CHAPTER 2

Background

2.1 Mechanical vibratory systems

Controlling vibration of mechanical systems has been an area of research interest for many decades. Vibration can cause damage to a machine or the environment. It can also been detrimental to machine performance. Thus, to reduce vibration by suppression, isolation, cancelation or complete elimination is often desirable. Examples of real-world system, where vibration is unwanted and must be dealt with through active or passive control methods include, hard disk drives, overhead cranes, vibration isolation systems, spacecraft structures, etc. However, in some applications the occurrence of vibration is desirable and preferred to be maximized e.g. tuning fork, mobile phone and audio speaker. There are also situations, like energy harvesting, where vibration of a specific frequency is more desirable to maximize energy conversion.

Mechanical structures are never rigid and, when moved, the whole structure may vibrate. However, it is often possible to derive a model of the whole system which allows the vibration behavior to be precisely described. There may be numerous physical parameters that determine the dynamic properties of a flexible structure, such as the mass, shape, material stiffness, etc. However, the basic vibration behaviors may be captured by two key parameters which are the natural frequency ω_n and the damping ratio ζ. These parameters relate to the frequency and decay-rate for free vibration of the structure, respectively. For a continuous structure there will usually be more than one (up to an infinite number of) distinct values for ω_n and ζ, depending on the number of modes of vibration that are included in the model.

2.1.1 Mathematical model of a flexible structure

In many situations, a flexible structure may be modeled as a Linear Time-Invariant (LTI) system using state-space equations having the form

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector having m components and t is time. Through an appropriate choice of states, the characteristic matrix $A \in \mathbb{R}^{nxn}$ can be expressed in block diagonal form:

$$A = blkdiag\,(A_0, A_1, ..., A_N)\,.$$

The matrix $B \in \mathbb{R}^{nxm}$ distributes the input u (which often represents an applied force) and will have the form

$$B = \begin{bmatrix} B_0^T & B_1^T & ... & B_N^T \end{bmatrix}^T$$

Here, the variables in the state vector can be ordered so that the first two states represent the overall motion of the mechanical system i.e. the motion of the rigid body mode. For the single input ($m = 1$) case, and with the first two states chosen as displacement and velocity:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & c \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \tag{2.2}$$

Here, c is a time constant and relates to the effect of viscous friction on net motion. The scalar b_0 is a coefficient that scales the control input and is normally associated with the overall inertia of the system. The constant c may be zero if friction is neglected or not present. In systems driven by a DC motor under voltage control, the effect of the back electromotive force (back EMF) may act in an equivalent way to viscous friction and may be taken into account with a model having the same form [46].

The sub matrices $A_i,\ i = 1, 2, ..., N$ capture each of the N modes of vibration for the structure. Again, through appropriate choice of state variables, each A_i can be written as

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix}, \quad i = 1,\, 2,\, \dots,\, N \tag{2.3}$$

where ω_i and ζ_i represent natural frequency and damping ratio of each flexible mode with $\omega_1 < \omega_2 < ... < \omega_N$.

To demonstrate how the model of the flexible structure in the form (2.3) may be obtained, let us start by considering a simple two-mass spring damper system as shown in Fig. 2.1. The vibratory characteristics of such a system are similar to those of a flexible structure and such a model may be used as an equivalent 'lumped mass' representation. Let the control input be the force acting on the mass A. By using Newton's laws, the

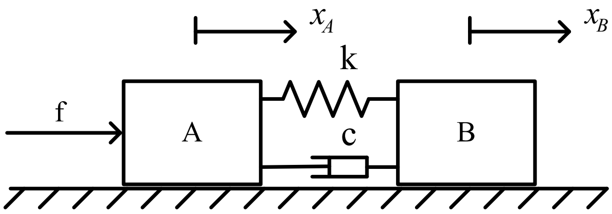


Figure 2.1: Two-mass spring damper model

differential equation of motion for the relative displacement $x_{AB} = x_A - x_B$ can be obtained as

$$\ddot{x}_{AB} + 2c\left(\frac{1}{m_A} + \frac{1}{m_B}\right)\dot{x}_{AB} - 2k\left(\frac{1}{m_A} + \frac{1}{m_B}\right)x_{AB} = \frac{1}{m_A}f. \tag{2.4}$$

Equation (2.4) can be transformed into the second order transfer function:

$$\frac{X_{AB}(s)}{F(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2.5}$$

where $\omega_n = \sqrt{2k\left(\frac{1}{m_A} + \frac{1}{m_B}\right)}$ and $\zeta = \frac{c}{\sqrt{2k}}.\sqrt{\left(\frac{1}{m_A} + \frac{1}{m_B}\right)}$. It is well-known that the free response of this system involves oscillation with exponential decay. By letting the relative displacement x_{AB} and its derivative $\dot{x}_{AB}$ be considered as the state variables of the system, the equation of motion (2.4) (and thus the transfer function model (2.5)) can be transformed into the state-space form of (2.3). Considering this form of model, some feedback control technique like pole placement may be implemented to reduce the vibration due to the effect of the complex pole of the system [27], [82]. However, such a control approach does not directly allow the specified boundary conditions to be exactly achieved.

