CHAPTER 2

Basic and Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Metric Spaces, Convex Metric Spaces and Convex Prestructure

Definition 2.1.1. Let X be a nonempty set and d be a real-valued function defined on $X \times X$ satisfying

(D1) $d(x, y) \geq 0$ for each $x, y \in X$;

(D2) $d(x, y) = 0$ if and only if $x = y$;

(D3) $d(x, y) = d(y, x)$ for each $x, y \in X$;

(D4) $d(x, y) \leq d(x, z) + d(z, y)$ for each $x, y, z \in X$.

Then d is called a *distance* or *metric* on X, and X together with d is called a *metric space* which will be denoted by (X, d).

Example 2.1.1.

1. The Euclidian space $\mathbb{R}^n$ with

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}},$$

for each $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, is a metric space. The metric d is called the *usual metric for* $\mathbb{R}^n$.

2. Let X be a nonempty set and for $x, y \in X$ define a metric d by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a metric space, called a *discrete space*.

3. Let X be the set of continuous functions from $[a, b]$ to $\mathbb{R}$. We define a metric d by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \text{ for all } f, g \in X.$$

Then (X, d) is a metric space and usually denoted by $C[a, b]$.

Definition 2.1.2. A sequence $\{x_n\}$ in a metric space (X, d) is said to be *convergent* if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. In this case, we write either $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 2.1.3. A sequence $\{x_n\}$ in a metric space (X, d) is said to be *Cauchy* if there exists a sequence $\{\alpha_n\}$ of nonnegative real numbers such that $d(x_m, x_n) \leq \alpha_n$ $(m > n)$ and $\lim_{n\to\infty} \alpha_n = 0$.

Definition 2.1.4. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Proposition 2.1.2. *Every convergent sequence in a metric space is a Cauchy sequence.*

Theorem 2.1.3 (Double Extract Subsequence Principle)**.** *Let $\{x_n\}$ be a sequence in a metric space (X, d) and $x \in X$. If for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x, then $\lim_{n\to\infty} x_n = x$.*

Proposition 2.1.4. *Let C be a nonempty subset of a metric space (X, d). Then*

1. *C is closed if and only if $\{x_n\} \subset C$ and $x_n \to x$ imply $x \in C$.*

2. *C is compact if and only if any sequence $\{x_n\}$ of C has a subsequence $\{x_{n_k}\}$ which converges to a point of C.*

Definition 2.1.5. Let X and Y be metric spaces and T be a mapping of X into Y. Then T is said to be *continuous* at x_0 in X if $x_n \to x_0 \Rightarrow Tx_n \to Tx_0$. A mapping T of X into Y is continuous if it is continuous at each x in X.

In 1970, Takahashi [19] introduced the concept of convex metric spaces by using the convex structure as follows:

Definition 2.1.6. Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda) d(z, y) \text{ for all } z \in X.$$

A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W).

A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. It is easy to see that open and closed balls are convex and the intersection of a family of convex subsets of X is also convex; see [19]. A function $f : X \to \mathbb{R}$ is said to be *convex* if $f(W(x, y, \lambda)) \leq \lambda f(x) + (1 - \lambda) f(y)$ for all $x, y \in X$ and $\lambda \in (0, 1)$.

Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold. (see more details in [19, 20].)

In 1979, Gudder [21] introduced the concept of convexity for any nonempty set.

Definition 2.1.7. A *convex prestructure* is a nonempty set S together with a map $F : [0, 1] \times S \times S \to S$ and a *convex structure* is a convex prestructure (S, F) in witch F satisfies the following five conditions.

(P1) $F(\lambda, x, y) = F(1 - \lambda, y, x)$ for every $\lambda \in [0, 1]$ and $x, y \in S$,

(P2) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1 - \lambda)\mu, F(\lambda(\lambda + (1 - \lambda)\mu)^{-1}, x, y), z)$ for every $\lambda, \mu \in [0, 1]$ with $\lambda + (1 - \lambda)\mu \neq 0$ and $x, y, z \in S$,

(P3) $F(\lambda, x, x) = x$ for every $\lambda \in [0, 1]$ and $x \in S$,

(P4) $F(0, x, y) = y$ for every $x, y \in S$,

(P5) If $F(\lambda, x, y) = F(\lambda, x, z)$ for some $\lambda \neq 1, x \in S$, then $y = z$.

