CHAPTER 3

Admissible Functions and Convergence Theorems

In this chapter, we study fixed point theorem for nonexpansive mapping and quasi-nonexpansive mapping in Banach spaces as follows: Section 3.1 contains the definition of admissible function and iterative algorithms in terms of admissible functions. Section 3.2 contains the convergence theorems for fixed point iterative methods defined by admissible function for nonexpansive mapping and quasi-nonexpansive mapping.

3.1 Admissible Functions and Iterative Algorithms in Terms of Admissible Functions

In the previous sections, we have introduced the admissible functions. We will now give some of their examples:

Example 3.1.1. Let $X = \mathbb{R}$ with usual metric d and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$G(x,y) = \begin{cases} x & \text{if } x = y \\ \frac{2x^2y}{x^2+y^2} & \text{if } x \neq y. \end{cases}$$

Then G is an admissible function.

Example 3.1.2. Let $(X, +, \mathbb{R})$ be a real vector space, $C \subseteq X$ a convex subset, $\lambda \in (0,1)$ and $G : C \times C \to C$ defined by

$$G(x,y) = (1-\lambda)x + \lambda y, \quad x, y \in C.$$

It is easy to see that G satisfies conditions G1 and G2, then G is an admissible function.

Example 3.1.3. Let $(X, +, \mathbb{R})$ be a real vector space, $C \subseteq X$ a convex subset, $\chi : C \times C \to (0,1)$ and $G : C \times C \to C$ defined by

$$G(x,y) = (1-\chi(x,y))x + \chi(x,y)y, \quad x, y \in X$$

It is clear that G is an admissible function.

Example 3.1.4. Let $(X, +, \mathbb{R})$ be a real vector space, $C \subseteq X$ a convex subset, $n \in \mathbb{N}$ and $G_n : C \times C \to C$ defined by

$$G_n(x, y) = (1 - \frac{1}{n})x + \frac{1}{n}y, \quad x, y \in C.$$

Thus, for each $n \in \mathbb{N}$, we have G_n is an admissible function.

Example 3.1.5. Let (X, d) be a metric space endowed with a W-convex structure of Takahashi (see Definition 2.1.6). Here $W : X \times X \times [0, 1] \to X$ is an operator with the following property

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \forall x, y, u \in X, \lambda \in [0, 1].$$

We additionally suppose that $\lambda \in (0, 1)$ and $G(x, y) := W(x, y, \lambda)$. Let $x, y \in X$ and $\lambda \in (0, 1)$, we have

$$d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, x) = d(u, x).$$

Choose $u = x$, then $d(x, W(x, x, \lambda)) = 0$. That is $G(x, x) = W(x, x, \lambda) = x$.
Now we suppose $x = G(x, y) = W(x, y, \lambda)$, then

$$d(u, x) = d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

Thus,

$$(1 - \lambda)d(u, x) = (1 - \lambda)d(u, y),$$

and choose $u = x$, then $d(x, y) = 0$. That is $x = y$.

Therefore $G(x, y) := W(x, y, \lambda)$ with $\lambda \in (0, 1)$ is an admissible function.

Example 3.1.6. Let X be a nonempty set endowed with an F-convex structure of Gudder (see Definition 2.1.7), where $F : [0, 1] \times X \times X \to X$ is an operator satisfies the conditions (P1)-(P5). We additionally suppose that $\lambda \in (0, 1)$ and $G(x, y) := F(\lambda, x, y)$. It is easy to see the conditions [P3] and [P5] implies G is an admissible function.

It is clear that the iterations in example 3.1.1, 3.1.2, 3.1.3, 3.1.5 and example 3.1.6 are GK-algorithms. Now we will introduce another representation of iterative algorithms in terms of admissible functions.

Definition 3.1.1. (GM-algorithm) Let $G_n : X \times X \to X$ be an admissible function for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$ and

$$x_{n+1} = G_n(x_n, T(x_n)), \quad n \in \mathbb{N} \tag{3.1}$$

is called *the Mann algorithm corresponding to* G_n or *the GM-algorithm.*

It is easy to see that example 3.1.4 is the GM-algorithm and in the particular case when C is a nonempty convex subset of a Banach space X and $G_n(x_n, Tx_n) = (1-\lambda_n)x_n + \lambda_n Tx_n$ with $\{\lambda_n\} \subseteq [0,1]$ for $n \in \mathbb{N}$, we have that $\{x_n\} \subseteq C$, where $x_{n+1} = G_n(x_n, Tx_n)$, is a usual Mann iteration.

Definition 3.1.2. (GH-algorithm) Let $G_n : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$, $u \in X$ and

$$x_{n+1} = G_n(u, Tx_n), \quad n \in \mathbb{N} \tag{3.2}$$

is called *the Halpern algorithm corresponding to G_n* or *the GH-algorithm.*

When C is a nonempty convex subset of a Banach space X, $u \in X$ and $G_n(x_n, Tx_n) = (1-\lambda_n)u + \lambda_n Tx_n$ with $\{\lambda_n\} \subseteq [0,1]$ for $n \in \mathbb{N}$, then we have $\{x_n\} \subseteq C$, where $x_{n+1} = G_n(u, Tx_n)$, is a usual Halpern iteration.