A closely related mechanical system for which the dynamic model has the same form as the flexible structure model (2.3) is the pendulum. This system is representative of an overhead crane where damping effects, e.g. due to the air resistance, are sometimes neglected [1], [2]. Although the exact equations of motion for a pendulum are non-linear, to acquire the linear model, a linearization around the equilibrium point may be made by using a small angle assumption.

By applying a state transformation, with proper transformation matrix M, defined by

$$M_i = \begin{bmatrix} \frac{1}{\omega_{d_i}} & 0 \\ -\frac{\zeta_i}{\sqrt{1-\zeta^2}} & 1 \end{bmatrix} \tag{2.6}$$

(so that the state variable $x(t)$ is transformed to $x(t) = M\tilde{x}(t)$) the flexible mode submatrices A_i in (2.3) can be converted to the form

$$\tilde{A}_i = M_i^{-1} A_i M_i = \begin{bmatrix} -\zeta_i \omega_i & \omega_{d_i} \\ -\omega_{d_i} & -\zeta_i \omega_i \end{bmatrix}, \quad i = 1, 2, \ldots, N \tag{2.7}$$

where $\omega_{d_i} = \omega_i \sqrt{1 - \zeta_i^2}$ are the damped natural frequencies. The model (2.7) is then called *modal form*. In this form, the matrix $\tilde{A}_i$ is anti-symmetric and the parameter ω_{d_i} appears in elements of $\tilde{A}_i$. This form will be seen to be useful for parameter-based optimization.

A more general model including Coulomb friction effects may also be considered. Coulomb friction arises due to the kinetic (sliding) velocity between surfaces and results in a force that acts in the opposite direction to the motion. Suppose that a friction force arises at a location where the sliding velocities can be expressed

$$V_c = Cx(t). \tag{2.8}$$

The corresponding forces will be

$$f_c = B_c \mathrm{sgn}\left(Cx(t)\right) \tag{2.9}$$

where B_c is a diagonal matrix with elements f_{c_i} that specify the magnitude of each friction force and sgn(x) is the signum function defined according to

$$sgn\left(x_i\right) = \begin{cases} 1, & x_i > 0 \\ 0, & x_i = 0 \\ -1, & x_i < 0 \end{cases} \tag{2.10}$$

The system equation has an additional term (composed with (2.1)):

$$\dot{x}(t) = Ax(t) + Bu(t) + B_c \mathrm{sgn}\left(Cx(t)\right) \tag{2.11}$$

The magnitudes and location of Coulomb friction forces are accounted for in the definition of B_c. The non-linearity of the system dynamics is due to the presence of sgn in the state equation and this makes achieving effective and accurate control of notion more complicated.

2.1.2 Boundary conditions

Suppose a motion task involves a specified initial and final position of the system/structure. Appropriate boundary conditions must then be considered for the dynamic equations (2.1) or (2.11). These will match the behavior of the system at the initial and final time according to the given tasks. Achieving a rest-to-rest motion is a common objective for motion control applications. To achieve rest-to-rest motion for a vibratory or flexible structure it may be assumed that the system starts from complete rest, i.e. there is no initial vibration. Similarly, at the end of motion, a complete stop is required with no persisting vibration. This condition is often called 'Zero-Residual-Vibration' (ZRV) and will correspond to the situation where all vibratory states of the system are zero-valued.

Consider the model of the flexible structure in the form (2.1) together with (2.2) and (2.3), the ZRV condition demands the all state variables, except possibly the first one (overall displacement), are zero-valued at the final time. If γ is the distance traveled (move distance) then the boundary conditions may be written

$$x(t_0) = [0\ 0\ 0\ ...\ 0]^T \qquad x(t_f) = [\gamma\ 0\ 0\ ...\ 0]^T \tag{2.12}$$

where t_0 and t_f are initial time and final time for the motion interval respectively.

