CHAPTER 5

Structural Tuning

Motivated by results in the previous chapter, the scope of investigation will now be expanded by considering the general problem of optimizing system parameters to achieve reductions in time-of-motion with respect to some nominal situation. The approach here will be a local, rather than global, optimization. For implementation, the optimization approach relies on gradient-descent based methods. This means the solution acquired at the end of the optimization will be a local minimum and will depend on how the algorithm converges from the initial nominal parameter values.

5.1 Problem formulation

Suppose that a system has a number (k) of design parameters that can be tuned/ optimized and these are denoted $p \in \mathcal{P} \subset \mathbb{R}^k$. It will be assumed that the linear dynamics of the system depend on p only through the A matrix in the state space equation:

$$\dot{x}(t) = A(p)x(t) + Bu(t). \tag{5.1}$$

As an integrated design optimization problem the objective is to achieve the optimal final time over p and u

$$t_f^* = \min_{u\in\left[\underline{U},\overline{U}\right]\ \ p\in\mathcal{P}} t_f \tag{5.2}$$

subject to given boundary conditions. To tackle this problem, we will consider a constrained perturbation of system parameters Δp away from some nominal values. This seems reasonable for practical situation because the problem is similar to design process of the structure where the first design is determined then it is altered to be suitable for the task. Consequently, the optimization can be based on gradient-descent methods, reliant only on local continuity of $y(t)$ with respect to the system and control parameters.

Based on the time-optimal control formulation and calculation method described in chapter 3, a control solution may be obtained for a nominal set of system parameter values. The objective now is to determine a change in system parameters values Δp that will allow a reduction in final time ($\Delta t_f < 0$). To maintain the same boundary conditions, a change in the control input will also be required. Within the optimization it would be preferable to directly determine the update in the initial co-state η, so that resolving for u^* from scratch, e.g. via the convex optimization algorithm, is not required.

5.2 Continuity-based optimization

Suppose that a bang-bang time-optimal control solution u^* with l switches at times $t_1, t_2, ..., t_m$ has been calculated for a nominal linear system in the form of (5.1). Now consider the small change $p \to p+dp$ which implies that $A \to A+dA$. The corresponding change in state transition matrix $X(t) \to X(t) + dX(t)$ is given by

$$dX(t) = e^{-(A+dA)t} - e^{-At}. \tag{5.3}$$

In the limit $dp \to 0$, this may be expressed

$$dX(t) = Z_1(t)dp_1 + Z_2(t)dp_2 + \cdots + Z_k(t)dp_k, \tag{5.4}$$

where Z_i is the sensitivity matrix that specifies the change in $X(t)$ due to a small change in p_i.

For a rest-to-rest maneuver, the calculation of initial state-values (assuming final state values are zero) is made from $x_0 = -y(t_f)$ where $y(t_f)$ is given by (3.17):

$$y(t_f) = \int_0^t X(\tau)Bu(\tau)d\tau$$

Consider the perturbed system, also with a perturbation in input and related switching times ($t_i \to t_i + dt_i$), an optimal point in the neighborhood of $y(t_f)$ follows as

$$y'(t_f + dt_f) = \sum_{i=0}^{l}(-1)^i \int_{t_i+dt_i}^{t_{i+1}+dt_{i+1}} (X(t) + dX(t))\, Bdt \tag{5.5}$$

where $dt_0 = 0$. By using the mean value theorem, we have

$$\begin{aligned}
y'(t_f + dt_f) \;&=\; \sum_{i=0}^{l}(-1)^i \int_{t_i}^{t_{i+1}} (X(t) + dX(t))\, Bdt \\
&\quad +2\left(X(t_1) + dX(t_1)Bdt_1\right) - 2\left(X(t_2) + dX(t_2)Bdt_2\right) + ... \\
&\quad (-1)^{l+1}\left(X(t_f) + dX(t_f)Bdt_f\right) \\
&= \sum_{i=0}^{l}(-1)^i \int_{t_i}^{t_{i+1}} (X(t) + dX(t))\, Bdt \\
&\quad +2\sum_{i=1}^{m}(-1)^{i+1}\left(X(t_i) + dX(t_1)\right) Bdt_i + (-1)^{l+1}\left(X(t_f) + dX(t_f)Bdt_f\right)
\end{aligned}$$