2.2 Banach Spaces and Useful Properties

The concept of a norm comes from thinking of vectors as follows. A norm on a vector space is function that assigns th each vector a length. There are some obvious properties that such a function should be required to have. Here are the definitions.

Definition 2.2.1. Let X be a vector space. A *norm* on X is a real-valued function $\| \cdot \|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(N1) $\|x\| \geq 0$,

(N2) $\|x\| = 0$ if and only if $x = 0$,

(N3) $\|\alpha x\| = |\alpha| \|x\|$,

(N4) $\|x + y\| \leq \|x\| + \|y\|$.

A norm on X defines a metric d on X which is given by

$$d(x,y) = \|x-y\|$$

and is called the *metric induced by the norm.* The normed space just defined by $(X, \|\cdot\|)$ or simply by X. A *Banach space* is a complete normed space (complete in the metric defined by the norm).

For later use we notice that (N4) implies

$$|\|x\| - \|y\|| \leq \|x-y\|.$$

This formula implies an important property of the norm, that is, $x \mapsto \|x\|$ is continuous mapping of normed space X into $\mathbb{R}$.

Example 2.2.1.

1. **Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$.** They are Banach spaces with norm defined by
$$\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}},$$
for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbb{C}^n$.
In fact, $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete with metric induced by the norm
$$d(x,y) = \|x-y\| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}},$$
for each $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\mathbb{C}^n$.

2. **Space C[a,b].** Let $C[a,b]$ be the set of continuous functions from $[a,b]$ to $\mathbb{R}$. This space is a Banach space with norm given by
$$\|f\| = \max_{x\in[a,b]} |f(x)|, \text{ for each } f \in C[a,b].$$
Then $C[a,b]$ is complete with metric induced by the norm
$$d(f,g) = \|f-g\| = \max_{x\in[a,b]} |f(x) - g(x)|, \text{ for all } f, g \in C[a,b].$$

3. **Space l^p.** Let $p \geq 1$ be a fixed real number. By definition, each element in the space l^p is a sequence $x = (x_1, x_2, \ldots)$ such that
$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

It is a Banach space with norm given by

$$\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}}.$$

In fact, this norm induces the metric

$$d(x,y) = \|x - y\| = \left[\sum_{i=1}^{\infty} |x_i - y_i|^p \right]^{\frac{1}{p}}, \text{ for all } x, y \in l^p.$$

Definition 2.2.2. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in the norm*) if there exists a point $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In this case, we write either $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ and we say that $\{x_n\}$ converges strongly to x.

Definition 2.2.3. A nonempty subset C of a normed space X is said to be *convex* if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in (0,1)$.

Definition 2.2.4. Let X be a normed space. A function $f : X \to (-\infty, \infty)$ is said to be *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in X$ and $t \in (0,1)$.

Of special interest are operators which "preserve" the two algebraic operations of vector space, in the sense of the following definition.

Definition 2.2.5. (Linear Operator) Let X and Y be vector spaces. A *linear operator* from X into Y is a function $T : X \to Y$ such that the following two condition are satisfied whenever $x, y \in X$ and $\alpha \in \mathbb{F}$:

1. $T(x+y) = T(x) + T(y)$
2. $T(\alpha x) = \alpha T(x)$.

If $\mathbb{F}$ is viewed as a one-dimentional vector space, then a linear operator from X into $\mathbb{F}$ is called a *linear functional* or *linear form on X*. Observe the notation; we write Tx instead of $T(x)$; this simplification is standard in function analysis.

Example 2.2.2.

1. **Identity operator.** The identity operator $I_x : X \to X$ is defined by $I_x(x) = x$ for all $x \in X$. We also write simply I for I_x.
2. **Zero operator.** The zero operator $0 : X \to Y$ is defined by $0(x) = 0$ for all $x \in X$.

3. **Differentiation.** Let X be the vector space of all polynomials on $[a, b]$. We may define a linear operator T on X by setting

$$Tx(t) = x'(t).$$

Definition 2.2.6. Let X, Y be a normed spaces and $T : X \to Y$ a linear operator. The operator T is said to be *bounded* if there is a real number $c > 0$ such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|.$$

The collection of all bounded linear operators from X into Y is denoted by $B(X, Y)$ and $B(X, \mathbb{F})$ for the collection of all bounded linear functional from X into $\mathbb{F}$.

