CHAPTER 4

Limits of Performance for time-optimal motion of flexible structures

When solving the time-optimal control problem for a given flexible structure system and given control input constraints, the final time achieved indicates the least time required to complete a motion. In another sense, this time-optimal motion is the fastest overall speed that can be achieved for a certain move distance. In general, the achieved speed, measured as overall distance divided by final time, could be used as the indicator of how good the performance of the controlled system is. The only possibility to improve (reduce) this measure of performance is to deal with the physical design of the system since the control being applied is already optimum. One possibility is to increase the capacity of the actuator so that the limit of input is increased. However, this will usually require an actuator that is larger, heavier and more expensive. Thus, practical and economic factors may prohibit this approach. The other possibility is to alter the system design parameters to be better matched with the assigned motion task.

To understand more about these issues, this chapter will further examine the set of time-optimal motion solutions for a range of flexible structure systems and for a range of final time value t_f. All solutions will be calculated using the convex optimization algorithm described in Chapter 3. The main focus here will be on the relation between actuation capacity and achievable speed of motion but it will also be considered how this relation is dependent on dynamic properties of the structure. By considering the results obtained, it is also hoped to get to a better understanding of how dynamic properties of the structure (particularly the natural frequencies of vibration) will affect the achievable time-optimal performance.

4.1 Rigid-body motion

Let us consider the rest-to-rest motion of the rigid-body with mass $m = 1$ and with no friction involved, where the maximum magnitude of control input $u(t)$ is $|u(t)| \leq U$. In order to complete the rest-to-rest motion from one position to another position in minimum time, first, it is necessary to apply the maximum acceleration U to the body. Then at the time of half motion interval $t_f/2$, the control input is switched to maximum deceleration $-U$ to achieve the complete stop at final time t_f. Considering this optimal control input with switching time at $t_f/2$ provides the distance travel γ in terms of the final time t_f as:

$$\gamma(t_f) = 2\int_0^{t_f/2}\int_0^t U dt = \frac{U}{4}(t_f)^2. \tag{4.1}$$

An example time-optimal motion of a rigid-body can be seen in terms of the state variables x_1 and x_2 shown in Fig. 4.1. Note that x_1 and x_2 correspond to the displacement and velocity respectively. The final time is $t_f = 2$ seconds and the switching time is at $t_f/2$. In this case, the distance travel is $\gamma = 1$. In another sense, γ can be considered as the furthest distance the system can travel within time t_f.

For motion-to-rest cases, it is fairly intuitive that the time-optimal motion for a rigid body is simply to drive the mass to a completely stop using maximum retarding force. This makes the relation between t_f and initial velocity ν a straight line. With structural flexibility taken into account, an interval with opposite input must be introduced. This yields a slower motion than for the rigid-body case. The relation of ν and t_f for the rigid-body case is a direct integration of t_f:

$$\nu(t_f) = \int_0^{t_f} U dt = Ut_f \tag{4.2}$$

The value ν can also be considered as the distance traveled for the 3-state model with velocity as input, as mentioned in single-mode cases in section 3.3, and the switch implies the change in direction of motion.

With the structural flexibility taken into account, extra switches in control input value are required to ensure the excited vibration is canceled at the end of the motion interval. It would seem fairly intuitive that the minimum time required to move a rigid-body will be less that that required to move a flexible structure of the same mass and under that same conditions. Thus, for the same value of γ in rest-to-rest motion task and ν in motion-to-rest, the achievable minimum-time t_f for the rigid-body structure may be considered as the lower limit for the minimum-time for flexible structures.

