \\
M_{99}^* &= -L_3 + \epsilon E_4^TE_4.
\end{aligned}$$

7.2 Numerical examples

In this section, we now provide examples to show the effectiveness of theoretical result.

Example 7.2.1 Consider the following neutral-type neural networks with time-varying discrete and neutral delays

$$\begin{aligned}
\dot{x}(t) - D_1\dot{x}(t-\eta(t)) &= -Ax(t) + W_0f(x(t)) + W_1f(x(t-h(t))) \\
&\quad +W_2f(\dot{x}(t-\eta(t))) \qquad (7.28)
\end{aligned}$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, F_1^- = F_2^- = F_1^+ = F_2^+ = 1,$$

$f_1(x(t)) = 0.5(|x_1(t)+1| - |x_1(t)-1|)$, $f_2(x(t)) = \tanh x_2(t)$. It is worth noting that, the delay function $h(t) = 0.1 + 0.1|\sin t|$ is non-differentiable. Moreover, the delay function $\eta(t) = \cos^2 e^t$ is bounded but the derivative of $\eta(t)$ is unbounded. Therefore, the methods used in [37,46,48,69] are not applicable to this system. We have $h_1 = 0.1, h_2 = 0.2$. Given $\alpha = 0.1$ and any initial function $\phi(t) = C^1([-0.2, 0], \mathcal{R}^2)$. By solving LMIs, we obtain

$$
\begin{aligned}
Q &= \begin{bmatrix} 5.1022 & -0.0352 \\ -0.0352 & 6.4839 \end{bmatrix}, \quad R = \begin{bmatrix} 2.9003 & 0.0566 \\ 0.0566 & 2.3908 \end{bmatrix}, \\
U &= \begin{bmatrix} 2.1591 & -0.0020 \\ -0.0020 & 2.0116 \end{bmatrix}, \quad H = \begin{bmatrix} 9.0302 & -0.0000 \\ -0.0000 & 9.0302 \end{bmatrix}, \\
L_1 &= \begin{bmatrix} 6.9237 & -1.0508 \\ -1.0508 & 6.4539 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 5.6888 & -0.0922 \\ -0.0922 & 5.8291 \end{bmatrix}, \\
L_3 &= \begin{bmatrix} 5.9651 & 0.0345 \\ 0.0345 & 5.9178 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 6.5649 & -0.5881 \\ -0.5881 & 6.1553 \end{bmatrix}, \\
P_4 &= \begin{bmatrix} 5.9136 & 0.4291 \\ -0.9162 & 6.0064 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1454 & -4.9315 \\ -5.1995 & -5.7929 \end{bmatrix}, \\
P_3 &= \begin{bmatrix} -0.3527 & 0.4746 \\ 0.8719 & 1.0132 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 9.2979 & -0.3096 \\ -0.3096 & 9.7321 \end{bmatrix}, \\
P_5 &= \begin{bmatrix} -0.1469 & 0.1824 \\ 0.0717 & 0.1361 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 1.3776 & -1.2832 \\ 0.0099 & 0.9253 \end{bmatrix}, \\
P_7 &= \begin{bmatrix} 5.6413 & 0.6167 \\ -0.2062 & 3.9149 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 0.7374 & -0.9594 \\ -1.4314 & -1.2410 \end{bmatrix}, \\
P_9 &= \begin{bmatrix} 1.5449 & 0.1516 \\ 0.2017 & 1.1209 \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0.4361 & 0.1274 \\ 0.1349 & 0.2299 \end{bmatrix}, \\
P_{11} &= \begin{bmatrix} 1.7026 & 0.9594 \\ 1.2535 & 2.5465 \end{bmatrix}, \quad S = \begin{bmatrix} 6.6337 & -9.5723 \\ -9.5723 & 18.3093 \end{bmatrix}, \\
P_1 &= \begin{bmatrix} 11.5638 & -1.8325 \\ -1.8325 & 7.5768 \end{bmatrix}.
\end{aligned}
$$

Thus, the system (7.2), is 0.1-exponentially stable. Given $\alpha > 0$, we will give the values of the maximum allowable upper bounds of neural-type neural networks with time-varying delay for different decay rates $0.1 \leq \alpha \leq 0.4$. From Theorem 7.1.5, we obtain the maximum allowable upper bound of the time-varying delay h_2, as shown in Table 7.1.