Example 2.2.3.

1. **Identity operator.** The identity operator $I : X \to X$ on a normed space $X \neq \{0\}$ is bounded and has $\|I\| = 1$.

2. **Zero operator.** The zero operator $0 : X \to Y$ on a normed space X is bounded and has norm $\|0\| = 0$.

3. **Differentiation.** Let X be the normed space of all polynomials on $[0, 1]$ with norm given $\|x\| = \max |x(t)|, t \in [0, 1]$. A differentiation operator T is defined on X by

$$Tx(t) = x'(t),$$

where the prime denotes differentiation with respect to t. This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in \mathbb{N}$. Then $\|x_n\| = 1$ and

$$Tx_n(t) = nt^{n-1},$$

so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|x_n\| = n$. Since $n \in \mathbb{N}$ is arbitrary, this shows that there is no fixed number c such that $\|Tx_n\|/\|x_n\| \leq c$.

Definition 2.2.7. Let X be a normed space. The *dual space of* X is the normed space $B(X, \mathbb{F})$ of all bounded linear functional on X with the norm defined by

$$\|T\| = \sup_{x \in X, x \neq 0} \frac{|Tx|}{\|x\|}, \text{ for each } T \in B(X, \mathbb{F})$$

This space is denoted by X^*.

If X is a normed space and Y is a Banach space, then $B(X, Y)$ is a Banach space. Letting $Y = \mathbb{F}$ in this result produces this theorem

Theorem 2.2.4. ([22]) *If X is a norm space, then X^* is a Banach space.*

Example 2.2.5.

1. The dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$.

2. The dual space of l^1 is l^∞.

3. The dual space of l^p is l^q, where $1 < p < \infty$ and $1/p + 1/q = 1$

The topology induced by a norm is quiet strong in the sense that it has many open sets. Indeed, in order that each bounded sequence in X has a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces which are related to the linear structure of the spaces to search for subsequential extraction principles. So it is worthwhile to define the weaker topology for a normed space X.

Definition 2.2.8. A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there exists a point $x \in X$ such that $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in X^*$. In this case, we write either $w - \lim_{n\to\infty} x_n = x$ or $x_n \rightharpoonup x$ and we say that $\{x_n\}$ converges weakly to x.

Proposition 2.2.6. ([23]) *Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \to x$. Then $x_n \rightharpoonup x$.*

Lemma 2.2.7. ([24]) *Let $\{x_n\}$ be a weakly convergent sequence in a normed space X, say, $x_n \to x$. Then:*

1. *The weak limit x of $\{x_n\}$ is unique.*

2. *Every subsequence of $\{x_n\}$ converges weakly to x.*

3. *The sequence $\{\|x_n\|\}$ is bounded.*

Definition 2.2.9. Let X be a normed space. The *second dual space or double dual space of X* is the dual space $(X^*)^*$ of X^* and denoted by X^{**}.

To each $x \in X$ and $g \in X^{**}$, which is a linear function defined on X^*, by choosing a fixed $x \in X$ and setting

$$g(f) = g_x(f) = f(x), \text{ for each } f \in X^*.$$

This defines a mapping

$$C : X \to X^{**}$$

$$x \mapsto g_x.$$

C is called the *canonical mapping* of X into X^{**}. It can be shown that the canonical mapping C is isomorphism of X onto the range $\mathfrak{R}(C) \subseteq X^{**}$.

Definition 2.2.10. A normed space X is said to be *reflexive* if

$$\mathfrak{R}(C) = X^{**},$$

where $C : X \to X^{**}$ is the canonical mapping.

