CHAPTER 2

Preliminaries

In this chapter, we begin with some basic knowledges of semigroup theory and some concepts of identities, hyperidentities and the monoid of generalized hypersubstitutions that will be used throughout this thesis.

2.1 Semigroups

2.1.1 Elementary Concepts of Semigroup Theory

A *groupoid* $(S, \cdot)$ is defined as a non-empty set S on which a *binary operation* "$\cdot$" (by which we mean a map $\cdot : S \times S \rightarrow S$) is defined. We called $(S, \cdot)$ is a *semigroup* if the operation $\cdot$ is *associative*, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in S$. For convenience, we write ab replacements of $a \cdot b$.

A semigroup S is called a *monoid* if a binary operation is defined on S that has an *identity*, i.e., there exists an element e in S such that $ae = a = ea$ for all a in S. Clearly, if a binary operation has an identity, then that identity is unique.

For any monoid S, an element $u \in S$ is called *unit* if there exists $u^{-1} \in S$ such that $uu^{-1} = e = u^{-1}u$ where e is the identity element of S and called u^{-1} is an *inverse* of u. In general, the set of all unit elements of S is denoted by $U(S)$. Obviously, if S is a monoid and $u \in S$ has an inverse in S, then that inverse is unique. A monoid S will be called a *group* if every element of S has an inverse in S. It is clear that $U(S)$ is a group and then it is called the group of units of S.

Let S be a semigroup, an element $e \in S$ is called *idempotent* if $e^2 = ee = e$ and the set of all idempotent elements of a semigroup S is denoted by $E(S)$.

Let a be an element of a semigroup S, then

a is called *regular* if $a = axa$ for some $x \in S$ and

a is called *completely regular* if $a = axa$ and $ax = xa$ for some $x \in S$.

A *regular* [*completely regular*] *semigroup* is a semigroup in which every element is regular [completely regular].

An element a of a monoid S is called *unit-regular* if there exists $u \in U(S)$ such that $a = aua$. The monoid S is called *unit-regular* if all its elements are unit-regular.

An element a of a semigroup S is called *left* [*right*] *regular* if $a \in Sa^2$ [$a \in a^2S$] and a is called *intra-regular* if $a \in Sa^2S$. The semigroup S is *left regular* [*right regular*, *intra-regular*] if all its elements are left regular [right regular, intra-regular].

Example 2.1.1. $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a commutative semigroup under multiplication with the identity 1. $E(\mathbb{Z}_{10}) = \{0, 1, 5, 6\}$. It is clear that for every element in $E(\mathbb{Z}_{10})$ is both left regular and right regular element in $\mathbb{Z}_{10}$. Consider

$$\begin{aligned} 2 &= 2^2 \cdot 8 = 8 \cdot 2^2, & 3 &= 3^2 \cdot 7 = 7 \cdot 3^2, \\ 4 &= 4^2 \cdot 4 = 4 \cdot 4^2, & 7 &= 7^2 \cdot 3 = 3 \cdot 7^2, \\ 8 &= 8^2 \cdot 2 = 2 \cdot 8^2, & 9 &= 9^2 \cdot 9 = 9 \cdot 9^2. \end{aligned}$$

Then $\mathbb{Z}_{10}$ is both left regular and right regular semigroup. It is clear that $\mathbb{Z}_{10}$ is intra-regular semigroup, since $1 \in \mathbb{Z}_{10}$.

$6\mathbb{Z}_{10} = \{0, 2, 4, 6, 8\}$ is a subsemigroup of $\mathbb{Z}_{10}$ and $E(6\mathbb{Z}_{10}) = \{0, 6\}$. We see that $6\mathbb{Z}_{10}$ is both left regular and right regular semigroup. It is clear that $0, 6$ are intra-regular elements in $6\mathbb{Z}_{10}$. Consider

$$2 = 2 \cdot 2^2 \cdot 4, \quad 4 = 4 \cdot 4^2 \cdot 6 \quad \text{and} \quad 8 = 2 \cdot 8^2 \cdot 6.$$

Then $6\mathbb{Z}_{10}$ is an intra-regular semigroup.

