CHAPTER 1

Introduction

Fixed point theory plays an important role in nonlinear analysis. This is because many practical problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings.

1.1 The Background of Fixed Point Theory

Definition 1.1.1. Let X be a nonempty set and $T : X \to X$ be a function. A *fixed point* of T is an element $x \in X$ which satisfies $T(x) = x$. The set of fixed points of T denoted by $F(T)$ or $Fix(T)$.

The study of fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the mapping $T : X \to X$, in order to obtain results on:

• the existence and the uniqueness of fixed points;
• the structure of fixed point sets;
• the approximation of fixed points.

For any given $x \in X$, we define $T^n x$ inductively by $T^0 x = x$ and $T^{n+1}x = TT^n x$; we call $T^n x$ the *iterate of* x *under* T. For $n \geq 1$, the mapping T^n is called the n^{th} *iterate* of T. For any $x_1 \in X$, the sequence $\{x_n\}$ given by

$$x_{n+1} = Tx_n = T^n x_1 \text{ for all } n \in \mathbb{N}$$

is called *the sequence of successive approximations* or *Picard iteration.* Many researchers concentrate in obtaining (additional) conditions on T and X as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T.

In 1922, the Polish mathematician Stefan Banach established a unusual fixed point theorem known as the "Banach Contraction Principle" which is one of the most important results of analysis and considered as the main source of metric fixed point theory. Banach [1] proved the famous theorem in fixed point theory for a contraction as follows:

Theorem 1.1.1. *(The Banach Contraction Principle) Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a self-map. Assume that there exists a nonnegative number* $k < 1$ *such that*

$$d(Tx, Ty) \leq kd(x, y), \qquad \textit{for all}\, x, y \in X.$$

Then T *has a unique fixed point* x *in* X*. Moreover, for each* $x \in X$*, the sequence* $\{T^n x\}$ *converges strongly to* x*.*

Since this theorem was proved by Banach, many researchers have used this theorem to show the existence and uniqueness of solutions for differential and integral equations.

Definition 1.1.2. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

The famous fixed point theorem for nonexpansive mappings have first studied by Browder [2] and Göhde [3] in Banach spaces as follows:

Theorem 1.1.2. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed convex bounded subset of* X*. Then every nonexpansive mapping* $T : C \to C$ *has a fixed point.*

However, the Picard iteration does not converge in general or, even if it converges, its limit is not a fixed point of T

Example 1.1.3. Let $C = [0, 1]$ and $T : C \to C$ defined by $T(x) = 1 - x$ for all $x \in [0, 1]$. Then T is nonexpansive, T has a unique fixed point, $F(T) = \{1/2\}$. But, for any $x_0 = a \neq 1/2$, the Picard iteration yields an oscillatory sequence $a, 1 - a, a, 1 - a, \ldots$.

Many researchers have been trying to find conditions to guarantee the existence of fixed points for nonexpansive mappings. Moreover, the fixed point theory for mappings which are more general than nonexpansive mappings is also interesting.

We now consider a class of mappings that properly includes the class of nonexpansive mappings with fixed points.

Definition 1.1.3. Let C be a nonempty subset of Banach spaces X and $T : C \to C$ be an operator that has at least one fixed point $p \in C$. Then T is said to be a *quasi-nonexpansive* mapping if

$$||Tx - p|| \leq ||x - p|| \quad \text{for all } x \in C, p \in F(T).$$

It is easy to see that a nonexpansive mapping with at least one fixed point is quasi-nonexpansive and a linear quasi-nonexpansive is nonexpansive.

Construction of fixed point iteration methods of nonlinear mappings is an important subject in the theory of nonlinear mappings, and finds application in a number of applied areas. Now, fixed point iteration methods for approximating fixed point of nonexpansive mappings and generalized nonexpansive mappings in various spaces have been studied by many mathematicians.

The following classical iteration methods are often used to approximate a fixed point of a mapping T. In 1953, Mann [4] introduced the following iteration method which was referred to as *Mann iteration* for approximating a fixed point of T.

Let C be a nonempty subset of Banach spaces X and $T : C \to C$ be a self-map. The sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence of real number in $[0, 1]$. He proved a weak convergence for a nonexpansive mapping under the control conditions $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. For $\alpha_n = \lambda$ (constant), the Mann iteration (1.1) reduces to the so-called *Krasnoselskij iteration* [5] that is

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

In 1967, Halpern [6] introduced the modified Mann iteration as follows: a sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ defined by $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)u + \alpha_n T x_n, \text{ for all } n \in \mathbb{N}, \tag{1.3}$$

where $u \in C$ are arbitrarily chosen and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Such a iteration is called the *Halpern Iteration*. He proved, in a real Hilbert space, the sequence $\{u_n\}$ converges strongly to a fixed point of T where $\alpha_n := n^{-a}$, $a \in (0, 1)$. In 1977, Lions [7] obtained a strong convergence provide the sequence $\{\alpha_n\}$ satisfies the control conditions $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$. However, The concept of Halpern iteration has been widely used to approximate the fixed points of nonexpansive mappings (see, for instance, [8, 9, 10, 11, 12]).

In 1974, Ishikawa [13] introduced a generalization of Mann iteration, which is called the *Ishikawa iteration*, as follows: a sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ defined by $x_1 \in C$ and

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. The Ishikawa iteration was first used to establish the strong convergence to a fixed point for a Lipschitzian, pseudo-contractive and nonexpansive mapping of a compact convex subset of a Hilbert space.