To first order, the change in optimal point is

$$\begin{aligned} dy &= y'(t_f + dt_f) - y(t_f) \\ &= \sum_{i=0}^{l} \int_{t_l}^{t_{l+1}} dX(t)Bdt + 2\sum_{i=1}^{l}(-1)^{i+1}X(t_i)Bdt_i + \\ &\quad (-1)^{l+1}X(t_f)Bdt_f \end{aligned} \tag{5.6}$$

To remain at the same final point (to preserve boundary conditions), it is require that $dy = 0$. At the same time, to maintain optimality of the solution (i.e. to stay on the surface of the reachable set), it is also required that optimality condition (3.7) must hold:

$$\eta^T X(t_i)B = 0 \;\; i = 1, 2, ..., l$$

Premultiplying (5.6) by η^T and applying the optimality condition (3.7) gives

$$\eta^T dy = \eta^T Jdp - \eta^T X(t_f)Bdt_f = 0 \tag{5.7}$$

where J is defined by

$$J = [j_1 \; j_2 \; \cdots \; j_k], \;\; j_i = \int_0^{t_f} Z_i(t)Bu(t)dt \tag{5.8}$$

This is the Jacobian/sensitivity matrix relating change in dy to the change in dp. If $\eta^T J \neq 0$ then an update for p allowing a decrease in t_f can be calculated as

$$dp = -\delta J^T \eta\eta^T X(t_f)B \tag{5.9}$$

where $\delta > 0$ is a suitably small step length. By substitute dp back in (5.7), the update of t_f corresponding to the change dp can be obtained

$$dt_f = \frac{\eta^T Jdp}{\eta^T X(t_f)B} = \delta\eta^T JJ^T\eta \tag{5.10}$$

which is negative if $\delta > 0$ and $\eta^T J \neq 0$

It is of interest to note that a reduction in final time depends on the component of dy in a direction normal to the surface of the reachable set i.e. in the direction η. If $y(t_f)$ can be moved outside the current reachable set by a change in p then a corresponding reduction in t_f can be achieved in accordance with (5.7). The component of dy orthogonal to η depends on the switch times for u. Thus u must now be updated to recover $y(t_f) = -x_0$

subject to maintaining the optimality condition (3.7). To maintain optimality of u, changes in η and $t_1, ..., t_m$ are constrained according to

$$(\eta^T + d\eta^T)\,(X(t_i + dt_i) + dX(t_i))\,B = 0, \quad i = 1, ..., l$$

where $X(t + dt) = (1 - dtA)X(t)$. Therefore,

$$(\eta^T + d\eta^T)\,((1 - Adt_i)X(t_i) + dX(t_i))\,B = 0$$
$$(\eta^T + d\eta^T)(1 - Adt_i)X(t_i)B) + (\eta^T + d\eta^T)dX(t_i)B = 0$$

By applying the optimality condition (3.7) and neglecting higher order terms, we have

$$-dt_i\eta^T AX(t_i)B + d\eta^T X(t_i)B + \eta^T dX(t_i)B = 0$$

which gives

$$dt_i = \frac{d\eta^T X(t_i)B}{\eta^T AX(t_i)B} + \frac{\eta^T dX(t_i)B}{\eta^T AX(t_i)B}. \tag{5.11}$$

Equation (5.11) describes (to first order) how changes in switching times relate to changes in A and/or η. Using this to substitute for $dt_1, ..., dt_l$ in (5.6) gives

$$dy = Jdp + Fdp + Wd\eta - X(t_f)Bdt_f \tag{5.12}$$

where

$$F = [f_1\ f_2\ \cdots\ f_k]\,, \quad f_i = \sum_{i=1}^{l}(-1)^{i-1}\frac{2X(t_i)B\eta^T Z_k(t_i)B}{\eta^T AX(t_i)B} \tag{5.13}$$

and

$$W = \sum_{i=1}^{l}(-1)^{i-1}\frac{2X(t_i)BB^T X(t_i)^T}{\eta^T AX(t_i)B}. \tag{5.14}$$

By using $dy = 0$ together with the update of dp from (5.9) and dt_f from (5.10), the update in η can be found as

$$d\eta = W^\dagger\,[-(J + F)dp + X(t_f)Bdt_f] \tag{5.15}$$

where $W^\dagger$ is the (Moore-Penrose) pseudo-inverse of W. The pseudo-inverse is unavoidable because W is formed from the column space of V and so will be rank-deficient.