Table 7.1: Maximum allowable upper bounds h_2 of neutral-type neural networks with time-varying delay for different values of h_1 and decay rates α.

	$h_1 = 0.1$	$h_1 = 0.2$	$h_1 = 0.3$	$h_1 = 0.4$
$\alpha = 0.1$	2.0879	2.2010	2.2548	2.3385
$\alpha = 0.2$	1.8065	1.8910	2.0110	2.0314
$\alpha = 0.3$	1.6107	1.6797	1.7492	1.8191
$\alpha = 0.4$	1.4710	1.5287	1.5943	1.6605

Figure 7.1. shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system (7.28) with time-varying delay $h(t) = 0.1 + 0.1|\sin t|$ and $\eta(t) = \cos^2 e^t$, $\phi(t) = [0.1 + \sin t, -0.9\cos t)]$, $\forall t \in [-1, 0]$.

Example 7.2.2 Consider the following system which was considered in [47]. The system matrices and parameters of delayed neutral-type neural network (7.2) are as follows:

$$A = \begin{bmatrix} 2.7644 & 0 & 0 \\ 0 & 1.0185 & 0 \\ 0 & 0 & 10.2716 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.2651 & -3.1608 & -2.0491 \\ 3.1859 & -0.1573 & -2.4687 \\ 2.0368 & -1.3633 & 0.5776 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -0.7727 & -0.8370 & 3.8019 \\ 0.1004 & 0.6677 & -2.4431 \\ -0.6622 & 1.3109 & -1.8407 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2076 & 0.0631 & 0.3915 \\ -0.0780 & 0.3106 & 0.1009 \\ -0.2763 & 0.1416 & 0.3729 \end{bmatrix},$$

$$W_2 = 0, F_1^- = F_2^- = F_3^- = 0, F_1^+ = 0.1019, F_2^+ = 0.3419, F_3^+ = 0.0633, h_1 = 0.$$

By solving the LMIs (7.4)-(7.5) given in Theorem 7.1.5, the maximum allowable bound h_2 is 8.6275 which is much better than the value $h_2 = 1.728$, obtained in [47]. Moreover, the differentiability of the time delay $h(t)$ is not required in Theorem 7.1.5.

Example 7.2.3 Consider the following neural networks with time-varying discrete delay

$$\dot{x}(t) \quad = \quad -Ax(t) + W_0 f(x(t)) + W_1 f(x(t - h(t))) \tag{7.29}$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad F_1^- = F_2^- = 0,$$