Theorem 2.1.2 ([18])**.** *An element a of a semigroup S is completely regular if and only if a is both left regular and right regular.*

Proof. Let a be a completely regular element in a semigroup S. Then there exists $x \in S$ such that $a = axa$ and $ax = xa$. So $a = axa = a^2x \in a^2S$ and $a = axa = xa^2 \in Sa^2$, i.e. a is both left regular and right regular.

Conversely, if a is both left regular and right regular element in a semigroup S, then $a \in a^2S \cap Sa^2$. So $a = a^2x$ and $a = ya^2$ for some $x, y \in S$. Consider

$$aya = ay(a^2x) = a(ya^2)x = aax = a^2x = a,$$

$$axa = (ya^2)xa = y(a^2x)a = yaa = ya^2 = a$$

$$\text{and} \quad ax = ya^2x = ya.$$

Hence $a(yax)a = (aya)xa = axa = a$ and $a(yax) = (aya)x = ax = ya = y(axa) = (yax)a$. Therefore a is completely regular. $\square$

Theorem 2.1.3 ([18])**.** *An element a of a semigroup S is a completely regular if and only if $a \in a^2Sa^2$.*

Proof. Let a be a completely regular element in a semigroup S. Then there exists $x \in S$ such that $a = axa$ and $ax = xa$. So

$$a = axa = (axa)x(axa) = (aax)x(xaa) = a^2(xxx)a^2 \in a^2Sa^2$$

Conversely, if $a \in a^2Sa^2$, then $a \in a^2Sa^2 \subseteq a^2S \cap Sa^2$. So a is both left regular and right regular. By Theorem 2.1.2, a is completely regular. $\square$

Theorem 2.1.4 ([18])**.** *Let S be a semigroup and $a \in S$. If a is completely regular, then a is intra-regular.*

Proof. Let a be completely regular. Then there exists $b \in S$ such that $a = aba$ and $ab = ba$. So $a = aba = a(ab) = aba(ab) = (ab)a^2(b) \in Sa^2S$. $\square$

2.1.2 Factorisation on Semigroups

Let S be a semigroup and let $E(S)$ be the set of all idempotent elements of S. We say S is *left* [*right*] *factorisable* if $S = GE(S)$ [$S = E(S)H$] for some subgroup G [H] of S. S is *factorisable* if S is both left and right factorisable.

Example 2.1.5. Consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a semigroup under multiplication. $G = \{1, 5\}$ is a subgroup of $\mathbb{Z}_6$ and $E(\mathbb{Z}_6) = \{0, 1, 3, 4\}$. Then $GE(\mathbb{Z}_6) = \mathbb{Z}_6$ and $E(\mathbb{Z}_6)G = \mathbb{Z}_6$. Hence $\mathbb{Z}_6$ is factorisable.

Theorem 2.1.6 ([2])**.** *A monoid S is factorisable if and only if it is unit-regular.*

Proof. If S is unit-regular, then for each $x \in S$, $x = xux$ for some unit u in S. Thus $xu, ux \in E(S)$ and so $x = xuu^{-1} \in E(S)U(S)$ and $x = u^{-1}ux \in U(S)E(S)$ Then $S = E(S)U(S) = U(S)E(S)$. So S is factorisable.

Conversely, if S is factorisable and $x \in S$, say $x = ea$ for some $e \in E(S)$ and for some $a \in U(S)$, then

$$xa^{-1}x = (ea)a^{-1}(ea) = (eaa^{-1})ea = eea = ea = x.$$

Therefore x is unit-regular and hence S is unit-regular. $\square$

Example 2.1.7. $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a semigroup under multiplication with the identity 1, $G = \{1, 5\} = U(\mathbb{Z}_6)$ and $E(\mathbb{Z}_6) = \{0, 1, 3, 4\}$. By Example 2.1.2, $\mathbb{Z}_6$ is factorisable. Consider

$$0 \cdot 1 \cdot 0 = 0, \quad 1 \cdot 1 \cdot 1 = 1,$$
$$2 \cdot 5 \cdot 2 = 2, \quad 3 \cdot 1 \cdot 3 = 3,$$
$$4 \cdot 1 \cdot 4 = 4, \quad 5 \cdot 5 \cdot 5 = 5.$$

Then $\mathbb{Z}_6$ is unit-regular.