With the direction given in (5.9), together with the updates of p, t_f and η in (5.9),(5.10) and (5.13), standard methods to deal with parameter constraints in gradient-based optimization can also be applied here. The equation ca be used to create a *simultaneous structural and control optimization* strategy with iterations involving updates dp until the change in value is sufficiently small enough. The stopping criterion can be a fairly simple condition, $dt_f < \epsilon_0$ where ϵ_0 is some small number.

It is recognized that an incorrect progression may arise because of the neglecting higher order terms. This will cause the newly generated point $\tilde{y}(t_f)$, which is generate from new value of p, t_f and η via the mapping (3.6), to deviate from $-x_0$ and thus lead to $dy \neq 0$. In this case, the further update to u^* can be made by considering (5.6) with $dX = 0$, because the deviation occurs due to the update of t_f and η. Therefore, we may apply

$$dy = W d\eta - X(t_f) B dt_f = -x_0 - \tilde{y}(t_f). \tag{5.16}$$

Multiplying this equation by η provides the required update to the values of t_f and consequently η:

$$\begin{aligned} dt_f &= \left(\eta^T X(t_f) B\right)^{-1} \eta^T \left(x_0 + \tilde{y}(t_f)\right) & (5.17) \\ d\eta &= W^{\dagger} \left(X(t_f) B dt_f + x_0 + \tilde{y}(t_f)\right) & (5.18) \end{aligned}$$

Note that this update can be repeated to eliminate errors as many times as needed, i.e. until $\tilde{y}(t_f)$ is sufficiently close enough to $-x_0$, meaning $\|\tilde{y}(t_f) + x_0\| \leq \epsilon_1$, where ϵ_1 is some small number.

Further considerations may be required when the number of switches m changes during optimization of p. This can lead to singularity issues for W associated with two distinct cases:

1. Convergence of adjacent switch times leading to $\eta^T A X(t_i) B \to 0$ in (5.13) and (5.14).

2. Surplus rank-deficiency of W when $m < n - 1$. This corresponds to when $y(t_f)$ is a singular point on the surface of the reachable set (i.e. where the surface is non-smooth).

These situations are fairly easy to detect by the failure of convergence of $\tilde{y}(t_f)$ to $-x_0$. Both issues can be circumvented through a recalculation of η and u^* using the convex optimization method.

The procedures for the proposed structure and control optimization are summarized in the flowchart presented in Fig. 5.1.