A different control task is one that involves changing from an in-motion state to a rest state. Suppose the system is initially moving without vibration and with a given velocity ν. At the final time, the ZRV condition is still required to be satisfied and so $x\,(t_f)$ is specified as in (2.12). Here the initial state is

$$x(t_0) = [0\ \nu\ 0\ ...\ 0]^T \tag{2.13}$$

Note that, in this case, γ may be unspecified since the task is simply to bring the system to rest as quickly as possible. Therefore, the first (displacement) state of the rigid body mode in (2.2) can be neglected completely and γ obtained later by integration of the velocity solution.

In general, if the final state values are an equilibrium then the system states can be shifted (transformed) to make the final point the origin. Suppose new states are defined as

$$\tilde{x} = x + d \tag{2.14}$$

where d is a constant vector then

$$\dot{\tilde{x}} = A\tilde{x} - Ad + Bu \tag{2.15}$$

With $d = -x_f$, the new boundary conditions for $\tilde{x}$ are

$$\tilde{x}_0 = x_0 - x_f, \qquad \tilde{x}_f = 0 \tag{2.16}$$

and if x_f is an equilibrium then $Ad = -Ax_f = 0$.

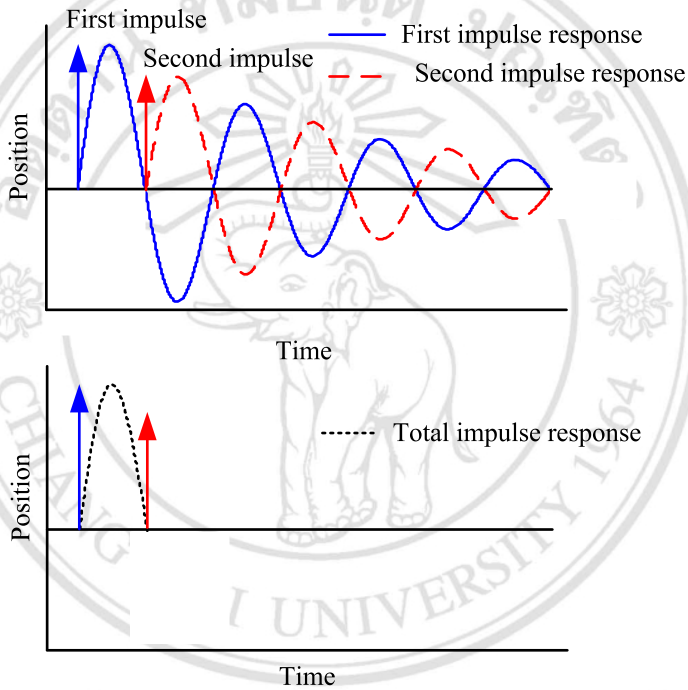
2.1.3 Motion control of flexible structures

Controlling the motion of a flexible structure undergoing overall positional changes has been of interest to many researchers [10]. The challenge of the problem is that one not only has to control the motion of the rigid-body mode but also the vibratory modes related to the structural deformation [109]. From previous researches, control techniques that limit the unwanted transient deflection and/or residual vibration of the flexible system can be broadly separated into two categories: 1) Feedback control, and 2) Input shaping [96].

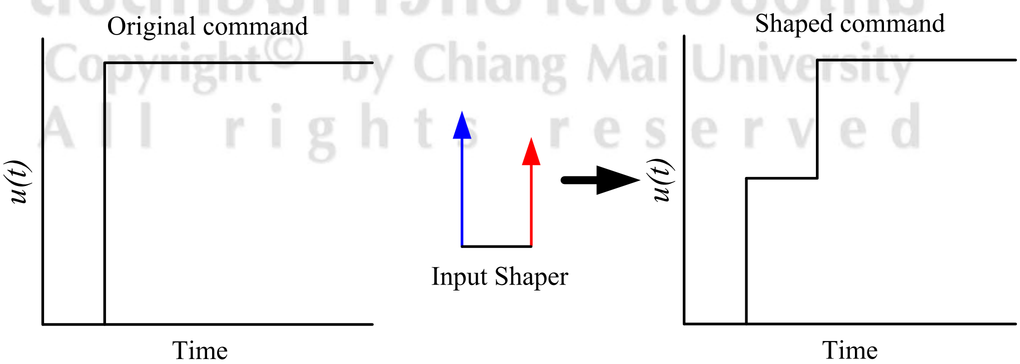
The feedback control techniques require the knowledge of the current state of the system to eliminate the undesired error in the chosen output states. A dynamic operation on the state/output error may be made before being fed back as the control input [114]. If the output signal or system states cannot be directly obtained by measurement then a state estimator is necessary and this increases the complexity of the problem and may lead to issue of control robustness. In this case, observability and controllability criteria must be satisfied for controller design to be possible [8]. However, feedback control can have advantages in term of accuracy and stability [10], [107], as well as robustness due to the uncertainties and disturbances [62], [115].