However, without damping present in the model, it is possible that the required input for the the flexible structure is equal to the rigid-body structure with same mass and for the given γ. This occurs when the final time equals an integer multiple of the natural period $t_f = \frac{2\pi}{\omega_n}, \frac{4\pi}{\omega_n}, \frac{6\pi}{\omega_n}$, ... (according to (4.1) also an order of $\sqrt{\gamma}$). This can happen because

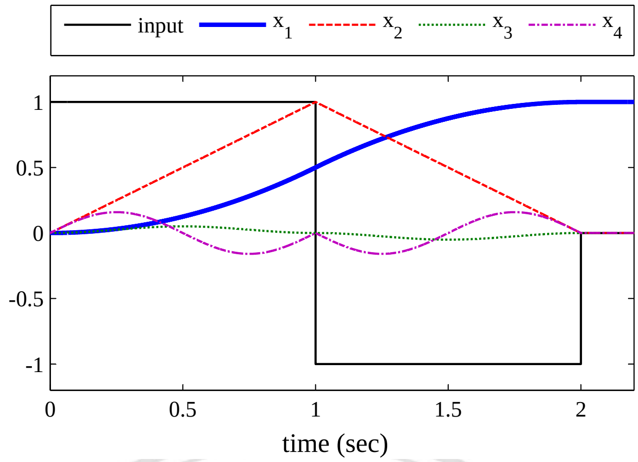


Figure 4.1: Time-optimal motion of undamped flexible structure

the motion stops when the oscillation and the corresponding derivatives all complete their periods at the same time. Fig. 4.1 gives an example for such a case where the rigid-body input gives rest-to-rest motion for a single-mode flexible structure system. The systems parameters in this case are $\omega_1 = 2\pi$ rad/s, $\zeta = 0$ and $t_f = 1$ sec. It can be seen that only one switch is needed at the middle of the motion interval. For non-zero damping system, it is impossible to reach the same final time as the rigid body case for any value of γ. Since having only one switch in the middle of the interval cannot cancel the decaying oscillation, extra switches are required and that slows down the overall motion.

4.2 Comparison of solutions

In order to investigate the relation between t_f, γ and the natural frequencies ω_n, the proposed algorithm in chapter 3 was used to find the time-optimal solutions for a certain range of γ. The time required to complete the rest-to-rest and motion-to-rest tasks t_f for each natural frequencies can be compared with the minimum-time required in rigid body case with the same mass and the same γ.

Figure. 4.2a, shows the set of time-optimal solutions for rest-to-rest motion of a single-mode flexible structure systems with natural frequencies $\omega_1 = 2\pi,\ 4\pi,\ 6\pi$ and 8π rad/sec and damping ratio $\zeta = 0.01$. It can be seen that, for a fixed system model, the minimum time required to complete a rest-to-rest motion increases monotonically with the distance γ. This can be deduced from the fact that the reachable set is always expanding over time. However, for a given travel distance and total mass, the optimum t_f may vary significantly with the system parameter values i.e, the natural frequencies ω_n. The nature of this relation can be seen more clearly in motion-to-rest situation as evident in Fig. 4.2b.

