CHAPTER 3

Characterization of Some Special Elements in $Hyp_G(\tau)$

In the semigroup theory, the special elements in semigroup have studied diverse such as regular element, quasi-regular element and idempotent element. In Chapter 2, we have $(Hyp(\tau), \circ_h, \sigma_{id})$ and $(Hyp_G(\tau), \circ_G, \sigma_{id})$ are a monoids. So we can characterized these special elements of $Hyp(\tau)$ and $Hyp_G(\tau)$. Th. Changphas characterize idempotent elements and regular elements of the monoid of all hypersubstitutions of type τ [7]. W. Puninagool and S. Leeratanavalee characterized some special elements of the monoid of all generalized hypersubstitutions of type τ. Such as the following:

(i) Characterize the set of all idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ [22].

(ii) Characterize the set of all regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ [20].

(iii) Characterize the set of all idempotent and regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ [21].

Furthermore, all idempotent and regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (3)$ was studied by S. Sudsanit and S. Leeratanavalee [23]. In 2014, S. Sudsanit, S. Leeratanavalee and W. Puninagool characterized left-right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ [24].

The main results of this thesis, we study on the factorisable monoid of generalized hypersubstitutions of type τ. We know that a semigroup is factorisable if and only if it is unit-regular semigroup. So in this chapter, at first we characterize the set of all unit elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Then we used the concepts of unit element and regular element as tools to determine the set of all unit-regular of the monoid of all generalized hypersustitutions of type $\tau = (2)$ and type $\tau = (n)$, respectively.

Moreover, we characterize the set of all completely regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ and we have that a completely regular element is both left regular and right regular element of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Finally, we show that the set of all completely regular elements and the set of all intra-regular elements of type $\tau = (2)$ are the same.

From now on, we introduce some notations which will be used throughout of this thesis. Let $\tau = (n)$ be a type, that means we have only one n-ary operation, say f and let $t \in W_{(n)}(X)$, we denote

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t) :=$ the set of all variables occurring in the term t,

$vb^t(x):=$ the number of occurrences of a variable x in t.

3.1 All Unit Elements in $Hyp_G(n)$

In this section, we characterize all unit elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

We fix a type $\tau = (n)$, i.e. we have only one n-ary operation, say f.

Lemma 3.1.1. *Let* $\sigma_t \in Hyp_G(n)$ *where* $t = f(t_1, t_2, ..., t_n) \in W_{(n)}(X)$. *If* $t_i \in W_{(n)}(X) \backslash X$ *for some* $i \in \{1, 2, ..., n\}$, *then* σ_t *is not unit.*

Proof. Let $t = f(t_1, ..., t_i, ..., t_n) \in W_{(n)}(X)$ where $t_i \in W_{(n)}(X) \backslash X$ for some $i \in \{1, 2, ..., n\}$. Let $\sigma_s \in Hyp_G(n)$ and $s = f(s_1, s_2, ..., s_n)$ where $s_i \in W_{(n)}(X)$ for all $i \in \{1, 2, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2, ..., s_n)] \\ &= S^n(f(t_1, ..., t_i, ..., t_n), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]) \\ &= f(S^n(t_1, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), ..., S^n(t_i, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), \\ &\quad ..., S^n(t_n, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n])).\end{aligned}$$

Since $t_i \in W_{(n)}(X) \backslash X$, so $\widehat{\sigma}_t[s_j] \in W_{(n)}(X) \backslash X$ for all $j \in \{1, 2, ..., n\}$. Then $(\sigma_t \circ_G \sigma_s)(f) \neq f(x_1, x_2, ..., x_n) = \sigma_{id}(f)$. Hence $\sigma_t \circ_G \sigma_s \neq \sigma_{id}$ for all $\sigma_s \in Hyp_G(n)$. Therefore σ_t is not unit in $Hyp_G(n)$. □

Example 3.1.2. Let $\tau = (2)$ and $t = f(x_1, f(x_2, x_3))$. For each $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2)] \\ &= S^2(f(x_1, f(x_2, x_3)), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \quad \text{where } \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2] \in W_{(2)}(X) \\ &= f(\widehat{\sigma}_t[s_1], f(\widehat{\sigma}_t[s_2], x_3)) \\ &\neq f(x_1, x_2) = \sigma_{id}(f).\end{aligned}$$

Hence $\sigma_t \notin U(Hyp_G(2))$.