Since the feedback control technique requires the realization of current state variables such as the vibratory state for vibration control, then sensors are required. For example, in previous studies the position of the end-tip of the flexible link manipulator was measured by a camera [103]. Alternatively, acceleration of the endpoints of the flexible link was measured by an accelerometer [116]. For flexible structures, feed forward control may be implemented together with feedback control in the form of multiple loops, that combine the rigid body motion control loop with others for flexible structure motion [14], [64], [78], [115].

'Input Shaping' is a technique to design control input profiles that avoid exciting structure vibration and can be categorized as a feedforward control method. The technique proposed by Singer and Seering in 1990 [92] as an extension of the Posicast method, is based on the impulse response of a second order linear time invariant system. Since then, the technique has became a popular method in flexible structure motion control [96]. The idea behind the method is to determine at least two impulses with proper magnitude and timing whose vibration response will completely cancel each other (Fig. 2.2a). The sequence of impulses is called the *shaper*. The shaper may be convolved with any command input to generate the newly shaped input that will result in zero residual vibration, as shown in Fig. 2.2b. This Zero-Vibration (ZV) shaper allows not only the elimination of residual vibration in point-to-point motion but also more precise trajectory tracking control [3], [112].



(a) Concept of vibration cancellation



(b) Process of input shaping

Figure 2.2: Vibration suppression using input shaping technique

Like other feedforward control techniques, the basic ZV input shaper can have poor robustness to errors in the system model. To deal with residual vibration due to changes/ errors in system parameters, (natural frequencies), an equivalent mathematical constraint was set up in the form of the derivative of the vibration sum with respect to frequencies [92]. The trade off is that the rise time of the response is significantly increased. Such a shaper, with extra constraints, is called Zero-Vibration-Derivative (ZVD) shaper. Rather than completely suppress the vibration, other designs of shaper allow a small percentage of vibration over a specified range of natural frequency and define an acceptable level of vibration [99]. The versatility of the technique allows input shaping to deal with more complicated systems such as third-order systems [38], linear-time-varying (LTV) systems [79], and simple non-linear systems [104] by solving two-point boundary value problem [102]. One disadvantage of the technique is that it extends the overall time-of-motion. In original form, the magnitude of every impulse was constrained to be positive with the summation equal to one so that the maximum and minimum value of the shaped input is the same or less than the unshaped one. By allowing a negative-valued impulse, the Specified-Negative-Amplitude (SNA) input shaper allows a shorter time of motion to be achieved and can guarantee that the input signal will not exceed the limit given but the limit must be greater than the maximum value of the original input signal [97]. To acquire a Time-Optimal (TO) shaper with an admissible shaped control input, the specified series of impulse is provided although certain conditions on the original input signal which must be satisfied to ensure saturation limits are not exceed [100], [101]. The TO-shaper, however, does not achieve the true time-optimal motion. A further possibility is to specify the magnitudes of the impulses to switching between 1 and -1. This is called the Unity-Magnitude (UM) input shaper [33], [32], [75]. The SNA, TO and UM shapers can be very sensitive to modeling errors but the rise time and settling time are better than those of the ZV, ZVD and EI shapers [32]. For further improvement in accuracy, input shaping can also be implemented together with a feedback control loop [103].

There are various general approaches to acquire the appropriate number, amplitude and timing of impulses for an input shaper. Besides solving the system of non-linear equations directly by using a numerical package [75], vector diagrams for the vibration can be used to determine the sequence of the impulses [99]. To be able to deal with modeling error, the probability distribution of uncertainties of system parameter might be taken into account in the process of designing the shaper [70]. An alternative approach using the PWM technique for UM shaper is also recently proposed in [33].

2.2 General optimal control problem

A general dynamical system model may be defined in the form

$$\dot{x}(t) = F\left(x, u, t\right), \quad t \in [t_0, t_f] \tag{2.17}$$

where the time interval of interest is $[t_0, t_f]$. Usually, the initial time t_0 is set to be equal to zero whereas the final time t_f can be either fixed or free.