Theorem 2.1.8 ([14]). *If S is factorisable then S has an identity and $S = GE(S) = E(S)G$ where G is a group of units of S. Moreover, if S has an identity and $S = E(S)G$ where G is a group of units of S then $S = GE(S)$.*

Proof. Suppose that $S = GE(S)$ for some subgroup G of S and let e be the identity of G. Then for each $x \in S$ there exists $a \in G$ and $f \in E(S)$ such that $x = af$ and $ex = (ea)f = af = x$. That is, S has a left identity and, by Duality, it has a right identity, and hence e is an identity of S. Suppose G is the group of units of S and $S = HE(S)$ for some subgroup H of S. If $x \in G$ then $x = af$ for some $a \in H$, $f \in E(S)$ and so

$$f = ef = x^{-1}xf = x^{-1}aff = x^{-1}x = e.$$

Hence, $x = ae = a \in H$. So $G \subseteq H$ and it follows that $G = H$. A dual argument shows that $S = E(S)G$ also. Finally, assume that S has an identity and $S = E(S)G$ where G is a group of units of S. Let $x \in S$. Then $x = fg$ for some $f \in E(S)$ and $g \in G$. Since $(g^{-1}fg)(g^{-1}fg) = g^{-1}fg$, we get $g^{-1}fg \in E(S)$. Thus

$$x = ex = (gg^{-1})fg = g(g^{-1}fg) \in GE(S).$$

This proves that $S = GE(S)$. □

2.2 Identities and Varities

In this section, we give the briefly concept of identities and hyperidentities.

Let $X := \{x_1, x_2, ...\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as *letters*, to X as an *alphabet*, and also refer to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an *n-element alphabet*. Let $\tau = (n_i)_{i \in I}$ be a type such that the set of operation symbols $\{f_i | i \in I\}$ is disjoint with X_n. An n-ary term, is defined inductively as follows:

(i) The variables $x_1, x_2, ..., x_n$ are n-ary terms.

(ii) If $t_1, t_2, ..., t_{n_i}$ are n-ary terms then $f_i(t_1, t_2, ..., t_{n_i})$ is an n-ary term.

The smallest set, which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii), is denoted by $W_\tau(X_n)$ and it is called the *set of all $n-$ary terms of type* τ. It is clear that every n-ary term is also an m-ary term for all $m \geq n$. Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$. It is called the set of all terms of type τ.

Example 2.2.1. Let $\tau = (2, 3)$. This means we have one binary operation symbol and one ternary operation symbol, say f and g respectively. These are some examples of ternary terms of type (2, 3): $x_1, x_2, x_3, f(x_3, g(x_1, x_3, x_3)), g(f(x_2, x_3), x_1,\ g(x_3, x_1, x_2))$.

The *complexity of terms* is a mapping $c : W_\tau(X) \to \mathbb{N} \cup \{0\}$ which is inductively defined by

(i) if $t = x_i \in X$ then $c(t) := 0$

(ii) if $t_1, t_2, ..., t_{n_i} \in W_\tau(X)$ and $max\{c(t_1), c(t_2), ..., c(t_{n_i})\} = m$
then $c(f_i(t_1, t_2, ..., t_{n_i})) := m + 1$.

Let $\tau = (n_i)_{i \in I}$ be a type with the sequence of operation symbol $(f_i)_{i \in I}$. Let $t \in W_\tau(X_n)$ for $n \in \mathbb{N}$ and $\mathcal{A} = (A, (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ. The *$n - ary$ term operation* $t^{\mathcal{A}} : A^n \to A$ of type τ is inductively defined by

(i) $t^{\mathcal{A}}(a_1, a_2, ..., a_n) := a_i$ if $t = x_i \in X_n$.

(ii) $t^{\mathcal{A}}(a_1, a_2, ..., a_n) := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, a_2, ..., a_n), t_2^{\mathcal{A}}(a_1, a_2, ..., a_n), ..., t_{n_i}^{\mathcal{A}}(a_1, a_2, ..., a_n))$
if t is a compound term $f_i(t_1, t_2, ..., t_{n_i})$.