Definition 3.1.3. (GI-algorithm) Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$ and

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(x_n, Ty_n), \ \ n \in \mathbb{N}, \end{cases} \tag{3.3}$$

is called *the Ishikawa algorithm corresponding to G_n^1 and G_n^2* or *the GI-algorithm.*

In this case, when C is a nonempty convex subset of a Banach space X, $G_n^2(x_n, Tx_n) = (1-\beta_n)x_n + \beta_n Tx_n$ and $G_n^1(x_n, T(G_n^2(x_n, Tx_n))) = (1-\alpha_n)x_n + \alpha_n T(G_n^2(x_n, Tx_n))$ with $\{\alpha_n\}, \{\beta_n\}$ are sequence of real number in $[0,1]$ for $n \in \mathbb{N}$. The sequence $\{x_n\} \subseteq C$ generated by $x_{n+1} = G_n^1(x_n, Ty_n)$, where $y_n = G_n^2(x_n, Tx_n)$ is a usual Ishikawa iteration.

Definition 3.1.4. (GS-algorithm) Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$ and

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(Tx_n, Ty_n), \ \ n \in \mathbb{N}, \end{cases} \tag{3.4}$$

is called *the S-algorithm corresponding to G_n^1 and G_n^2* or *the GS-algorithm.*

We see that when C is a nonempty convex subset of a Banach space X, $G_n^2(x_n, Tx_n) = (1-\beta_n)x_n + \beta_n Tx_n$ and $G_n^1(Tx_n, T(G_n^2(x_n, Tx_n))) = (1-\alpha_n)Tx_n + \alpha_n T(G_n^2(x_n, Tx_n))$ with $\{\alpha_n\}, \{\beta_n\}$ are sequence of real number in $[0,1]$ for $n \in \mathbb{N}$. The sequence $\{x_n\} \subseteq C$ generated by $x_{n+1} = G_n^1(Tx_n, Ty_n)$, where $y_n = G_n^2(x_n, Tx_n)$ is a S-iteration.

Definition 3.1.5. (GC-algorithm) Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $S, T : X \to X$ be operators. Then the iterative algorithm $\{x_n\} \subseteq X$ given by $x_1 \in X$ and

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(y_n, Sy_n), \ \ n \in \mathbb{N}, \end{cases} \tag{3.5}$$

is called *the Common algorithm corresponding to* G_n^1 *and* G_n^2 or *the GC-algorithm.*

If $G_n^2(x_n, T(x_n)) = (1-\beta_n)x_n + \beta_n Tx_n$ and $G_n^1(G_n^2(x_n, T(x_n)), S(G_n^2(x_n, T(x_n)))) = (1-\alpha_n)G_n^2(x_n, T(x_n)) + \alpha_n S(G_n^2(x_n, T(x_n)))$ with $\{\alpha_n\}, \{\beta_n\}$ are sequence of real number in $[0,1]$ for $n \in \mathbb{N}$ and C is a nonempty convex subset of a Banach space X. The sequence $\{x_n\} \subseteq C$ generated by $x_{n+1} = G_n^1(y_n, S(y_n))$, where $y_n = G_n^2(x_n, Tx_n)$ is a common fixed point iteration.

3.2 Convergence Theorems for Fixed Point Iterative Methods Defined by Admissible Function

In this section, we find control conditions for iterative methods defined by admissible function to converge to fixed points.

First, recall that let G be an admissible function on a normed space X. We say that G is affine Lipschitzian if there exist a constant $\mu \in [0,1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq \|\mu(x_1 - x_2) + (1-\mu)(y_1 - y_2)\|,$$

for all x_1, x_2, y_1, y_2 in X and it is clear that an admissible function in example 3.1.2 is affine Lipschitzian.

We begin with the GK-algorithm of nonexpansive mapping in a uniformly convex Banach space.

Theorem 3.2.1. *Let C be a closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ be a nonexpansive and demicompact mapping. If $G : C \times C \to C$ is an affine Lipschitzian admissible function which constant $\lambda \in (0,1)$. Then the GK-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges(strongly) to a fixed point of T in C.

Proof. By Theorem 1.1.2, $F(T)$ is nonempty set. Let $p \in F(T)$. We first show that the sequence $\{x_n - Tx_n\}$ converges strongly to zero. Since G is an affine Lipschitzian admissible function and T is nonexpansive, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|G(x_n, Tx_n) - G(p,p)\| \\ &\le \|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| \\ &\le \lambda\|(x_n - p)\| + (1-\lambda)\|(Tx_n - p)\| \\ &\le \lambda\|(x_n - p)\| + (1-\lambda)\|(x_n - p)\| \\ &= \|x_n - p\|. \end{aligned}$$

That is $\lim_{n\to\infty} \|x_n - p\|$ exist. Suppose that $\lim_{n\to\infty} \|x_n - p\| = a$, then

$$\limsup_{n\to\infty} \|Tx_n - p\| \le \lim_{n\to\infty} \|x_n - p\| = a$$

and since

$$a = \lim_{n\to\infty} \|x_{n+1} - p\| \le \lim_{n\to\infty} \|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| \le \lim_{n\to\infty} \|x_n - p\| = a,$$

we have

$$\lim_{n\to\infty} \|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| = a$$

By Theorem 2.2.16, we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This shows that $x_n - Tx_n \to 0$. Since T is demicompact and $\{x_n\}$ is bounded in C, it follows that there exist a subsequence $\{x_{n_k}\} \subseteq C$ of $\{x_n\}$ and $q \in C$ such that

$$\lim_{n\to\infty} x_{n_k} = q.$$

But T is nonexpansive, hence continuous. This implies

$$\lim_{n\to\infty} Tx_{n_k} = Tq.$$

That is

$$0 = \lim_{n\to\infty} (x_{n_k} - Tx_{n_k}) = q - Tq.$$

This means that q is a fixed point of T and since $\lim_{n\to\infty} \|x_n - q\| = \lim_{n\to\infty} \|x_{n_k} - q\| = 0$. Therefore, $\{x_n\}$ converges strongly to a fixed point of T in C. □

We now consider a class of mappings that properly includes the class of nonexpansive mappings with fixed points, that is quasi-nonexpansive mappings. The following example shows that there exists a nonlinear continuous quasi-nonexpansive mapping that is not nonexpansive.