The objective for the optimal control problem is to determine the admissible control input $u^*(t)$ (possibly as a function of the state variable $x(t)$) that will transfer the system from an initial state to a desired final state. The control law must minimize (or maximize) a chosen system performance measure, represented by a cost function which is a function of states and control variable. The general form of the cost function is

$$\mathcal{J}(x, u, t) = \Phi(x, u, t_f) + \int_0^{t_f} L(x, u, t)dt. \tag{2.18}$$

The final weighting function $\Phi(x, u, t_f)$ represents a penalty on state variable values at the final time and the weighting function $L(x, u, t)$ depends on the state variables and control input at intermediate times in $[0, t_f]$. Additionally, a final-state constraint may be included:

$$\psi\left(x(t_f), t_f\right) = 0. \tag{2.19}$$

Here $*$ denotes the optimum values. The control input u^* that minimizes $\mathcal{J}(x, u, t)$ is called an *optimal control* and x^* is an *optimal trajectory*.

Analytical methods for solving the optimal control problem may involve techniques such as dynamic programming or the calculus of variations. However, a guess for a suitable form of the control solution may still need to be made [44]. For some certain classes of optimal control problem, such as the Linear Quadratic Regulator (LQR) problem, the specific feedback form of the analytical solution can be acquired by solving an Algebraic Ricatti Equation as shown in [12],[44].

Numerical methods used to solve the optimal control problem by directly searching to find the optimal control input are considered to be direct methods. Mostly these use gradient based searching methods such as Steepest Descent [31], Conjugate Gradient Method [49],[68] and Genetic Algorithm (GA) searching method [60], [61]. Another interesting technique is to represent the optimal control problem as a set of Linear Matrix Inequalities (LMI) [89]. This approach can allow the synthesis of multiobjective controllers like

H_2/H_∞ controller whose cost functions involve the 2-norm and ∞-norm for the system transfer functions. Disturbance and uncertainty effects may be embedded within a Lyapunov stability condition, which leads to a set of LMIs that can be solved numerically.

The indirect method involves developing and solving a set of necessary conditions proposed by Pontryagin that will be discussed in detail in the next section. The following section focuses on the specific form of the optimal control problem, where the main point of interest lies, which is the time-optimal control problem.

2.3 Time-optimal control

Finding a solution to time-optimal control problems for motion of mechanical structures and mechanisms has drawn interest from many researchers during the passed few decades. There is a wide variety of practical uses and possible implementations. The common application of solutions to the time-optimal control problem for mechanical systems include the disk-drive [113], flexible robot manipulator [26], [58], overhead crane [63] and spacecraft structures [25], [51].

2.3.1 Problem formulation

Consider the performance index for the general optimal control problem (2.18) with $\Phi(x, u, t_f) = 0$ and $L(x, u, t) = 1$, the cost function is then reduced to

$$\begin{aligned} \mathcal{J}(x,u,t) &= \int_0^{t_f} 1dt \\ &= t_f \end{aligned} \tag{2.20}$$

Thus, minimizing this cost function will minimize the final time for the control interval. The problem of finding the control input $u \in \mathcal{U}$ that drives a system from one state to another desired final state while minimizing (2.20) is called the *time-optimal control problem*.

It is necessary to define the set of admissible control $\mathcal{U}$ which determines the limitation for each control input component $u_i, \;\; i = 1, 2, ..., m$. Usually these need to be finite ($< \infty$), although the allowed range of values will depend on the physical limitations of the actuators. The limitation on control input could be written as

$$\underline{U} \le u_i \le \overline{U}, \quad i = 1, 2, ..., m \tag{2.21}$$

Without loss of generality, for a linear system (2.1) where the superposition properties holds, the input can be shifted and scaled so that the control input bounds are given by

$$|u_i| \leq 1, \quad i = 1, 2, ..., m. \tag{2.22}$$

Similar to the general optimal control problem, methods for solving time-optimal control problems can be classified into two main categories, which are the direct and indirect methods. The direct methods involve using numerical algorithms to find a solution for the problem. An indirect method will make use of the "Pontryagin's Minimum Principle" to formulate the optimal solution, or possible solution, analytically.

For the direct method, the problem may be treated as an optimization problem whose objective is to achieve a minimized value for the final time

$$t_f^* = \min_{u \in [\underline{U}, \overline{U}]} \quad t_f \tag{2.23}$$

subject to the boundary conditions $x = x_0$, $x(t_f) = x_f$. Typically, the problem may be numerically solved e.g. by using gradient based searching method, after the form of the solution has been deducted or guessed [58].

In terms of robustness to errors in the system model, both the time-optimal control and its equivalent minimum-time input shaper [21], [50] are relatively poor (i.e. sensitive to model error) compared with various robust input shaping techniques [98].