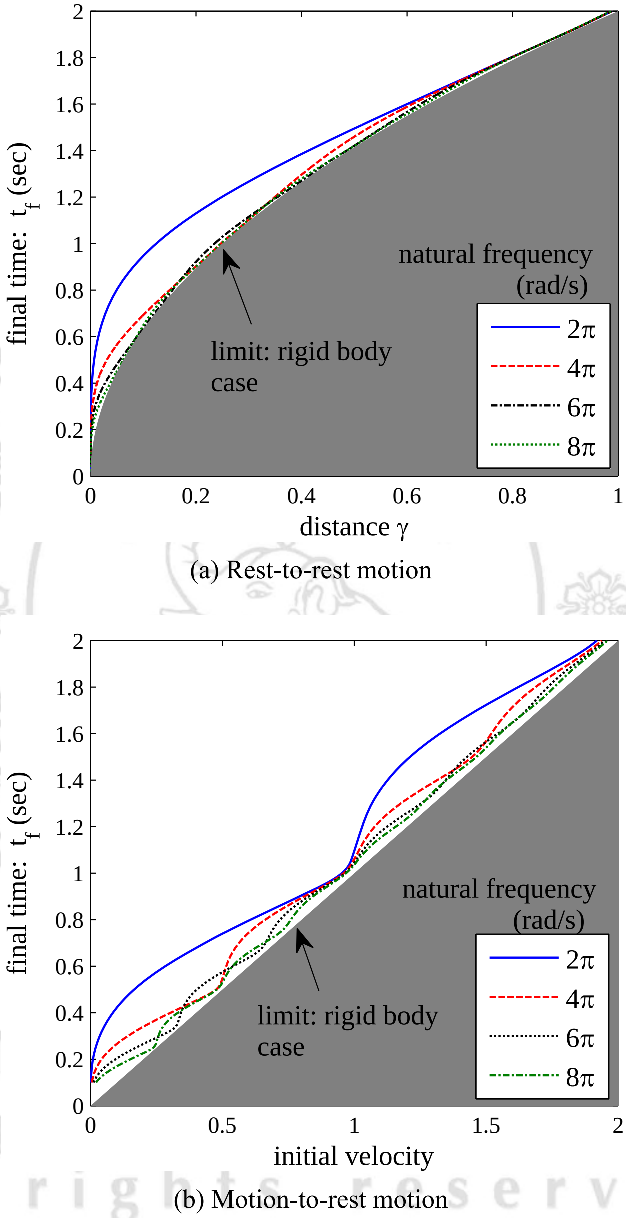


Figure 4.2: Solution sets of time-optimal motion task involving the single mode structures with various natural frequencies

There are points where the time required to complete the motion for flexible structure case is equal to the rigid-body case. For example, let us consider the case where $\omega_1 = 4\pi$ rad/sec or 2 Hz. Then the completed cycle of the oscillation can be complete with either t_f equal to 1 second or 2 seconds. Therefore if the required distance γ correspond to these t_f there will be no residual vibration. Following (4.1), the γ that is require t_f for 1 sec and 2 sec to complete the rest-to-rest motion are $\sqrt{0.125}$ and 1 meter respectively.

The points where t_f required for each γ in flexible structures cases equal to the limit corresponding to rigid-body cases are simpler in motion-ro-rest cases because the dynamic relation between t_f and ν is linear according (4.2). For example, when mass $m = 1$ and natural frequency $\omega_1 = 8\pi$ rad/sec (8 Hz), the distance travel $\gamma = 0.25, 0.5, 0.75$ and 1 meter, require final time $t_f = 0.25, 0.5, 0.75$ and 1 sec respectively to complete the motion-to-rest motion with no residual vibration. These are also the times when the oscillations complete their cycle.

4.3 Performance metrics for actuation capacity and speed of motion

Some key metrics may be defined for the controlled system as follows:

Relative actuation capacity. Each solution obtained corresponds to a pair of values for t_f and γ. In normal circumstances, it is usual to interpret the set of solutions as being for fixed maximum input $|U|(= 1)$ and varying over the actual distance traveled. For $|U| \neq 1$, due to linear dynamic properties, it is possible to normalize the distance with $|U|$. Therefore, the solutions may also be as all being for a move distance of 1 and varying over $|U|$. With this idea the relative actuation capacity may be defined as

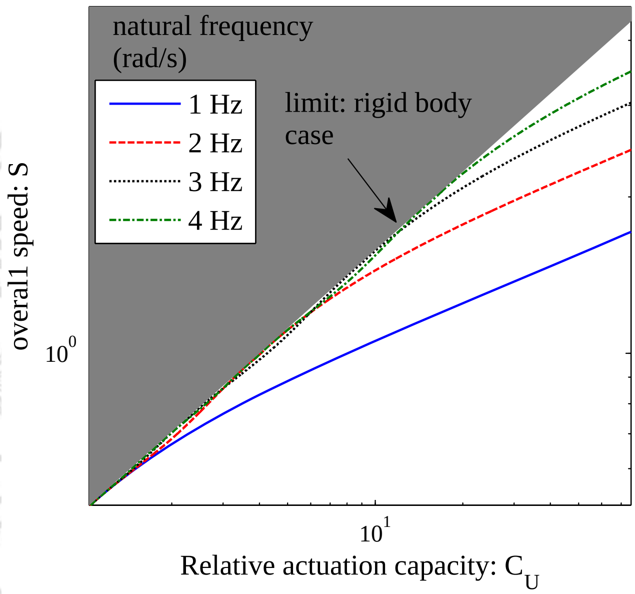
$$C_U \cong \frac{1}{\gamma}. \tag{4.3}$$

This definition could apply for both rest-to-rest motion and motion-to-rest motion.