Example 3.2.2. Let $X = l_\infty$ with $\|x\|_\infty = sup_{i\in\mathbb{N}}|x_i|$, $C = B_X = \{x \in l_\infty : \|x\|_\infty \leq 1\}$ and $T : C \to C$ a mapping define by

$$Tx = (0, x_1^2, x_2^2, x_3^2, \dots) \text{ for } x = (x_1, x_2, x_3, \dots) \in C.$$

It is clear that T is continuous mapping with unique fixed point 0 in C. Moreover,

$$\|Tx - 0\|_\infty = \|(0, x_1^2, x_2^2, x_3^2, \dots)\|_\infty \leq \|(0, x_1, x_2, x_3, \dots)\|_\infty = \|x - p\|_\infty,$$

for all $x \in C$. Then T is quasi-nonexpansive. However, for $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ and $y = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \dots)$, we have

$$\|Tx - Ty\|_\infty = \|(0, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \dots)\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty.$$

Then T is not nonexpansive mapping.

A condition that ensures strong convergence of iterative sequences to fixed points of quasi-nonexpansive type mappings was introduced in [23].

Definition 3.2.1. Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$. Then T is said to satisfy *Condition I* if there exist a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > t$ for $t \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Example 3.2.3. Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

for all $x, y \in C$, where $a, b, c \geq 0$ with $a + b + c \leq 1/2$ and $F(T) \neq \emptyset$.
Let $p \in F(T)$, then

$$\|Tx - p\| \leq a\|x - p\| + b\|x - Tx\| \leq \|x - p\| + b(\|x - p\| + \|p - Tx\|),$$

which implies that

$$\|Tx - p\| \leq \frac{a + b}{1 - b}\|x - p\| \leq \|x - p\|.$$

Hence T is quasi-nonexpansive. Observe that

$$\|Tx - p\| \geq |\|Tx - x\| - \|x - p\|| \geq \|x - p\| - \|x - Tx\|$$

That is

$$a\|x-p\|+b\|x-Tx\| \geq \|x-p\|-\|x-Tx\|,$$

which gives

$$\|x-Tx\| \geq \frac{1-a}{1+b}\|x-p\|.$$

Therefore, T is satisfies Condition I, where $f(\|x-p\|)=\frac{1-a}{1+b}\|x-p\|$.

Next, we introduce a new property for the algorithms.

Definition 3.2.2. Let $G_n : X \times X \to X$ be an admissible function on a normed space X for $n \in \mathbb{N}$. We say that $\{G_n\}$ is *sequentially affine Lipschitzian* if there exist a sequence of real number $\{\alpha_n\}$ in $[0,1]$ such that

$$\|G_n(x_1,y_1)-G_n(x_2,y_2)\| \leq \|\alpha_n(x_1-x_2)+(1-\alpha_n)(y_1-y_2)\|,$$

for all x_1, x_2, y_1 and y_2 in X.

It is easy to see that admissible functions in example 3.1.2 and 3.1.4 are sequentially affine Lipschitzian. In the particular case when $G_n(x,y)=(1-\alpha_n)x+\alpha_n y$ with $\{\alpha_n\} \subseteq [0,1]$ and $n \in \mathbb{N}$, we have $\{G_n\}$ is sequentially affine Lipschitzian.

We prove the strong convergence of the GM-iteration for quasi-nonexpansive mappings satisfying Condition I.

Theorem 3.2.4. *Let C be a closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a continuous quasi-nonexpansive mapping with satisfies Condition I. If $\{G_n\}$ is sequentially affine Lipschitzian with a sequence $\{\alpha_n\}$ which is bounded away from 0 and 1. Then the GM-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1}=G_n(x_n,Tx_n), \quad n \in \mathbb{N}$$

converges(strongly) to a fixed point of T in C.

Proof. Let $p \in F(T)$. Since $\{G_n\}$ is sequentially affine Lipschitzian and T is quasi-nonexpansive, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|G_n(x_n,Tx_n)-G_n(p,p)\| \\
&\leq \|\alpha_n(x_n-p)+(1-\alpha_n)(Tx_n-p)\| \\
&\leq \alpha_n\|(x_n-p)\|+(1-\alpha_n)\|(Tx_n-p)\| \\
&\leq \alpha_n\|(x_n-p)\|+(1-\alpha_n)\|(x_n-p)\| \\
&= \|x_n-p\|.
\end{aligned}$$

Using the same proof as in Theorem 3.2.1, we can show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Because for $p \in F(T)$, $\|x_{n+1} - p\| \leq \|x_n - p\|$, it follows that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Since T satisfies condition I, we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))), n \geq 0.$$

This implies, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Then for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \epsilon/2 \text{ for all } n \geq n_0.$$

Consider, for $n, m \geq n_0$. So there is a $p \in F(T)$ such that $d(x_{n_0}, p) < \epsilon/2$, we have

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq 2\|x_{n_0} - p\| < \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence and by completeness of X, we have $\lim_{n\to\infty} x_n = q$ for some $q \in C$. Since T is continuous and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Therefore, $q \in F(T)$ implies that $\{x_n\}$ converges strongly to a fixed point of T in C. $\square$

Now, we will work on Hilbert space with GH-algorithm and find a new condition for convergence of this algorithm.