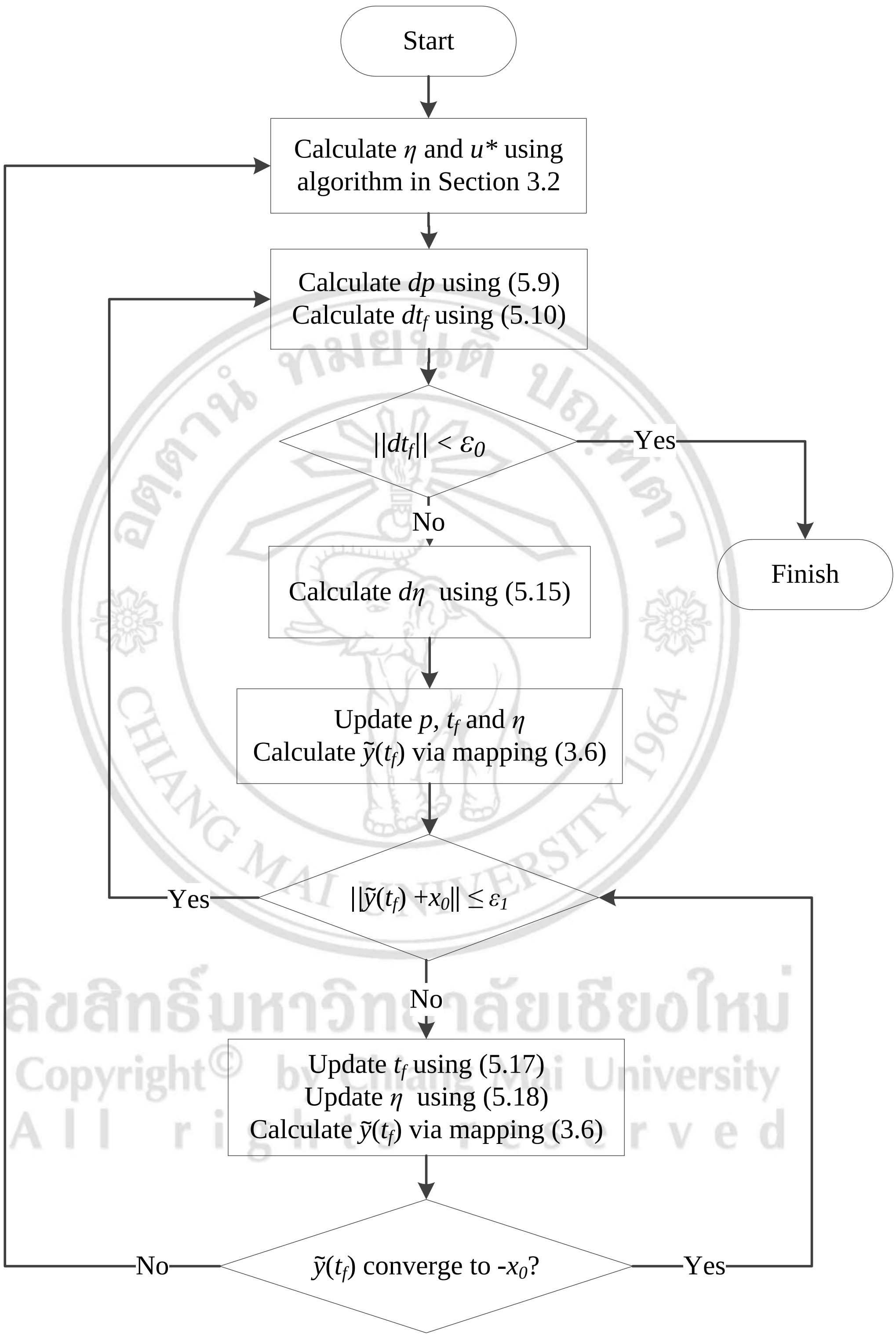

Figure 5.1: Flowchart of structure and control optimization procedures

5.3 Flexible structure tuning

It has been shown in Chapter 4 that the natural frequencies for structural vibration have a key influence on the minimum time for motion of a flexible body. Following from the previous discussions, cases will be considered where the damped natural frequencies are treated as free parameters: $p_i = \omega_{d_i}$, i = 1,...,n. From (2.7), we then have

$$A_i + dA_i = A_i + dp_i \begin{bmatrix} -a_i & 1 \\ -1 & -a_i \end{bmatrix} \tag{5.19}$$

where a_i are correlation factors relating the change in real and imaginary parts of the system pole. Thus, there is an assumption that these parameters cannot be selected independently but that a linear relation is valid (at least locally).

For the case (5.19), A_i and dA_i commute and a corresponding change in state transition sub-matrix $X_i(t) = e^{-A_i t}$ can be computed from (5.3):

$$X_i(t) + dX_i(t) = e^{-(A_i + dA_i)t} = (I - t dA_i)X(t).$$

Using a Taylor's series expansion of $e^{-(A_i + dA_i)t}$ around A_i, gives

$$dX_i(t) = Z_i(t) dp_i, \quad Z_i(t) = t \begin{bmatrix} a_i & -1 \\ 1 & a_i \end{bmatrix} X_i(t), \tag{5.20}$$

The calculation of the sensitivity matrix J follows by analytical integration of (5.8).

5.4 Numerical examples

This section will examine the results from the numerical optimization approach when applied to a two-mode flexible structure model. A system is considered for which the natural frequencies may be treated as independently tunable parameters, having unoptimized values $\omega_1 = 6.28$ rad/s and $\omega_2 = 30.3$ rad/s. Noting that these values correspond closely to lateral vibration of a uniform cantilever beam, a realization of a tuned system could notionally involve a non-uniform beam design. Damping ratios for both modes are taken as fixed values of $\zeta = 0.05$. As a test case, a time-optimal rest-to-rest motion which for the unoptimized system requires a total time $t_f = 1$ s is considered. Correlation coefficients relating real and imagine parts of pole values a_1 and a_2 are chosen as $1/10$ and $1/20$ respectively.