In 2007, Agarwal, ORegan and Sahu [14] introduced the *S-iteration* process in a Banach space, a sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ defined by $x_1 \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these for asymptotically nonexpansive mappings. (see [14], Proposition 3.1).

The problem of finding common fixed points is now has been extensively studied by mathematicians. To deal with a fixed point problem of two nonlinear mappings, there have been several ways appeared in the literature.

In 2011, Suthep Suantai [15] introduced the following iteration. Let C be a nonempty subset of Banach spaces X and $S, T : C \to C$ be a self-map. The sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ defined by $x_1 \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n S y_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They proved some weak and strong convergence results for approximating common fixed points of two nonexpansive self-mappings.

1.2 Some Convergence Theorems for Fixed Point Iterative Methods Defined by Admissible Function

In the previous section, we stated elementary concepts, notations and a brief history of fixed point theory of nonexpansive mappings and quasi-nonexpansive mapping.

This section, we talk about a new approach of fixed point iterative methods, based on the concept of *admissible functions* (see Definition 1.2.1) of a self operator. The theory of admissible functions of an operator opened a new direction of research and unified the most important aspects of the iterative approximation of fixed point for single valued self operators. A general fixed point iterative method defined by means of the new concept of admissible function was introduced by Rus in 2012 ([16]).

Definition 1.2.1. ([16, 17, 18]) Let X be a nonempty set. A mapping $G : X \times X \to X$ is called an *admissible function* if it satisfies

(G1) $G(x,x)=x$, for all $x \in X$ and

(G2) $G(x,y)=x$ implies $y=x$, for $x,y \in X$.

In 2013, Berinde [17] introduced a iterative algorithm in terms of admissible functions, which is called *the Krasnoselskij algorithm corresponding to G* or *the GK-algorithm.*

Definition 1.2.2. Let X be a nonempty set, $G : X \times X \to X$ be an admissible function and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$ and

$$x_{n+1} = G(x_n, T(x_n)), \quad n \in \mathbb{N}. \tag{1.7}$$

is called *the Krasnoselskij algorithm corresponding to G* or *the GK-algorithm.*

He proved some strong and weak convergence theorems for a Krasnoselskij type fixed point iterative method defined by admissible function for nonexpansive mapping on Hilbert spaces. He obtained the following result (Theorem 1.2.1 and 1.2.2).

Definition 1.2.3. ([17]) Let C be a nonempty subset of Banach space X and $T : C \to C$ be a self-map. A mapping T is called *demicompact* if every bounded $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ is strongly convergent, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is strongly convergent.

Definition 1.2.4. ([17]) Let H be a Hilbert space and $T : H \to H$ be an operator with $Fix(T) \neq \emptyset$. We say that the admissible mapping $G : H \times H \to H$ has *the property (C)* with respect to T if there exists $\lambda \in (0,1)$ such that

$$\|G(x,Tx) - p\|^2 \leq \lambda^2 \|x-p\|^2 + (1-\lambda)^2 \|Tx - p\|^2 + 2\lambda(1-\lambda) < Tx - p, x - p >$$

for all $x \in X$ and $p \in Fix(T)$.

Theorem 1.2.1. ([17]) *Let C be a closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a nonexpansive and demicompact operator. Then the set $Fix(T)$ is a nonempty convex set. Moreover, if $G : H \times H \to H$ is an admissible function which has the property (C), then the GK-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges(strongly) to a fixed point of T in C.

In next theorem, he removed the demicompactness assumption and a defined new property, which is call *affine Lipschitzian*, and have a new result for the GK-algorithm.

Definition 1.2.5. ([18]) Let $G : X \times X \to X$ be an admissible function on a normed space X. We say that G is *affine Lipschitzian* if there exist a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \le \|\mu(x_1 - x_2) + (1 - \mu)(y_1 - y_2)\|,$$

for all x_1, x_2, y_1 and y_2 in X.

Theorem 1.2.2. ([17]) *Let C be a closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a nonexpansive operator. If $G : H \times H \to H$ is a affine Lipschitzian admissible function which has the property (C), then the GK-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges weakly to a fixed point of T in C.

The following year, Berinde proved some convergence theorems for a GK-algorithm of a nonlinear φ-pseudocontractive operator defined on a closed convex subset of a Hilbert space and obtain the result in [18].

Definition 1.2.6. ([18]) Let H be a real Hilbert space and C be a nonempty subset of H. An operator $T : C \to C$ is said to be (strictly) *φ-pseudocontractive* if, for given $a, b, c \in \mathbb{R}^+$ with $a + b + c = 1$, there exist a (comparison) function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$a \cdot \|x - y\|^2 + b < Tx - Ty, x - y > +c\|Tx - Ty\|^2 \le \varphi^2(\|x - y\|),$$

holds, for all $x, y \in C$.

Theorem 1.2.3. ([18]) *Let C be a nonempty closed and convex subset of a real Hilbert space C and $T : C \to C$ a strictly φ-pseudocontractive. Then*

1. *T has a unique fixed point p in C;*
2. *If $G : H \times H \to H$ is a affine Lipschitzian admissible function which constant $\lambda \in (0, 1)$, then the GK-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges (strongly) to p, for any $x_1 \in C$

This result unifies and generalizes many convergence theorems in the existing literature.

The thesis consists of four chapters. Chapter 1 introduces the concepts and the aim of the research. Chapter 2 deals with some basic concepts, preliminaries and some useful results that will be used in the later chapter. Chapter 3 states and proves the main results of the research and the concluded in Chapter 4.