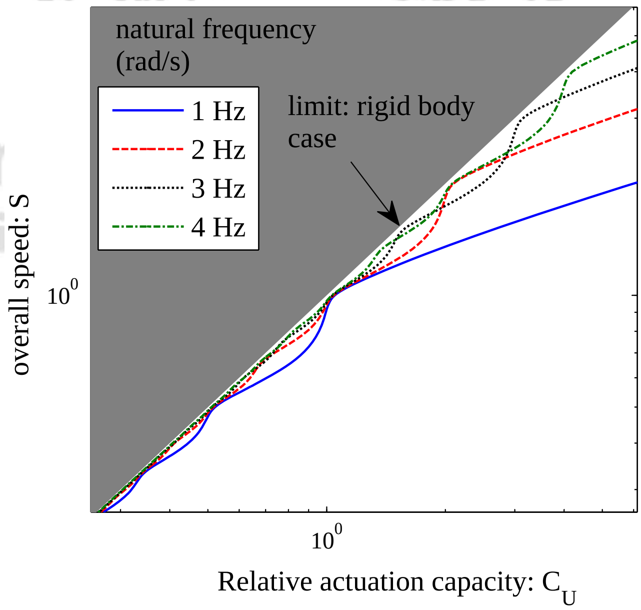
Overall speed of motion. The overall speed of motion or average speed can be calculated by γ/t_f. Corresponding to the actuation capacity where the distance is assumed fixed, the overall speed then needs to be normalized by the distance as well. Therefore, the overall speed of motion may be defined as

$$S \cong \frac{1}{t_f}. \tag{4.4}$$

Figure 4.3 shows the set of results obtained for a single mode flexible structure, where the overall speed measure S is plotted against the relative actuation capacity C_U. We can see that, when the overall speed of motion is relatively slow, the overall speed for the flexible structure is close to the rigid-body case. However, when the actuation capacity is increased, the average speed can deviate significantly from the rigid-body line. This effect is most significant when the time interval of motion is of the order of the natural period of vibration of the system i.e. when $t_f \approx \frac{2/pi}{\omega_n}$, or less for rest-to-rest motion (and $t_f \approx \frac{1}{\omega_n}$ for motion-to-rest motion). These points are corresponding to the critical values of the actuation capacity above which speeds close to the rigid-body case can no longer be obtained. The critical value for is given approximately by $C_U^{crit} = 2/\omega_n^2$ for rest-to-rest motion and $C_U^{crit} = 1/\omega_n$ for motion-to-rest case.