Definition 3.2.3. Let $G_n : X \times X \to X$ be a admissible function on a normed space X for $n \in \mathbb{N}$. We say that G_n has *the property (C*)* if there exist a sequence of real number α_n in $[0, 1]$ such that

$$\|G_n(x_1, y_1) - G_{n-1}(x_2, y_2)\| \leq \|(\alpha_n x_1 - \alpha_{n-1} x_2) + ((1 - \alpha_n)y_1 - (1 - \alpha_{n-1})y_2)\|,$$

for all x_1, x_2, y_1 and y_2 in X.

It is clear that admissible functions in example 3.1.2 and 3.1.4 have the property (C*) and if $\alpha_n = \lambda$(constant), then the property (C*) is an affine Lipschitzian property.

Theorem 3.2.5. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $G_n : H \times H \to H$ is an admissible function which has the property (C*) for each $n \in \mathbb{N}$ and $\{G_n\}$ is sequentially affine Lipschitzian with $\{\alpha_n\}$ satisfying the following conditions:*

1. $\lim_{n\to\infty} \alpha_n = 0$,

2. $\Sigma_{n=0}^{\infty} \alpha_n = \infty$,

3. $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0.$

Then the GH-algorithm $\{x_n\}_{n=1}^{\infty}$ *given by* $x_1 \in C$, $u \in C$ *and*

$$x_{n+1} = G_n(u, Tx_n), \quad n \in \mathbb{N},$$

converges(strongly) to a fixed point of T *in* C.

Proof. Let $q \in F(T)$ and since $\{G_n\}$ is sequentially affine Lipschitzian, we get

$$\begin{aligned} \|x_{n+1} - q\| &= \|G_n(u, Tx_n) - G_n(q, q)\| \\ &\leq \|\alpha_n(u - q) + (1 - \alpha_n)(Tx_n - q)\| \\ &\leq \alpha_n\|(u - q)\| + (1 - \alpha_n)\|(x_n - q)\| \\ &\leq \alpha_n(\max\{\|u - q\|, \|x_n - q\|\}) + (1 - \alpha_n)(\max\{\|u - q\|, \|x_n - q\|\}) \\ &\leq \max\{\|u - q\|, \|x_n - q\|\}. \end{aligned}$$

Since

$$\|x_n\| - \|q\| \leq \|x_{n+1} - q\| \text{ and } \|Tx_n\| - \|q\| \leq \|Tx_{n+1} - q\| \leq \|x_{n+1} - q\|.$$

Then by induction, we have

$$\|x_{n+1} - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\}, \quad n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ and $\{Tx_n\}$ are bounded.

We next use Lemma 2.3.8 to show $\limsup_{n\to\infty}\langle u - p, z_n - p\rangle \leq 0$, where $z_n = \alpha_n u + (1 - \alpha_n)Tx_n$ and $p = \lim_{t\to 0} z_t$ such that $z_t = tu + (1 - t)Tz_t, t \in (0, 1)$.

The boundedness of $\{Tx_n\}$ also implies that $\{z_n\}$ is bounded and since T nonexpansive, thus $\{Tz_n\}$ is bounded. By Lemma 2.3.7, we have

$$
\begin{aligned}
\|z_t - z_n\|^2 &= \|t(u - z_n) + (1-t)(Tz_t - z_n)\|^2 \\
&\leq (1-t)^2\|Tz_t - z_n\|^2 + 2t\langle u - z_n, z_t - z_n\rangle \\
&= (1-t)^2\|Tz_t - Tz_n + Tz_n - z_n\|^2 + 2t\langle u - z_n + z_t - z_t, z_t - z_n\rangle \\
&\leq (1-t)^2(\|Tz_t - Tz_n\| + \|Tz_n - z_n\|)^2 + 2t(\|z_t - z_n\|^2 + \langle u - z_t, z_t - z_n\rangle) \\
&= (1 - 2t + t^2)(\|Tz_t - Tz_n\|^2 + 2\|Tz_t - Tz_n\|\|Tz_n - z_n\| + \|Tz_n - z_n\|^2) \\
&\quad +2t\|z_t - z_n\|^2 + 2t\langle u - z_t, z_t - z_n\rangle \\
&\leq (1 - 2t + t^2)(\|z_t - z_n\|^2 + 2\|z_t - z_n\|\|Tz_n - z_n\| + \|Tz_n - z_n\|^2) \\
&\quad +2t\|z_t - z_n\|^2 + 2t\langle u - z_t, z_t - z_n\rangle \\
&= (1 + t^2)\|z_t - z_n\|^2 + 2(1-t)^2\|z_t - z_n\|\|Tz_n - z_n\| + (1-t)^2\|Tz_n - z_n\|^2 \\
&\quad +2t\langle u - z_t, z_t - z_n\rangle \\
&= (1 + t^2)\|z_t - z_n\|^2 + (1-t)^2\|Tz_n - z_n\|(2\|z_t - z_n\| + \|Tz_n - z_n\|) \\
&\quad +2t\langle u - z_t, z_t - z_n.\rangle
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle u - z_t, z_n - z_t\rangle &\leq \frac{t}{2}\|z_t - z_n\|^2 + \frac{(1-t)^2}{2t}\|Tz_n - z_n\|(2\|z_t - z_n\| + \|Tz_n - z_n\|) \\
&\leq \frac{t}{2}\|z_t - z_n\|^2 + \frac{1}{2t}\|Tz_n - z_n\|(2\|z_t - z_n\| + \|Tz_n - z_n\|). \qquad (3.6)
\end{aligned}
$$