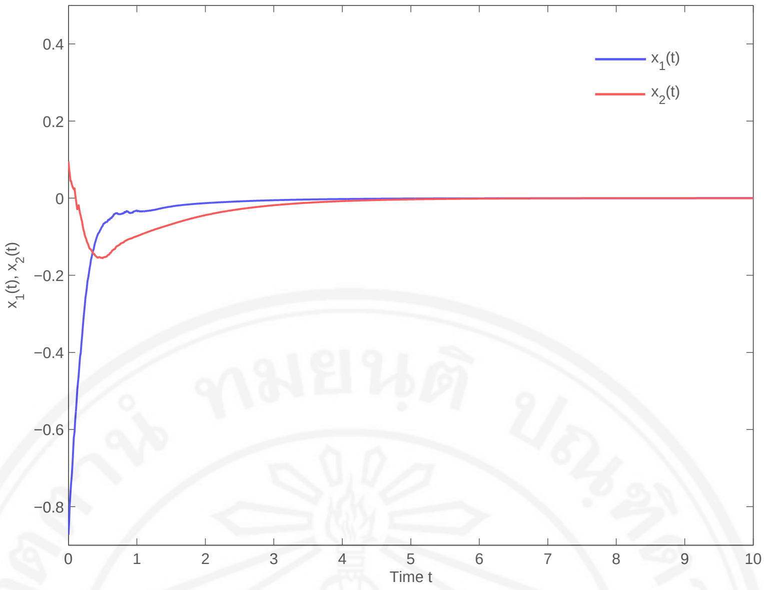

Figure 7.1: The trajectories of $x_1(t)$ and $x_2(t)$ of the system (7.28) with time-varying delay $h(t) = 0.1 + 0.1|\sin t|$ and $\eta(t) = \cos^2 e^t$.

$F_1^+ = 0.4,\ F_2^+ = 0.8$. Table 7.2 provides comparison of maximum allowable value of h_2 for (7.29) obtained in Corollary 7.1.6. We see that, the maximum allowable bounds for h_2 obtained from Corollary 7.1.6 are much better than that obtained in [8, 15–18, 23, 24, 62, 64, 71, 74]. The results obtained in [8, 15, 17, 18, 24, 62] may not be used for the case when $h_1 \neq 0$. Moreover, the differentiability of the time delay $h(t)$ is not required in Corollary 7.1.2.

Example 7.2.4 Consider neutral-type neural networks with time-varying discrete and neutral delays with the following parameters :

$$\begin{aligned}\dot{x}(t) - (D_1 + \Delta D_1(t))\dot{x}(t - \eta(t)) &= -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0(t))f(x(t)) \\ &\quad +(W_1 + \Delta W_1(t))f(x(t - h(t))) \qquad (7.30) \\ &\quad +(W_2 + \Delta W_2(t))f(\dot{x}(t - \eta(t)))\end{aligned}$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, W_1 = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

Table 7.2: Maximum allowable upper bounds h_2 of neutral-type neural networks with time-varying delay for different values of h_1.

h_1	Methods	restriction on $\dot{\eta}(t)$: $\dot{\eta}(t) \leq 0.8$	$\dot{\eta}(t) \leq 0.9$	no restriction on $\dot{\eta}(t)$
0	[16–18, 74]	1.2281	0.8636	0.8298
	[8]	1.2459	0.8827	0.8259
	[15, 23]	1.6831	1.1493	1.0880
	[64]	2.3534	1.6050	1.5103
	[71]	2.8654	1.9508	1.7809
	[24]	2.8854	1.9631	1.7810
	[62]	3.0640	2.0797	1.9207
	Corollary 3.2	≥ 3.0640	≥ 2.0797	56.0988
1	[23]	2.5967	2.0443	1.9621
	[64]	3.2575	2.4769	2.3606
	[71]	3.8359	2.9234	2.7532
	[62]	4.0414	3.0250	2.8573
	Corollary 3.2	≥ 4.0414	≥ 3.0250	57.0664
100	[23]	101.5946	101.0443	100.921
	[64]	102.2552	101.4769	101.3606
	[71]	102.8335	101.9234	101.7532
	[62]	103.0385	102.0250	101.8573
	Corollary 3.2	≥ 103.0385	≥ 102.0250	151.4999

$$D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 0.1 & 0.6 \\ 0.6 & -0.3 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -0.1 & 0.4 \\ 0.4 & -0.1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$
$$E_5 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad F(t) = \begin{bmatrix} \sin^2(t) & 0 \\ 0 & \cos^2(t) \end{bmatrix}, F_1^- = F_2^- = F_1^+ = F_2^+ = 1,$$