Theorem 2.2.8. ([22]) *If a norm space X is reflexive, then X is a Banach space.*

A subset C of X is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of all of its weakly convergent sequences. The weakly open sets are now taken as those sets whose complements are weakly closed. The resulting topology on X is called the *weak topology on X*. Sets which are compact in this topology are said to be *weakly compact.*

Theorem 2.2.9. ([25]) *Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset of X is weakly compact.*

Lemma 2.2.10. ([23]) *Let X be a Banach space. Then the following are equivalent.*

1. *X is reflexive.*
2. *X^* is reflexive.*
3. *Every bounded sequence in X has a weakly convergent subsequence.*
4. *Whenever $\{C_n\}$ is a sequence of nonempty bounded closed convex sets in X such that $C_{n+1} \subset C_n$ for each n, it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.*

The basic property of a norm of a Banach space X is that it is always convex, i.e.,

$$\|(1-\lambda)x + \lambda y\| \leq (1-\lambda)\|x\| + \lambda\|y\|, \text{ for all } x, y \in X \text{ and } \lambda \in [0,1].$$

A number of Banach spaces do not have equality when $x \neq y$, i.e.,

$$\|(1-\lambda)x + \lambda y\| < (1-\lambda)\|x\| + \lambda\|y\|, \text{ for all } x, y \in X, x \neq y \text{ and } \lambda \in [0,1].$$

We use S_x to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ on Banach space X. If $x, y \in S_X$ with $x \neq y$, then we have

$$\|(1-\lambda)x + \lambda y\| < 1, \text{ for all } \lambda \in (0,1),$$

which say that the unit sphere S_X contains no line segments. This suggests strict convexity of norm.

Definition 2.2.11. A Banach space X is said to be *strictly convex* if $x, y \in S_X$ with $x \neq y$, then

$$\|(1-\lambda)x + \lambda y\| < 1, \text{ for all } \lambda \in (0,1).$$

This says that the midpoint $(x+y)/2$ of two distinct points x and y in the unit sphere S_X of X does not lie on S_X. In other words, if $x, y \in S_X$ with $\|x\| = \|y\| = \|(x+y)/2\|$, then $x = y$.

Example 2.2.11.

1. Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|x\|_2$ defined by

$$\|x\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{2}}, \text{ for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

   Then X is strictly convex.

2. Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|x\|_1$ defined by

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|, \text{ for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

   Then X is not strictly convex. To see it, let

$$x = (1, 0, 0, \ldots, 0) \text{ and } y = (0, 1, 0, \ldots, 0).$$

   It easy to see that $x \neq y$, $\|x\|_1 = 1 = \|y\|_1$, but $\|x+y\|_1 = 2$.

3. Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|x\|_\infty$ defined by

$$\|x\|_\infty = max_{1 \leq i < n}|x_i|, \text{ for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

   Then X is not strictly convex. Indeed, for $x = (1, 0, 0, \ldots, 0)$ and $y = (1, 1, 0, \ldots, 0)$. we have, $x \neq y$, $\|x\|_\infty = 1 = \|y\|_\infty$, but $\|x+y\|_\infty = 2$.

In such spaces, we have no information about $1 - \|(x+y)/2\|$, the distance of midpoints from the unit sphere S_X. A stronger property than strict convexity that provides information about the distance $1 - \|(x-y)/2\|$ is uniform convexity.

Definition 2.2.12. A Banach space X is said to be *uniformly convex* if for any ϵ, $0 < \epsilon \leq 2$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$ imply there exists a $\delta = \delta(\epsilon) > 0$ such that $\|(x+y)/2\| \leq 1-\delta$.

Example 2.2.12.

1. The Banach space l_p with $1 < p < \infty$ is uniformly convex.

2. The spaces l_1,l_∞ and $C[a,b]$ are not uniformly convex.

We derive some consequences from the definition of uniform convexity.

Theorem 2.2.13. ([23]) *Every uniformly convex Banach space is strictly convex.*

For the class of uniform convex Banach spaces, we have the following important results:

Theorem 2.2.14. ([23]) *Every uniformly convex Banach space is reflexive.*

Theorem 2.2.15. ([26]) *Let $r > 0$ be a fixed real number. Then a Banach space X is uniformly convex if and only if there is a continuous strictly increasing convex map $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in B_r = \{x \in X : \|x\| \leq r\}$,*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $\lambda \in [0,1]$.