The set of all $n-$ary term operations of the algebra $\mathcal{A}$ denoted by $W_\tau(X_n)^{\mathcal{A}}$ and denote the set of all (finitary) term operations on $\mathcal{A}$ by $W_\tau(X)^{\mathcal{A}}$. Make a remark that the element of $W_\tau(X_n)^{\mathcal{A}}$ are also called $n-$ary term operations induced by terms from $W_\tau(X_n)$.

Let $X = \{x_1, x_2, ...\}$ be a countably infinite set of variables. Let $\tau = (n_i)_{i \in I}$ be a type and $\mathcal{A}$ be an algebra of type τ.

An *equation* of type τ is a pair of terms (s, t) from $W_\tau(X)$; such pairs are more commonly write as $s \approx t$. The set of all equations of type τ is denoted by $E_\tau(X)$.

An equation $s \approx t$ is said to be an *identity* in an algebra $\mathcal{A}$ of type τ if $s^{\mathcal{A}} = t^{\mathcal{A}}$, that is, if the term operations induced by s and t on the algebra $\mathcal{A}$ are equal. In this case we also say that the equation $s \approx t$ is *satisfied* or *modelled* by the algebra $\mathcal{A}$, and we write $\mathcal{A} \models s \approx t$.

We now consider the class $Alg(\tau)$ of all algebras of type τ. Let K be a class of algebra of type τ. The class K satisfies an equation $s \approx t$, if for every $\mathcal{A} \in K$, $\mathcal{A} \models s \approx t$, and we write $K \models s \approx t$.

Let Σ be a set of equations of type τ. A class K of algebras of type τ is said to satisfies Σ, if $K \models s \approx t$ for every $s \approx t \in \Sigma$, and we write $K \models \Sigma$. Let

$$\begin{aligned} IdK &:= \{s \approx t \in E_\tau(X) | K \models s \approx t\}, \\ Mod\Sigma &:= \{\mathcal{A} \in Alg(\tau) | \mathcal{A} \models \Sigma\}. \end{aligned}$$

be the set of all identities satisfied in K and the class of all algebras satisfied Σ, respectively. We obtain the following theorem.

Theorem 2.2.2 ([11]). *Let $K, K_1, K_2 \subseteq Alg(\tau)$ and $\Sigma, \Sigma_1, \Sigma_2 \subseteq E_\tau(X)$. Then*

(i) $K \subseteq ModIdK$ and $\Sigma \subseteq IdMod\Sigma$,

(ii) if $K_1 \subseteq K_2$ then $IdK_2 \subseteq IdK_1$ and, if $\Sigma_1 \subseteq \Sigma_2$ then $Mod\Sigma_2 \subseteq Mod\Sigma_1$.

A class Σ of equations of type τ is called an *equational theory* if $\Sigma = IdMod\Sigma$. A class V of algebra of type τ is called a *variety* if $V = ModIdV$.

Theorem 2.2.3 ([11]). *A non-empty subclass V of $Alg(\tau)$ is a variety if and only if there exists $\Sigma \in E_\tau(X)$ such that $V = Mod\Sigma$.*

2.3 Hyperidentities and Hypervarities

The notions of hyperidentities and hypervarieties of a given type τ without nullary operations originated by J.Aczèl [1], V.D. Belousov [3], W.D. Neumann [17] and W. Taylor [25]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [9].

Let $\tau = (n_i)_{i \in I}$ be a type with the sequence of operation symbols $(f_i)_{i \in I}$. A *hypersubstitution* of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps n_i-ary operation symbols to n_i-ary terms. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ.

For all $\sigma \in Hyp(\tau)$ induces a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows, for any $t \in W_\tau(X)$, $\widehat{\sigma}[t]$ is inductively defined by

(i) $\widehat{\sigma}[t] := t$ if $t \in X$.