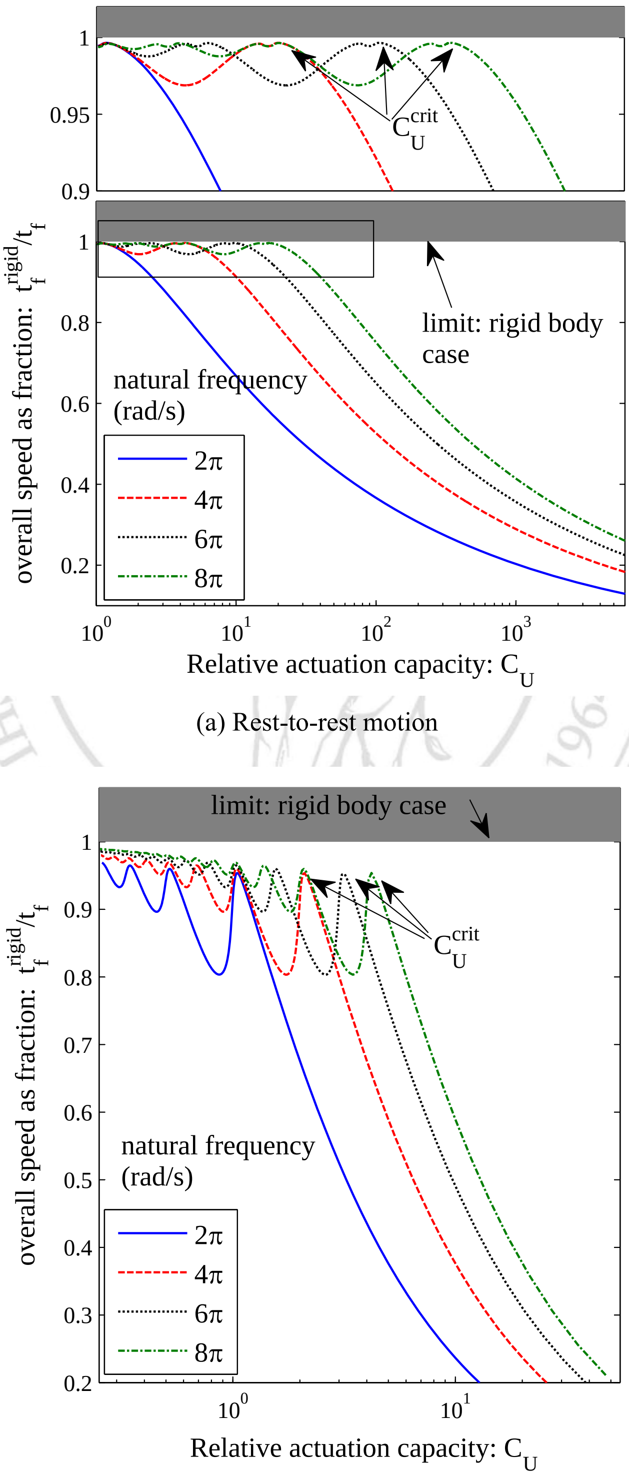
(a) Rest-to-rest motion



(b) Motion-to-rest motion

Figure 4.3: Increasing of the overall speed of motion with larger relative actuation capacity

Overall speed as fraction of rigid body case. To show more clearly how natural frequency influences the relationship between speed of motion and move distance, an alternative presentation of the solution sets is given in Fig. 4.4. To show more clearly



(a) Rest-to-rest motion

(b) Motion-to-rest motion

Figure 4.4: Speed as fraction of the rigid-body case

how natural frequency influences the relationship between speed of motion and move distance, an alternative presentation of the solution sets is given in Fig. 4.4. The curves in this figure show the quantity of overall speed as a fraction of the rigid body case, calculated as t_f^{rigid}/t_f for each γ. For small actuation capacity the overall speed of motion fluctuates with move distance and natural frequency but remains close to the rigid-body case. When C_U increases greater than the critical value, the fractional speed of motion decreases monotonically. These critical points for $\omega_n = 4\pi,\ 6\pi$ and 8π rad/s are pointed out by arrows in Fig. 4.4.

A further implication here is that increasing structural stiffness (natural frequency) can improve speed of motion. This behavior may be examined for the illustrative case $C_U = 10$ in Fig.4.4a. For this condition, if $\omega_n = 2\pi$, a fractional speed of motion 0.67 is achievable. Increasing natural frequency to 4π and 6π would result in an improved speeds of 0.93 and 0.995 respectively. However, increasing natural frequency further to 8π would then reduce the overall speed of motion to 0.97. Clearly, an exact optimization of natural frequency could be usefully applied.

Equivalent results for the case of motion-to-rest with no initial or residual vibration are shown in Fig. 4.4b. These solution sets have similar characteristics to the rest-to-rest motion cases, although the fluctuations in time of motion with natural frequency are more pronounced.

The conclusion for this chapter can be drawn here that the relation between speed of motion and relative actuation capacity, (as determined by t_f and γ) varies with the natural frequency of the structure but the overall trend is similar. It can be pointed out that increasing structural stiffness (natural frequency) can improve speed of motion, but only up to a certain point. Further increases in natural frequency then produce fluctuations in time of motion and only certain values allow performance close to the rigid-body case. This clearly motivates the idea of tuning natural frequencies to match a required motion task in order to achieve the further reduction in final time.