CHAPTER 1

Introduction

Let A be a non-empty set and $n \geq 1$ be a natural number. Define $A^n = \{(a_1, a_2, ..., a_n) |\, a_1, a_2, ..., a_n \in A\}$. An $n-$ *ary operation* on A is a function $f^A : A^n \rightarrow A$ and the natural number n is called the *arity* of f^A.

Let I be some non-empty index set, and let $(f_i^A)_{i\in I}$ be a function which assigns to every element of I an n_i-ary operation f_i^A defined on A. Then the pair $\mathcal{A} = (A, (f_i^A)_{i\in I})$ is called an *algebra*. The set A is called the *base* or *carrier set* or *universe* of $\mathcal{A}$, and $(f_i^A)_{i\in I}$ is called the *sequence of fundamental operation* of $\mathcal{A}$. The sequence $\tau := (n_i)_{i\in I}$ for all the arities is called the *type* of the algebra $\mathcal{A}$. We denote $Alg(\tau)$ for the class of all algebras of a given type τ.

For examples: A semigroup $(S, \cdot)$ is an algebra of type (2). A ring $(R, +, \cdot)$ is an algebra of type $(2, 2)$.

To study Universal Algebra, the first main approach is to produce new algebras of the same type from given ones. This concept appear in specific algebraic theories such as group theory, ring or field theory. The second main approach is to classify algebras into collections called *varieties*. We use identities to classify algebras into varieties.

An *identity* is a formula with equality whose the variables are bound by the universal quantifier as an example:

$$\forall x_1 \forall x_2 \forall x_3 (x_1 \cdot (x_2 \cdot x_3) \approx (x_1 \cdot x_2) \cdot x_3)$$

where we call the *associative law*.

The associative law is satisfied(true) in an algebra $\mathcal{A} = (A, f^A)$ with a binary fundamental operation f^A if when we substituted any elements of A for x_1, x_2, x_3 and the operation symbol $\cdot$ is substituted by f^A, the resulting elements from A are equal.

If we substitute not only elements for the variables but also term operations for the operation symbols, the resulting elements from A are equal, we call it *hyperidentity*.

Hyperidentities are used to classify varieties into collections, called hypervarities. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarities. Hyperidentities and hypervarieties of a given type τ without nullary operations were first introduced by J.Aczèl [1], V.D. Belousov [3], W.D. Neumann [17] and W. Taylor [25].

The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [9].

A hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps n_i-ary operation symbols to n_i-ary terms. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. For all $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$. It turns out that $(Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and σ_{id} is the identity element.

In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [16]. A generalized hypersubstitution of type τ, for simply, a generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to a term of this type in $W_\tau(X)$ which does not necessarily preserve the arity and the set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. For all $\sigma \in Hyp_G(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$. It turns out that $(Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid where $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, σ_{id} is the identity element and $(Hyp(\tau), \circ_h, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau), \circ_G, \sigma_{id})$.

Let S be a semigroup. S is factorisable if $S = GE(S) = E(S)H$ for some subgroups G, H of S. In 1979, Y. Tirasupa proved that every factorisable semigroup is a regular semigroup [26]. Later in 1980, H. D'Alarcao showed that a moniod S is factorisable if and only if it is unit-regular [2].

A number of authors study factorisable on different semigroups. S.Y. Chen and S.C. Hsieh have studied factorisable inverse semigroups [8]. Y. Tirasupa have studied factorisable transformation semigroups [26]. In 2001, P. Jampachon, M. Saichalee and R.P. Sullivan used the concept of factorisable to study locally factorisable transformation semigroups [14]. The main purpose of this thesis, is to study on the factorisable monoid of generalized hypersubstitutions of type τ.

The study material has been organized in five chapters. Chapter 1 is the introduction. The basic concepts of semigroup theory and generalized hypersubstitutions and some useful results, which are used in later chapters, are contained in Chapter 2. In Chapter 3, we characterize all unit elements of the set of all generalized hypersustitutions of type $\tau = (n)$, all unit-regular elements of the set of all generalized hypersustitutions of type $\tau = (2)$ and all unit-regular elements of the set of all generalized hypersustitutions of type $\tau = (n)$. We characterize all completely regular elements of the set of all generalized hypersustitutions of type $\tau = (n)$ and show that the set of all completely regular elements and the set of all intra-regular elements of the set of all generalized hypersustitutions of type $\tau = (2)$ are the same. In Chapter 4, we find the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (2)$ and the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (n)$. The last chapter of this thesis is the conclusion.