The cost-surface for parameter optimization, generated point-wise using the algorithm in Section 3.2, is shown in Fig. 5.2. Two numerical cases are shown where the structural tuning algorithm has been used to optimize ω_1 and ω_2. For Case 1, only the first natural frequency of the system is varied, and the optimization converges on the point A, giving a final time of 0.7186 s. For Case 2, where both natural frequencies are optimized, the algorithm converges on the local minimum point B, giving a final time of 0.7024 s. The optimization paths were calculated using (5.9), and can be seen to correspond with the steepest decent direction. This results confirm the applicability of the algorithm to a general multi-mode problem, though it would seem that the benefit from tuning a natural frequency tends to diminish for higher frequency modes.

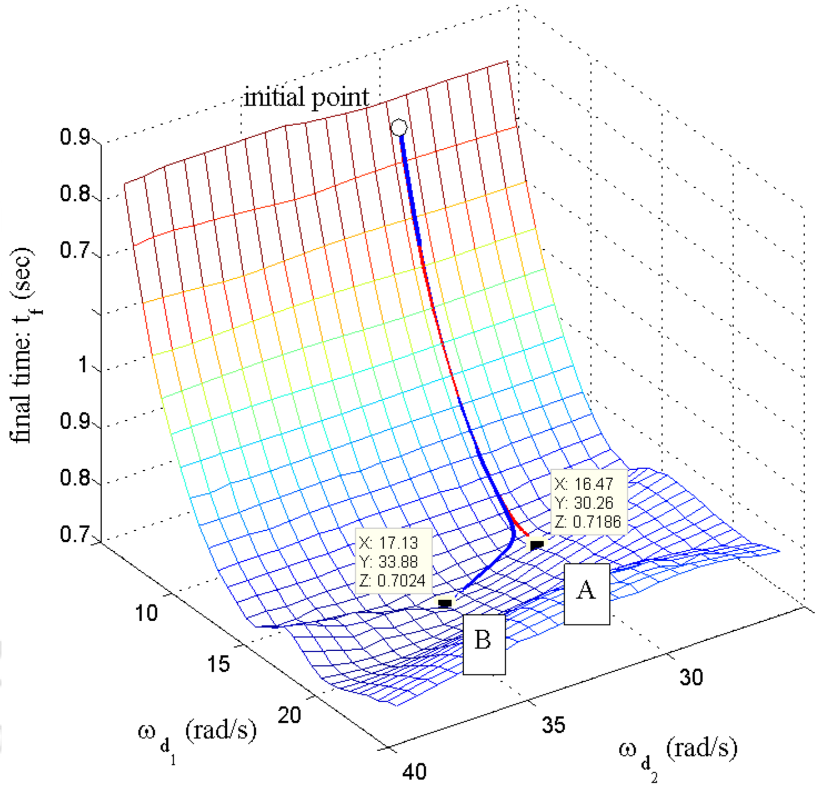

Figure 5.2: Cost surface for a two-mode tunable structure. Optimization paths are shown for two cases: 1) single mode optimization 2) two-mode optimization

These numerical results illustrate that utilization of the optimization method can allow the local minimum for the final time to be obtained, together with the correct update of the optimal control input corresponding to the tuned system. By monitoring the change in system parameters, some issues can arise that are explained here:

- The optimization process for multi-mode system required very long time compared with the single-mode system. Increasing step size during early steps could sometimes speed up the process. However too big step size can cause the algorithm to fail to converge.

- When the updates (5.17) and (5.18) cannot produce convergence $\tilde{y}(t_f)$ to $-x_0$, the error $\|\tilde{y}(t_f) - x_0\|$ will either increase or not decrease. This can easily be detected by monitoring the error during each iteration. Decrease in step size could sometimes solve the problem and let the optimization process continue. However, recalculating the time-optimal solution using the convex optimization algorithm is much more reliable but will make the process consume much more time.