Since G_n has the property (C*) and $\{G_n\}$ is sequentially affine Lipschitzian, we have that

$$
\begin{aligned}
\|x_{n+1} - Tx_n\| &= \|G_n(u, Tx_n) - G_n(Tx_n, Tx_n)\| \\
&\leq \|\alpha_n(u - Tx_n) + (1-\alpha_n)(Tx_n - Tx_n)\| \\
&= \alpha_n\|u - Tx_n\| \to 0, \text{ as } n \to \infty \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|G_n(u, Tx_n) - G_{n-1}(u, Tx_{n-1})\| \\
&\leq \|(\alpha_n u - \alpha_{n-1}u) + ((1-\alpha_n)Tx_n - (1-\alpha_{n-1})Tx_{n-1}))\| \\
&= \|(\alpha_n - \alpha_{n-1})(u - Tx_{n-1}) + (1-\alpha_n)(Tx_n - Tx_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u - Tx_{n-1}\| + (1-\alpha_n)\|Tx_n - Tx_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n\beta_n,
\end{aligned}
$$

where $M := \sup_{n\geq 1}\|u - Tx_{n-1}\|$ and $\beta_n := M\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}$. By assumptions and $\{Tx_n\}$ is bounded, then $M < \infty$ and $\beta_n \to 0$. Hence by Lemma 2.3.9 we get $\|x_{n+1} - x_n\| \to 0$. This together with (3.7) implies that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Consider

$$\begin{aligned}
\|z_n - Tz_n\| &= \|\alpha_n u + (1-\alpha_n)Tx_n - Tz_n\| \\
&\leq \alpha_n\|u - Tx_n\| + \|Tx_n - Tz_n\| \\
&\leq \alpha_n\|u - Tx_n\| + \|x_n - z_n\| \\
&= \alpha_n\|u - Tx_n\| + \|x_n - \alpha_n u - (1-\alpha_n)Tx_n\| \\
&\leq 2\alpha_n\|u - Tx_n\| + \|x_n - Tx_n\|. \qquad (3.8)
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, we have $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$.
By taking lim sup as $n \to \infty$ in (3.6), we can conclude that

$$\limsup_{n\to\infty}\langle u - z_t, z_n - z_t\rangle \leq \limsup_{n\to\infty}\frac{t}{2}\|z_t - z_n\|^2,$$

and since $\lim_{t\to 0} z_t = p$, we get

$$\limsup_{n\to\infty}\langle u - p, z_n - p\rangle \leq 0. \qquad (3.9)$$

Lastly, we show that $\{x_n\}$ converges strongly to p. By Lemma 2.3.8, implies p is fixed point of T. So

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|G_n(u, Tx_n) - G_n(p,p)\|^2 \\
&\leq \|\alpha_n(u-p) + (1-\alpha_n)(Tx_n - p)\|^2 \\
&\leq (1-\alpha_n)^2\|Tx_n - p\|^2 + 2\alpha_n\langle u - p, (1-\alpha_n)(Tx_n - p) + \alpha_n(u-p)\rangle \\
&= (1-\alpha_n)^2\|Tx_n - p\|^2 + 2\alpha_n\langle u - p, \alpha_n u + (1-\alpha_n)Tx_n - p\rangle \\
&= (1-\alpha_n)^2\|Tx_n - p\|^2 + 2\alpha_n\langle u - p, z_n - p\rangle \\
&\leq (1-\alpha_n)\|Tx_n - p\|^2 + 2\alpha_n\langle u - p, z_n - p\rangle,
\end{aligned}$$

for every $n \in \mathbb{N}$. Thus Lemma 2.3.9 and (3.9) imply $\lim_{n\to\infty} x_n = p$. □

We show that GI-algorithm with only sequentially affine Lipschitzian property converges weakly on Hilbert space.

Theorem 3.2.6. *Let C be a closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping. If $G_n^1, G_n^2 : H \times H \to H$ are admissible function for all $n \in \mathbb{N}$ and $\{G_n^1\}, \{G_n^2\}$ are sequentially affine Lipschitzian with $\{\alpha_n\}$ and $\{\beta_n\}$ respectively, and suppose that*

$$\limsup_{n\to\infty}\alpha_n < 1 \text{ and } 0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1.$$

Then the GI-algorithm $\{x_n\}_{n=1}^{\infty}$ *given by* $x_1 \in X$ *and*

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(x_n, Ty_n), \;\; n \in \mathbb{N}, \end{cases}$$

converges weakly to a fixed point of T*.*

Proof. By Theorem 1.1.2, $F(T)$ is nonempty and convex set. Let us consider $p \in F(T)$, $\{G_n^2\}$ is sequentially affine Lipschitzian with $\{\beta_n\}$ and T is nonexpansive mapping, we have

$$\begin{aligned} \|y_n - p\| &= \|G_n^2(x_n, Tx_n) - G_n^2(p,p)\| \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned}$$

and since $\{G_n^1\}$ is sequentially affine Lipschitzian with $\{\alpha_n\}$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|G_n^1(x_n, Ty_n) - G_n^1(p,p)\| \\ &\leq \|\alpha_n(x_n - p) + (1-\alpha_n)(Ty_n - p)\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \|x_n - q\|, \end{aligned}$$

which shows that $\lim_{n\to\infty}\|x_n - p\|$ is exist. By Lemma 2.3.6, we get

$$\begin{aligned} \|y_n - p\|^2 &= \|G_n^2(x_n, Tx_n) - G_n^2(p,p)\|^2 \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|G_n^1(x_n, Ty_n) - G_n^1(p,p)\|^2 \\ &\leq \|\alpha_n(x_n - p) + (1-\alpha_n)(Ty_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Ty_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2) \\ &= \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2. \end{aligned}$$