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Example 2.3.1. Let $\tau = (3)$ be a type, i.e. we have only ternary operation symbol, say f. Let $\sigma : \{f\} \longrightarrow W_\tau(X)$ be defined by $\sigma(f) = f(x_2, x_1, x_3) \in W_\tau(X_3)$. Then σ is a hypersubstitution of type $\tau = (3)$. Then we obtain

$$\begin{aligned}
\widehat{\sigma}[f(x_2, x_1, f(x_3, x_2, x_1))] &= \sigma(f)(\widehat{\sigma}[x_2], \widehat{\sigma}[x_1], \widehat{\sigma}[f(x_3, x_2, x_1)]) \\
&= f(x_2, x_1, x_3)(\widehat{\sigma}[x_2], \widehat{\sigma}[x_1], \widehat{\sigma}[f(x_3, x_2, x_1)]) \\
&= f(x_2, x_1, x_3)(x_2, x_1, \sigma(f)(\widehat{\sigma}[x_3], \widehat{\sigma}[x_2], \widehat{\sigma}[x_1])) \\
&= f(x_2, x_1, x_3)(x_2, x_1, f(x_2, x_1, x_3)(x_3, x_2, x_1)) \\
&= f(x_2, x_1, x_3)(x_2, x_1, f(x_2, x_3, x_1)) \\
&= f(x_1, x_2, f(x_2, x_3, x_1)).
\end{aligned}$$

By using the induced maps $\widehat{\sigma}$, a binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$. For any hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$, $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings.

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp(\tau)} = (Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

Let $\underline{M}$ be a submonoid of the monoid of all hypersubstitutions of type τ and let V be a variety of type τ. We called an identity $s \approx t \in IdV$ is $M-$*hyperidentity* if

$$\forall \sigma \in M (V \models \widehat{\sigma}[s] \approx \widehat{\sigma}[t]).$$

We called the variety V is $M-$*solid variety* if

$$\forall s \approx t \in IdV, \forall \sigma \in M (\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV).$$

If $\underline{M}$ is the monoid of all hypersubstitutions of type τ, then we called $M-$hyperidentity and $M-$solid variety is *hyperidentity* and *solid variety*, repectively.

2.4 The Monoid of all Generalized Hypersubstitutions

In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [16]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid.

Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. A *generalized hypersubstitution* of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on this set, we need the concept of *generalized superposition* of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ which is defined by the following steps:

(i) If $t = x_j$, $1 \le j \le m$, then $S^m(t, t_1, ..., t_m) = S^m(x_j, t_1, ..., t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(t, t_1, ..., t_m) = S^m(x_j, t_1, ..., t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, ..., s_{n_i})$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m))$.

Every generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, t_2, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\widehat{\sigma}[t_j]$, $1 \le j \le n_i$ are already defined.

Example 2.4.1. Let $\tau = (2)$, i.e. there is only one binary operation symbol, say f. Let $\sigma \in Hyp_G(2)$ where $\sigma(f) = f(x_2, f(x_3, x_2))$. Then

$$\begin{aligned}
\widehat{\sigma}[f(f(x_1, x_5), x_3)] &= S^2(\sigma(f), \widehat{\sigma}[f(x_1, x_5)], \widehat{\sigma}[x_3]) \\
&= S^2(f(x_2, f(x_3, x_2)), S^2(\sigma(f), \widehat{\sigma}[x_1], \widehat{\sigma}[x_5]), x_3) \\
&= S^2(f(x_2, f(x_3, x_2)), S^2(f(x_2, f(x_3, x_2)), x_1, x_5), x_3) \\
&= S^2(f(x_2, f(x_3, x_2)), f(x_5, f(x_3, x_5)), x_3) \\
&= f(x_3, f(x_3, x_3)).
\end{aligned}$$

Example 2.4.2. Let $\tau = (2, 3)$, i.e. we have one binary operation symbol and one ternary operation symbol, say f and g, respectively. Let $\sigma : \{f, g\} \longrightarrow W_{(2,3)}(X)$ where $\sigma(f) = f(x_1, g(x_3, x_1, x_2))$ and $\sigma(g) = f(x_4, x_1)$. Then σ is a generalized hypersubstitution of type $\tau = (2,3)$ which is not a hypersubstitution of type $\tau = (2,3)$. Then we have