Theorem 2.2.16. ([23] , Theorem 2.3.13) *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence of real numbers in $(0,1)$ bounded away from 0 and 1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq a, \limsup_{n\to\infty} \|y_n\| \leq a \text{ and } \limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$$

for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.2.13. Let C be a nonempty subset of Banach space X, $v \in X$ and $T : C \to X$ a mapping. Then T is said to be *demiclosed* at v if for any sequence $\{x_n\}$ in C the following implication holds:

$$x_n \rightharpoonup u \in C \text{ and } Tx_n \longrightarrow v \text{ imply } Tu = v.$$

Definition 2.2.14. A Banach space X is said to satisfy *the Opial condition* if whenever a sequence $\{x_n\}$ in X converges weakly to $x_0 \in X$, then

$$\limsup_{n\longrightarrow\infty} \|x_n - x_0\| < \limsup_{n\longrightarrow\infty} \|x_n - x\| \text{ for all } x \in X, x \neq x_0.$$

Theorem 2.2.17. ([27]) *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex subset of X. If $T : C \longrightarrow C$ is a nonexpansive mapping. Then $I - T$ is demiclosed with respect to zero.*

2.3 Hilbert spaces and Useful Lemmas

The spaces to be considered in this section are defined as follows.

Definition 2.3.1. Let X be a vector space. A *inner product* on X is a mapping of $X \times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, x \rangle$$

and is called inner product of x and y, such that for all members x, y and z of X and each scalar α we have

(H1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

(H2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(H3) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,

(H4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and a metric on X given by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

Hence inner product spaces are normed spaces, and *Hilbert space* is a complete inner product space, that is Hilbert spaces are Banach spaces.

In (H3), the bar denotes complex conjugation. Consequently, if X is a real vector space, we simply have

$$\langle x, y \rangle = \langle y, x \rangle$$

and from (H2) to (H3) we obtain the formula

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

2. $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$,

3. $\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$.

Example 2.3.1.

1. **Euclidean space $\mathbb{R}^n$.** The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.
In fact, we obtain

$$\|x\| = \sqrt{\langle x, x\rangle} = \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{2}},$$

and from this the Euclidean metric defined by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle} = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{\frac{1}{2}}.$$

2. **Unitary space $\mathbb{C}^n$.** The space $\mathbb{C}^n$ is a Hilbert space with inner product given by

$$\langle x, y\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$.
In fact, we obtain the norm defined by

$$\|x\| = \sqrt{\langle x, x\rangle} = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}.$$

3. **Space C[a,b].** The space $C[a, b]$ is not an inner product space, hence not a Hilbert space.

4. **Space l^2.** The space l^2 is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}.$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in l^2$.
In fact, the norm is defined by

$$\|x\| = \sqrt{\langle x, x\rangle} = \left[\sum_{i=1}^{\infty} |x_i|^2\right]^{\frac{1}{2}}.$$

5. **Space l^p.** The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.

Theorem 2.3.2. *(parallelogram law) For any inner product space X, the following holds:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all} x, y \in X.$$

Lemma 2.3.3. *(Schwarz inequality) Let X be a inner product space. If $x, y \in X$, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem 2.3.4. ([24]) *Every Hilbert space H is reflexive*

Theorem 2.3.5. ([24]) *Let $\{x_n\}$ be a sequence in Hilbert space H, $x_n \rightharpoonup x$ if and only if $\langle x_n, z\rangle \to \langle x, z\rangle$ for all z in H*

Lemma 2.3.6. ([28]) *Let x, y and z be points in a Hilbert space and $\lambda \in [0,1]$ then*

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Lemma 2.3.7. ([29]) *Let x, y be points in a real Hilbert space then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.3.8. ([30]) *Let H be a Hilbert space, C be a closed convex subset of H and $S : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{z_t\}$ be a sequence given by for a fixed number $t \in (0,1)$, a point $u \in C$ and*

$$z_t = tu + (1-t)Sz_t.$$

Then the strong $lim_{t\to 0} z_t$ exists and is a fixed point of S.

If X is a uniformly smooth Banach space, Reich [31] proved that strong $lim_{t\to 0} z_t$ exists and is a fixed point of S.

Lemma 2.3.9. ([32]) *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the condition:

1. $\{\alpha_n\} \subseteq [0,1]$ *and* $\Sigma_n \alpha_n = \infty$,

2. $\limsup_{n\to\infty} \beta_n \leq 0$

3. $\gamma_n \geq 0$ *and* $\Sigma_n \gamma_n < \infty$.

Then $lim_{n\to\infty} s_n = 0$.