$f_1(x(t)) = 0.5(|x_1(t)+1| - |x_1(t)-1|)$, $f_2(x(t)) = \tanh x_2(t)$. It is worth noting that, the delay functions $h(t) = 0.5 + |\sin t|$ is non-differentiable. Moreover, the delay function and $\eta(t) = \cos^2 e^t$ is bounded but the derivative of $\eta(t)$ is unbounded. Therefore, the methods used in [37, 46, 48, 69] are not applicable to this system. We have $h_1 = 0.5, h_2 = 1.5$. Given $\alpha = 0.4$ and any initial function $\phi(t) = C^1([-1.5, 0], \mathcal{R}^2)$. We obtain

$$Q = \begin{bmatrix} 30.1001 & 14.6797 \\ 14.6797 & 63.8758 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1399 & 0.0917 \\ 0.0917 & 0.2872 \end{bmatrix},$$
$$U = \begin{bmatrix} 6.1115 & 1.6000 \\ 1.6000 & 8.9398 \end{bmatrix}, \quad H = \begin{bmatrix} 48.2965 & 0.0000 \\ 0.0000 & 48.2965 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 110.5900 & 0 \\ 0 & 98.2462 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 22.3059 & 0 \\ 0 & 81.9352 \end{bmatrix},$$
$$L_3 = \begin{bmatrix} 45.1431 & 0 \\ 0 & 38.9857 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 5.0321 & 0 \\ 0 & 11.5383 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 19.5460 & -65.4903 \\ 75.4717 & 13.5740 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -2.8179 & -47.9049 \\ -39.5315 & -84.1958 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 11.4230 & 1.4375 \\ -9.7520 & 16.3431 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 58.2931 & 0 \\ 0 & 65.2384 \end{bmatrix},$$
$$P_5 = \begin{bmatrix} -8.5924 & 2.1423 \\ 4.2180 & -7.4852 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 42.6040 & 50.1722 \\ -78.7019 & 9.2284 \end{bmatrix},$$

Table 7.3: Maximum allowable upper bounds h_2 of uncertain neutral-type neural networks with time-varying delay for different values of the h_1 and decay rates.

$\alpha = 0.2$	h_1	0	0.5	1	2	4	6	10	14.250
	h_{2M}	1.696	2.070	2.452	3.231	4.860	6.574	10.197	14.250
$\alpha = 0.4$	h_1	0	0.5	1	2	4	6.137		
	h_{2M}	1.343	1.664	1.997	2.706	4.271	6.137		
$\alpha = 0.6$	h_1	0	0.5	1	2	3.108			
	h_{2M}	1.065	1.346	1.645	2.297	3.108			
$\alpha = 0.8$	h_1	0	0.5	1	1.737				
	h_{2M}	0.734	0.992	1.276	1.737				

$$
\begin{aligned}
P_7 &= \begin{bmatrix} 47.5216 & 8.5036 \\ 21.2216 & 69.1077 \end{bmatrix}, \quad P_8 = \begin{bmatrix} -4.8877 & -12.5067 \\ -12.4794 & -20.5132 \end{bmatrix}, \\
P_9 &= \begin{bmatrix} 29.8518 & 2.1611 \\ -0.4197 & 27.2787 \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 3.2049 & 2.2325 \\ 0.6709 & 5.3477 \end{bmatrix}, \\
P_{11} &= \begin{bmatrix} -16.3652 & 5.7603 \\ 17.8957 & 11.6807 \end{bmatrix}, \quad S = \begin{bmatrix} 16.2682 & 15.1591 \\ 15.1591 & 48.4851 \end{bmatrix}, \\
P_1 &= \begin{bmatrix} 1.8069 & -0.6699 \\ -0.6699 & 0.6782 \end{bmatrix}.
\end{aligned}
$$

Thus, the system (7.30), is 0.4-exponentially stable. Given $\alpha > 0$ and $\eta = 1$ we will give the values of the maximum allowable upper bounds of neural-type neural networks with time-varying delay for different decay rates $0.2 \leq \alpha \leq 0.8$. From Corollary 7.1.6, we obtain the maximum allowable upper bound of the time-varying delay h_2, as shown in Table 7.3. For presentation convenience, we denote h_{2M} for the maximum allowable upper bound of the time-varying delay h_2 as shown in Table 7.3.