$$\begin{aligned}
\widehat{\sigma}[f(g(x_3,x_2,x_1),x_5)] &= S^2(\sigma(f),\widehat{\sigma}[g(x_3,x_2,x_1)],\widehat{\sigma}[x_5]) \\
&= S^2(f(x_1,g(x_3,x_1,x_2)),S^3(\sigma(g),\widehat{\sigma}[x_3],\widehat{\sigma}[x_2],\widehat{\sigma}[x_1]),x_5) \\
&= S^2(f(x_1,g(x_3,x_1,x_2)),S^3(f(x_4,x_1),x_3,x_2,x_1),x_5) \\
&= S^2(f(x_1,g(x_3,x_1,x_2)),f(x_4,x_3),x_5) \\
&= f(f(x_4,x_3),g(x_3,f(x_4,x_3),x_5)).
\end{aligned}$$

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings.

Example 2.4.3. Let $\tau = (3)$ be a type with an operation symbol f. Let $\sigma, \rho \in Hyp_G(3)$ where $\sigma(f) = f(x_3, f(x_2,x_4,x_5),x_1)$ and $\rho(f) = f(f(x_4,x_2,x_5),x_1,x_2)$. Then

$$\begin{aligned}
(\sigma \circ_G \rho)(f) &= (\widehat{\sigma} \circ \rho)(f) \\
&= \widehat{\sigma}[f(f(x_4,x_2,x_5),x_1,x_2)] \\
&= S^3(\sigma(f),\widehat{\sigma}[f(x_4,x_2,x_5)],\widehat{\sigma}[x_1],\widehat{\sigma}[x_2]) \\
&= S^3(\sigma(f),S^3(\sigma(f),\widehat{\sigma}[x_4],\widehat{\sigma}[x_2],\widehat{\sigma}[x_5]),x_1,x_2) \\
&= S^3(\sigma(f),S^3(f(x_3,f(x_2,x_4,x_5),x_1),x_4,x_2,x_5),x_1,x_2) \\
&= S^3(\sigma(f),f(x_5,f(x_2,x_4,x_5),x_4),x_1,x_2) \\
&= S^3(f(x_3,f(x_2,x_4,x_5),x_1),f(x_5,f(x_2,x_4,x_5),x_4),x_1,x_2) \\
&= f(x_2,f(x_1,x_4,x_5),f(x_5,f(x_2,x_4,x_5),x_4))
\end{aligned}$$

and

$$\begin{aligned}
(\rho \circ_G \sigma)(f) &= (\widehat{\rho} \circ \sigma)(f) \\
&= \widehat{\rho}[f(x_3,f(x_2,x_4,x_5),x_1)] \\
&= S^3(\rho(f),\widehat{\rho}[x_3],\widehat{\rho}[f(x_2,x_4,x_5)],\widehat{\rho}[x_1]) \\
&= S^3(\rho(f),x_3,S^3(\rho(f),\widehat{\rho}[x_2],\widehat{\rho}[x_4],\widehat{\rho}[x_5]),x_1) \\
&= S^3(\rho(f),x_3,S^3(f(f(x_4,x_2,x_5),x_1,x_2),x_2,x_4,x_5),x_1) \\
&= S^3(\rho(f),x_3,f(f(x_4,x_4,x_5),x_2,x_4),x_1) \\
&= S^3(f(f(x_4,x_2,x_5),x_1,x_2),x_3,f(f(x_4,x_4,x_5),x_2,x_4),x_1) \\
&= f(f(x_4,f(f(x_4,x_4,x_5),x_2,x_4),x_5),x_3,f(f(x_4,x_4,x_5),x_2,x_4)).
\end{aligned}$$

We see that $\sigma \circ_G \rho \neq \rho \circ_G \sigma$, so $\circ_G$ does not satisfy commutative law.

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$. In [16], S. Leeratanavalee and K. Denecke proved that:

Theorem 2.4.4 ([16]). *For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$,*

(i) $S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]) = \widehat{\sigma}[S^n(t, t_1, ..., t_n)]$,

(ii) $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Theorem 2.4.5 ([16]). $\underline{Hyp_G(\tau)} = (Hyp_G(\tau), \circ_G, \sigma_{id})$ *is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of* $\underline{Hyp_G(\tau)}$.

Next, for more convenience we will write $Hyp_G(\tau)$ instead of the monoid $\underline{Hyp_G(\tau)}$.