2.3.2 Pontryagin's minimum principle

Pontryagin's minimum principle was first published in 1962 by the Soviet mathematician, L.S. Pontryagin, in *The mathematical theory of optimal process*. (For more information on the historical aspects see [66].) The main contribution of the theory is a set of necessary conditions for a control solution to be optimum. This provides knowledge about the form of the optimal control which can allow the solution to be determined by additional considerations, such as boundary conditions. An interesting version of the proof of the principle was derived in [37], [44] along with the proof of the existence and uniqueness of the time-optimal control solution [52], [66]. An alternative proof of the existence of the time-optimal control law, specified for a mechanical manipulator, was given in 1985 by Ailon [4] and Singh in 1987 [93].

For the general form of optimal control problem (2.18) with system model (2.17), the following set of equations describing the time-evolution of the system states and costates must be considered:

$$\begin{aligned}\dot{x}(t) &= \frac{\partial \mathcal{H}}{\partial \lambda}, \quad t \in [t_0, t_f] &(2.24)\\ \dot{\lambda}(t) &= -\frac{\partial \mathcal{H}}{\partial x}, \quad t \in [t_0, t_f] &(2.25)\end{aligned}$$

Here, $\mathcal{H}$ is the Hamiltonian defined as

$$\mathcal{H}(x,u,\lambda,t) = L(x,u,t) + \lambda^T(t)\left(F\left(x,u,t\right)\right) \tag{2.26}$$

and $\lambda(t)$ is called the *co-state variable*. Equation (2.25) is called the co-state equation or adjoint equation. $\lambda(t)$ can be considered as the Lagrange multiplier commonly used in constrained optimization problems.

Pontryagin's Minimum Principle states that *the Hamiltonian must be minimized over all admissible u for optimal values of the state and co-state*. In other words, if u^* is the optimal control for the problem, then

$$\mathcal{H}\left(x^*, u^*, \lambda^*, t\right) \leq \mathcal{H}\left(x^*, u, \lambda^*, t\right), \quad \forall u \in \mathcal{U}, \quad t \in [t_0, t_f] \tag{2.27}$$

The set of equations (2.24), (2.25) and (2.27) are necessary conditions for optimality but they are not, in general, sufficient.

With help of this principle and by using a calculus of variations approach, one way to find the unconstrained optimal input u^* is to solve the stationary condition

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \tag{2.28}$$

along with boundary condition

$$\left(\phi_x + \psi_x^T \nu - \lambda\right)^T |_{t_f} dx(t_f) + \left(\phi_t + \psi_t^T \nu + \mathcal{H}\right)|_{t_f} dt_f = 0 \tag{2.29}$$

where $x(0)$ and $x(t_f)$ is given and ν is a constant vector [52].

2.3.3 Bang-bang solution

For the time-optimal control problem (2.20) for the LTI system (2.1) with constraints on control input (2.21), the Hamiltonian is reduced to

$$\mathcal{H}(x,u,\lambda,t) = 1 + \lambda^T(t)\left(Ax(t) + Bu(t)\right), \tag{2.30}$$

and the corresponding co-state equation can be written as

$$\dot{\lambda}(t) = -A^T \lambda(t). \tag{2.31}$$

Then, with the help of Pontryagin Minimum Principle, the Hamiltonian (2.26) could be replaced with the one in (2.30). Thus, by collecting the terms of $u(t)$, we have

$$\lambda^{*T}(t)\left(Bu^*(t)\right) \leq \lambda^{*T}(t)\left(Bu(t)\right), \quad \forall u \in \mathcal{U}, \quad t \in [t_0, t_f] \tag{2.32}$$

Each component of the optimal control $u_i^*(t)$ forces the term $\lambda^{*T}(t)B_i u^*(t)$ to take the minimum value. Here B_i denotes the i^{th} column of the matrix B. In other words, the term $\lambda^{*T}(t)\left(Bu^*(t)\right)$ must be minimized with respect to $u(t)$. This forces the control $u^*(t)$ to take the extreme values and it can be assumed as:

$$u_i^*(t) = \begin{cases} \underline{U} & \text{for } \lambda^{*T}(t)B_i > 0 \\ \overline{U} & \text{for } \lambda^{*T}(t)B_i < 0 \\ \text{undefined} & \text{for } \lambda^{*T}(t)B_i = 0 \end{cases} \tag{2.33}$$

The equation (2.33) defines the so-called switching function and leads to a control input signal that has a finite number of switches between extreme values. This form of control is referred to as a *Bang-Bang* control.