Thus

$$\beta_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

By the control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, we get $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Next, we show that if $\{x_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to $p_0 \in C$, then p_0 is a fixed point of T. Consider

$$\|x_{n_j} - Tp_0\| \le \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - Tp_0\| \le \|x_{n_j} - Tx_{n_j}\| + \|x_{n_j} - p_0\|,$$

that is

$$\limsup_{j\to\infty}(\|x_{n_j} - Tp_0\| - \|x_{n_j} - p_0\|) \le \limsup_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0. \tag{3.10}$$

By definition of inner product, we have

$$\begin{aligned} \|x_{n_j} - Tp_0\|^2 &= \|(x_{n_j} - p_0) + (p_0 - Tp_0)\|^2 \\ &= \|x_{n_j} - p_0\|^2 + \|p_0 - Tp_0\|^2 + 2\langle x_{n_j} - p_0, p_0 - Tp_0\rangle, \end{aligned}$$

This together with $x_{n_j} \rightharpoonup p_0$ as $j \to \infty$, we obtain

$$\lim_{j\to\infty}(\|x_{n_j} - Tp_0\|^2 - \|x_{n_j} - p_0\|^2) = \|p_0 - Tp_0\|^2$$

Since C is bounded, the sequence $\{\|x_{n_j} - Tp_0\| - \|x_{n_j} - p_0\|\}, \{\|x_{n_j} - Tp_0\| + \|x_{n_j} - p_0\|\}$ are bounded, and by equation (3.10) we get

$$\|p_0 - Tp_0\|^2 = \lim_{j\to\infty}((\|x_{n_j} - Tp_0\| - \|x_{n_j} - p_0\|)(\|x_{n_j} - Tp_0\| + \|x_{n_j} - p_0\|)) \le 0.$$

Therefore, $p_0 \in F(T)$.

Lastly, we show that $\{x_n\}$ converges weakly to a fixed point of T. Let $g : F(T) \to [0,\infty)$ defined by

$$g(p) = \lim_{n\to\infty} \|x_n - p\|,$$

since $\|x_{n+1} - p\| \le \|x_n - p\|$ for each $n \in \mathbb{N}$, which shows that the function g is well defined and it's clear that g is a continuous convex function on $F(T)$. Let

$$d_0 = \inf\{g(p) : p \in F(T)\}.$$

For each $\epsilon > 0$, we define

$$F_\epsilon = \{y \in F(T) : g(y) \le d_0 + \epsilon\},$$

by definition of infimum and C is bounded, we have F_ϵ is nonempty closed convex bounded subset of H. Therefore $\bigcap_{\epsilon>0} F_\epsilon \neq \emptyset$ by Lemma 2.2.10. Let

$$F_0 = \{y \in Fix(T) : g(y) = d_0\},$$

it is easy to see that $F_0 = \bigcap_{\epsilon>0} F_\epsilon$. Moreover, F_0 contains exactly one point. Indeed, let $q_0, q_1 \in F_0$ such that $q_0 \neq q_1$,$\lambda \in (0,1)$ and $q_\lambda = (1-\lambda)q_0 + \lambda q_1$, that is

$$\begin{aligned}
(d_0)^2 &= (g(q_\lambda))^2 \\
&= \lim_{n\to\infty} \|q_\lambda - x_n\|^2 \\
&= \lim_{n\to\infty} \|\lambda q_1 + (1-\lambda)q_0 - x_n\|^2 \\
&= \lim_{n\to\infty} \|\lambda(q_1 - x_n) + (1-\lambda)(q_0 - x_n)\|^2 \\
&= \lim_{n\to\infty} (\lambda^2\|q_1 - x_n\|^2 + (1-\lambda)^2\|q_0 - x_n\|^2 + 2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n >) \\
&= \lim_{n\to\infty} (\lambda^2\|q_1 - x_n\|^2 + (1-\lambda)^2\|q_0 - x_n\|^2 + 2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|) \\
&\quad + \lim_{n\to\infty} (2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n > -2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|) \\
&= \lim_{n\to\infty} (\lambda\|q_1 - x_n\| + (1-\lambda)\|q_0 - x_n\|)^2 \\
&\quad + \lim_{n\to\infty} (2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n > -2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|) \\
&= (\lambda d_0 + (1-\lambda)d_0)^2 \\
&\quad + \lim_{n\to\infty} (2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n > -2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|) \\
&= (d_0)^2 + \lim_{n\to\infty} (2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n > -2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|).
\end{aligned}$$