For some cases there may be a finite interval of time called a singular interval when $\lambda^{*T}(t)B_i = 0$. For this interval, the optimal control $u_i^*(t)$ is not specified by (2.33) and so there may be freedom to choose any admissible control input to favor the system behavior and thus improve some others aspects of the performance. However, singular intervals will not exist if the system is 'Normal' (in the sense defined by Hermes and Lasalle [34]), or in other words, if the system is completely controllable from each input. In this thesis we will focus on systems and control problems in which the singular condition dose not arise.

In general, if the bounded control is scaled according to (2.22), i.e. $u(t) \in [-1, 1]$ the switching function (2.33) reduces to

$$u_i^*(t) = -\text{sgn}\left(\lambda^{*T}(t)B_i\right). \tag{2.34}$$

The time t when the optimal control input $u_i^*(t)$ makes a switch is called a *switching time*. Suppose that the total number of switches is l the switches will be at the time t_j when

$$\lambda^T(t_j)B_i = 0; \quad j = 1, 2, ..., l. \tag{2.35}$$

By recalling the costate function (3.1), the value of costate variables $\lambda(t)$ is defined by the its initial value λ_0 as

$$\lambda(t) = e^{-A^T t}\lambda_0. \tag{2.36}$$

Therefore, in order for the control input u_i to be optimum, there necessarily needs to be some λ_0 which can satisfy

$$\begin{bmatrix} B_i^T e^{-A^T t_1} \\ B_i^T e^{-A^T t_2} \\ \vdots \\ B_i^T e^{-A^T t_l} \end{bmatrix} \lambda_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.37}$$

for the corresponding set of switching times $t_1, t_2, ..., t_l$. Equation (2.37) can be consider as a necessary condition for the control $u_i(t)$ to be optimal.

The same approach of using Pontryagin's minimum principle can also be applied to similar types of optimal control problem. For example, let's consider the linear system minimum fuel problem with constraints on input as (2.21) whose cost function is in the form:

$$\mathcal{J}(x,u,t) = \int_0^{t_f} C^T \left| u(t) \right| dt \tag{2.38}$$

where $C = [c_1 \;\; c_2 \;\; ... \;\; c_m]^T$ is the weighting vector which allows the possibility of penalizing fuel burn for each of m input components. By applying the principle, the switching condition gives the form of control input solution as switches between -1, 1 and zero values. This form of solution is called *bang-off-bang* [44]. For a collection of related optimal control problems readers may refer to [20].

2.3.4 Methods of solving time-optimal control problems

To solve the time-optimal control problem for motion of a flexible structure, the bang-bang form of solution (as deduced from using Pontryagin's principle) may firstly be assumed. Then, the complete control solution profile may be acquired by solving the dynamic equations to find switching times that will achieve the required state transfer.

For a simple undamped single-mode flexible structure system the analytical solution is known [94]. The switching times for the optimal solution can be found by directly solving a system of non-linear equations which can be derived from the simple form of boundary conditions. Then, the solution may be checked with the necessary conditions

(2.37) [69]. An alternative approach is to consider the geometry of phase plane trajectories of the system [63], [106]. The optimal solution in this case consists of three switches in value and the overall form for $u(t)$ is anti-symmetric around the midpoint of the control profile [65], [94].

For systems with damping and with multiple vibratory modes, guessing the correct number of switching times is essential. Usually the number of switching times will be assumed to be $2N - 1$ where N is the number of flexible modes considered within the model [24], [67]. For a single flexible mode system, it is claimed that no more than three switches are needed [71]. Under these assumptions, the time-optimal control problem (2.23) could be tackled using various approaches such as linear programming [15], [25], non-linear programming [47] or as a Two Point Boundary Value Problem (TPBVP) [26], [28]. To solve the optimization problem, many numerical techniques have been developed and applied. The names of strategies used include the shooting method [5], [26], forward-backward method, branch-and-bound method [83] and bisection method [40]. For these iteration based methods, whether the solution is obtainable and whether the solution obtained is a global or local minimum, greatly depends on the initial guess [29], [76].

To avoid assuming an incorrect number of switching times, the problem may be discretized [118] and then a numerical routine used to search for the values of the input for each interval [69]. Another approach is to consider the phase plane trajectories of the vibratory states and specify the switching curve with respect to the boundary conditions and switching conditions [30], [91] (although this is only suitable for single-mode structures).

Poor robustness to errors in the system model is often a feature of minimum-time control [74], [87]. Robust time-optimal control can be achieved with the trade-off of increased final time [6], [25], [72]. One robust control technique is referred to as near-minimum-time control and allows easier calculation due to some approximations made [41]. Similar to the ZVD shaper design, addition conditions involving zero derivative constraints may be added to the time-optimal control problem to increase robustness [77], [95]. This results in additional switches in the bang-bang input [54], [55]. Various techniques have been employed together with the time-optimal feed forward command to improve the robustness of the overall control performance. Such techniques that require on-line monitoring and calculation are feedback control [39], input shaping technique [73] and model predictive control [13].

Another type of time-optimal control problem that should be mentioning is where the control input is applied prior to the output-transmission interval (i.e. $t < t_0$) for a further reduction in the time interval [16], [17], [18]. Moreover the post-actuation can also be applied after the system has come to a complete stop ($t > t_f$) and is useful for systems with additional zeros. The post actuation considered in [103] results in a control input whose shape is bang-bang control but also with a tail [111]. The technique is used to achieve an output transition and so states not affecting the output can be non-zero outside the interval of transition.

The extension of the problem to some classes of non-linear system has been investigated. The time-optimal control of unknown non-linear plant could be achieved numerically with iterative learning control method after the model has been roughly determined by experimental identification [110]. For a non-linearity due to effects of Coulomb friction, a standard optimization package could be used to search for the switching times [23]. A more analytical way of compensating friction effects has also been proposed but the constructed solution does not possess the bang-bang property which resulted in the input exceeding the specified limits [42], [43].

It should be remarked that the aforemention methods which use a switching time searching approach cannot guarantee the true optimality of the control solution unless the necessary conditions from Pontryagin's Principle have also been fulfilled by solving for the co-state variables (2.31) [9]. This requires complicated calculation [19], [76] and so is difficult to incorporate in the optimization/solution method directly. Thus, alternative approaches of determining the time-optimal control via calculation of the co-state have been investigated in the present work and a new approach to solve the time-optimal control problem developed. Aspects of this work are also presented in [105]. This approach will be presented in chapter 3

2.4 Structure/control optimization

During the design process for a machine or mechanism, structural design parameters may be chosen by considering physical and environmental requirements such as shape, weight, motion range, workspace, cost, etc. In principle it is also possible to design the structure so that the dynamic behavior is more compatible with achieving effective control. A successful simultaneous optimization of structure and controller can improve the performance over traditional controller-only design methods [56], [84], [85] and can also makes the controller design much easier [48]. Such a concept of simultaneous structure/ controller optimization is sometimes called *Integrated Design*.

The integrated structure/controller design for motion control systems can be usefully categorized into two distinct practical situations. One is where a system is to be created to perform a repetitive motion task with unchanging boundary conditions. In this case, optimum design parameter values can be calculated and a fixed design created accordingly. Example cases include the flexible link which is part of a robot manipulator [22], [35] and spacecraft structures [59]. The optimal designs obtained may involve a non-uniform link geometry which is more suitable for vibration reduction control [7] and high speed movement [80], [85].

A different situation is where the structure under control has variable properties that, although fixed during operation, can be changed beforehand according to the current task. Such tunable or adaptive structures have previously been considered for vibration control/ isolation applications, where shifts in resonant frequency can have beneficial effects [90]. Other examples are tunable stiffness devices, including bimorph piezoelectric beams, that help to maximize the harvesting of vibration energy [86] and tunable stiffness structures for improved isolation performance [53], [117], [119].

The methods used to solve the aforementioned integrated design problems have involved various numerical algorithms and no closed form solutions have been presented. For multiobjective control performance optimization, linear matrix inequality (LMI) methods and related numerical tools can be used to solve the integrated design problem. This has been reported for several types of controller such as PID controller [81] and H_2/H_∞ controller [36]. More generic approaches may be used to solve the same class of problem, such as random search optimization algorithm. This has been used for both PID [56] and reduce-order-H_∞ controller tuning [35]. Other examples are particle swarm optimization for model predictive control [108] and steepest descent method based on sensitivity Jacobian matrix [80], [84] which can be used together with singular value decomposition [85] for time-optimal control.

Distinction should be made between the situation that will be considered in this thesis and those involving semi-active structures where a system has passive elements with controllable properties that can be switched or varied during operation. Such systems would possess additional (but limited) forms of actuation. For example, variable rope length for an over head crane has made vibration control of the load easier and allows the controller to focus on controlling the overall motion of the system [1], [64] and thus improve the operation time. The type of system considered in this thesis will be one for which structural properties (or specific parameters) can be set prior to operation but do not change during a motion task.