Hence

$$\lim_{n\to\infty} (2\lambda(1-\lambda) < q_1 - x_n, q_0 - x_n > -2\lambda(1-\lambda)\|q_1 - x_n\|\|q_0 - x_n\|) = 0,$$

since $\lim_{n\to\infty} \|q_1 - x_n\| = d_0 = \lim_{n\to\infty} \|q_0 - x_n\|$, that is

$$\lim_{n\to\infty} < p_1 - x_n, p_0 - x_n >= (d_0)^2.$$

Therefore

$$\begin{aligned}
\|q_1 - q_0\|^2 &= \lim_{n\to\infty} \|(q_1 - x_n) + (x_n - q_0)\| \\
&= \lim_{n\to\infty} (\|p_1 - x_n\|^2 + \|x_n - p_0\|^2 - 2 < p_1 - x_n, p_0 - x_n >) \\
&= (d_0)^2 + (d_0)^2 - 2(d_0)^2 = 0,
\end{aligned}$$

giving a contradiction. Since $\{x_{n_j}\}$ converges weakly to $p \in F(T)$, we have

$$\|x_{n_j} - q_0\|^2 = \|x_{n_j} - p + p - q_0\|^2 = \|x_{n_j} - p\|^2 + \|p - q_0\|^2 - 2 < x_{n_j} - p, p - q_0 >,$$

thus

$$(d_0)^2 = \lim_{j\to\infty} \|x_{n_j} - q_0\|^2 = (g(p))^2 + \|p - q_0\|^2 - 0,$$

and since $(g(p))^2 \geq (d_0)^2$, then $\|p - q_0\|^2 = 0$, that is $p = q_0$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. □

Theorem 3.2.7. *Let X be a uniformly convex Banach space that satisfies Opials condition, C be a nonempty closed convex subset of X and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $\{G_n^1\}, \{G_n^2\}$ are sequentially affine Lipschitzian with $\{\alpha_n\}$ and $\{\beta_n\}$ respectively. Suppose that $\limsup_{n\to\infty} \alpha_n < 1$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then the GS-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in X$ and*

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(Tx_n, Ty_n), \ \ n \in \mathbb{N}, \end{cases}$$

converges weakly to a fixed point of T.

Proof. Let $p \in F(T)$. By using the same proof as in Theorem 3.2.6, we can show that $\|y_n - p\| \leq \|x_n - p\|$. Since $\{G_n^1\}$ is sequentially affine Lipschitzian with $\{\alpha_n\}$ and T is nonexpansive mapping, we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|G_n^1(Tx_n, Ty_n) - G_n^1(p,p)\| \\ &\leq \|\alpha_n(Tx_n - p) + (1-\alpha_n)(Ty_n - p)\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \|x_n - q\|, \end{aligned}$$

therefore, $\lim_{n\to\infty}\|x_n - p\|$ is exist and hence $\{x_n\}$ is bounded. By Theorem 2.2.15, there is a continuous strictly increasing convex mapping g with $g(0) = 0$ such that

$$\begin{aligned} \|y_n - p\|^2 &= \|G_n^2(x_n, Tx_n) - G_n^2(p,p)\|^2 \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|), \end{aligned}$$

and

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|G_n^1(Tx_n, Ty_n) - G_n^1(p,p)\|^2 \\
&\le \|\alpha_n(Tx_n - p) + (1-\alpha_n)(Ty_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|Tx_n - Ty_n\|) \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)) \\
&= \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)g(\|x_n - Tx_n\|).
\end{aligned}
$$

Thus

$$\beta_n(1-\alpha_n)(1-\beta_n)g(\|x_n - Tx_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies by the control conditions on α_n and β_n, we get

$$\lim_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Because $\{x_n\}$ is bounded in X, it follows that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to $p \in C$. By Theorem 2.2.17, $I - T$ is demiclosed at zero, from $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, we have $(I-T)p = 0$, so that $p \in F(T)$.

Next, we show that $\{x_n\}$ converges weakly to a fixed point of T. Suppose there exist another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $q \in F(T)$ such that $q \neq p$. Since X satisfies the Opials condition, we have

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{j\to\infty}\|x_{n_j} - p\| < \lim_{j\to\infty}\|x_{n_j} - q\| = \lim_{n\to\infty}\|x_n - q\|,$$

and

$$\lim_{n\to\infty}\|x_n - q\| = \lim_{j\to\infty}\|x_{n_j} - q\| < \lim_{j\to\infty}\|x_{n_j} - p\| = \lim_{n\to\infty}\|x_n - p\|,$$

a contradiction, hence $p = q$. Therefore $\{x_n\}$ converges weakly to $p \in F(T)$. □

We next consider common fixed point of two nonexpansive mappings.

Theorem 3.2.8. *Let C be a closed convex subset of a uniformly convex Banach space X. Let $S, T : C \to C$ be two nonexpansive mappings such that one of the mappings T and S satisfies Condition I and $F(S) \cap F(T) \neq \emptyset$. If $G_n^1, G_n^2 : H \times H \to H$ are admissible function and $\{G_n^1\}, \{G_n^2\}$ are sequentially affine Lipschitzian with $\{\alpha_n\}$ and $\{\beta_n\}$ respectively. Suppose that*

$$0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1 \text{ and } 0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1.$$

Then the GC-algorithm $\{x_n\}_{n=1}^{\infty}$ *given by* $x_1 \in X$ *and*

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(y_n, Sy_n), \ \ n \in \mathbb{N}, \end{cases}$$

converges(strongly) to a common fixed point of S *and* T.

Proof. Let $p \in F(S) \cap F(T)$. By using the same proof as in Theorem 3.2.6, we get $\|y_n - p\| \leq \|x_n - p\|$. Since T is nonexpansive mapping, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|G_n^1(y_n, Sy_n) - G_n^1(p, p)\| \\ &\leq \|\alpha_n(y_n - p) + (1 - \alpha_n)(Sy_n - p)\| \\ &\leq \alpha_n\|y_n - p\| + (1 - \alpha_n)\|y_n - p\| \\ &= \|y_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - p\|$ exist for any $p \in F(S) \cap F(T)$. By Theorem 2.2.15, there exist a continuous strictly increasing convex mapping g with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|G_n^1(y_n, Ty_n) - G_n^1(p, p)\|^2 \\ &\leq \|\alpha_n(y_n - p) + (1 - \alpha_n)(Ty_n - p)\|^2 \\ &\leq \alpha_n\|y_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|Tx_n - Ty_n\|) \\ &\leq \|y_n - p\|^2 \\ &= \|G_n^2(x_n, Tx_n) - G_n^2(p, p)\|^2 \\ &\leq \|\beta_n(x_n - p) + (1 - \beta_n)(Tx_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 - \beta_n(1 - \beta_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - p\|^2 - \beta_n(1 - \beta_n)g(\|x_n - Tx_n\|), \end{aligned}$$

which implies

$$\beta_n(1 - \beta_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

By definition of g and condition of $\{\beta_n\}$, we get $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose that $\lim_{n\to\infty} \|x_n - p\| = c$ and since $\|y_n - p\| \leq \|x_n - p\|$, we have

$$\limsup_{n\to\infty} \|y_n - p\| \leq c.$$

Also from $\|Sy_n - p\| \leq \|y_n - p\|$, implies that

$$\limsup_{n\to\infty} \|Sy_n - p\| \leq c.$$

Moreover,

$$\begin{aligned}\|x_{n+1}-p\| &= \|G_n^1(y_n,Sy_n)-G_n^1(p,p)\|\\ &\le \|\alpha_n(y_n-p)+(1-\alpha_n)(Sy_n-p)\|\\ &\le \|x_n-q\|.\end{aligned}$$

That is $\lim_{n\to\infty}\|\alpha_n(y_n-p)+(1-\alpha_n)(Sy_n-p)\|=c$. By Theorem 2.2.16, we have $\lim_{n\to\infty}\|y_n-Sy_n\|=0$. Since

$$\begin{aligned}\|y_n-x_n\| &= \|G_n^2(x_n,Tx_n)-G_n^2(x_n,x_n)\|\\ &\le \|\beta_n(x_n-x_n)+(1-\beta_n)(Tx_n-x_n)\|,\\ &= (1-\beta_n)\|Tx_n-x_n\|,\end{aligned}$$

it implies that $\lim_{n\to\infty}\|y_n-x_n\|=0$, and since

$$\begin{aligned}\|y_n-Tx_n\| &= \|G_n^2(x_n,Tx_n)-G_n^2(Tx_n,Tx_n)\|\\ &\le \|\beta_n(x_n-Tx_n)+(1-\beta_n)(Tx_n-Tx_n)\|,\\ &\le \beta_n\|x_n-Tx_n\|,\end{aligned}$$

we have $\lim_{n\to\infty}\|y_n-Tx_n\|=0$. Therefore

$$\begin{aligned}\|x_n-Sx_n\| &\le \|x_n-Tx_n\|+\|Tx_n-y_n\|+\|y_n-Sy_n\|+\|Sy_n-Sx_n\|\\ &\le \|x_n-Tx_n\|+\|Tx_n-y_n\|+\|y_n-Sy_n\|+\|y_n-x_n\|,\end{aligned}$$

and so $\lim_{n\to\infty}\|x_n-Sx_n\|=0$. Thus from S,T satisfy condition I, we get

$$\lim_{n\to\infty} f(d(x_n,F)) \le \lim_{n\to\infty}\|x_n-Tx_n\|=0$$

or

$$\lim_{n\to\infty} f(d(x_n,F)) \le \lim_{n\to\infty}\|x_n-Sx_n\|=0,$$

where $F=F(S)\cap F(T)$. In both the cases, we have

$$\lim_{n\to\infty} f(d(x_n,F))=0.$$

This implies $\lim_{n\to\infty} d(x_n,F)=0$. By using the same proof as in Theorem 3.2.4, we can conclude that $\{x_n\}$ converges(strongly) to a common fixed point of S and T. □

Theorem 3.2.9. *Let C be a closed convex subset of a uniformly convex Banach space X that satisfies Opials condition and let $S,T:C\to C$ be two nonexpansive mappings with $F(S)\cap F(T)\neq\emptyset$. If $\{x_n\}$ be a sequence as in Theorem 3.2.8, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F(S) \cap F(T)$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ as proved in Theorem 3.2.8. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. Let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Theorem 2.2.17, we get $I - T$ and $I - S$ are demiclosed at 0, therefore we obtain $Tu = u$ and $Su = u$. Similarly, we have $Tv = v$ and $Sv = v$. That is $u, v \in F(S) \cap F(T)$.

Next, we prove the uniqueness. Since X satisfies the Opials condition and suppose $u \neq v$, then

$$\lim_{n\to\infty} \|x_n - u\| = \lim_{j\to\infty} \|x_{n_j} - u\| < \lim_{j\to\infty} \|x_{n_j} - v\| = \lim_{n\to\infty} \|x_n - v\|,$$

and

$$\lim_{n\to\infty} \|x_n - v\| = \lim_{j\to\infty} \|x_{n_j} - v\| < \lim_{j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.$$

This is a contradiction, hence $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of S and T. $\square$