Lemma 3.1.3. *Let $\sigma_t \in Hyp_G(n)$ where $t = f(x_{m_1}, x_{m_2}, ..., x_{m_n}) \in W_{(n)}(X)$. If $m_i > n$ for some $i \in \{1, 2, ..., n\}$, then σ_t is not unit in $Hyp_G(n)$.*

Proof. Let $t = f(x_{m_1}, x_{m_2}, ..., x_{m_n})$ and $m_i > n$ for some $i \in \{1, 2, ..., n\}$. Then $x_{m_i} \in X \backslash X_n$. Let $\sigma_s \in Hyp_G(n)$ where $s = f(s_1, s_2, ..., s_n)$.
Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2, ..., s_n)] \\
&= S^n(f(x_{m_1}, x_{m_2}, ..., x_{m_n}), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]) \\
&= f(S^n(x_{m_1}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), S^n(x_{m_2}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], \\
&\quad ..., \widehat{\sigma}_t[s_n]), ..., S^n(x_{m_n}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n])).
\end{aligned}$$

Since $x_{m_i} \in X \backslash X_n$, so $S^n(x_{m_i}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]) = x_{m_i}$. Then $(\sigma_t \circ_G \sigma_s)(f) \neq f(x_1, x_2, ..., x_n) = \sigma_{id}(f)$, i.e. $\sigma_t \circ_G \sigma_s \neq \sigma_{id}$ for all $\sigma_s \in Hyp_G(n)$. Hence σ_t is not unit in $Hyp_G(n)$. □

Example 3.1.4. Let $\tau = (3)$ and $t = f(x_1, x_4, x_3)$. For each $s = f(s_1, s_2, s_3)$ where $s_1, s_2, s_3 \in W_{(3)}(X)$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2, s_3)] \\
&= S^3(f(x_1, x_4, x_3), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], \widehat{\sigma}_t[s_3]) \\
&\quad \text{where } \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], \widehat{\sigma}_t[s_3] \in W_{(3)}(X) \\
&= f(\widehat{\sigma}_t[s_1], x_4, \widehat{\sigma}_t[s_3]) \\
&\neq f(x_1, x_2, x_3) \\
&= \sigma_{id}(f).
\end{aligned}$$

Hence $\sigma_t \notin U(Hyp_G(3))$.

Theorem 3.1.5. *An element $\sigma_t \in U(Hyp_G(n))$ if and only if $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ where $\pi \in S_n$ and S_n is a set of all permutations of $\{1, 2, ..., n\}$.*

Proof. Assume that $\sigma_t \in U(Hyp_G(n))$, then there exists $\sigma_s \in U(Hyp_G(n))$ such that $\sigma_t \circ_G \sigma_s = \sigma_{id} = \sigma_s \circ_G \sigma_t$. By Lemma 3.1.1 and Lemma 3.1.3, if $t = f(t_1, t_2, ..., t_n)$ and $s = f(s_1, s_2, ..., s_n)$ then $t_1, ..., t_n, s_1, ..., s_n \in \{x_1, x_2, ..., x_n\}$. Let $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$

and $s = f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})$ where $\pi, \pi' : \{1, 2, ..., n\} \to \{1, 2, .., n\}$. Consider

$$\begin{aligned}
\sigma_{id}(f) &= (\sigma_t \circ_G \sigma_s)(f) \\
f(x_1, x_2, ..., x_n) &= \widehat{\sigma}_t[f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})] \\
&= S^n(f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}), x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)}) \\
&= f(x_{\pi'(\pi(1))}, x_{\pi'(\pi(2))}, ..., x_{\pi'(\pi(n))}) \\
&= f(x_{(\pi'\circ\pi)(1)}, x_{(\pi'\circ\pi)(2)}, ..., x_{(\pi'\circ\pi)(n)})
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{id}(f) &= (\sigma_s \circ_G \sigma_t)(f) \\
f(x_1, x_2, ..., x_n) &= \widehat{\sigma}_s[f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})] \\
&= S^n(f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)}), x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) \\
&= f(x_{\pi(\pi'(1))}, x_{\pi(\pi'(2))}, ..., x_{\pi(\pi'(n))}) \\
&= f(x_{(\pi\circ\pi')(1)}, x_{(\pi\circ\pi')(2)}, ..., x_{(\pi\circ\pi')(n)}).
\end{aligned}$$

Then $\pi \circ \pi' = (1) = \pi' \circ \pi$ and $\pi \circ \pi', \pi' \circ \pi$ are bijective. Next, we will show that π is bijective. Let $\pi(i) = \pi(j)$ for some $i, j \in \{1, 2, .., n\}$. Then

$$(\pi' \circ \pi)(i) = (\pi'(\pi(i)) = \pi'(\pi(j)) = (\pi' \circ \pi)(j).$$

Since $\pi' \circ \pi$ is one-to-one, $i = j$. Thus π is one-to-one. Let $i \in \{1, 2, ..., n\}$. Since $\pi \circ \pi'$ is onto, there exists $j \in \{1, 2, ..., n\}$ such that $(\pi \circ \pi')(j) = i$. Then $\pi(\pi'(j)) = i$ for some $\pi'(j) \in \{1, 2, ..., n\}$. Hence π is onto, so $\pi \in S_n$.

Conversely, let $\sigma_t \in Hyp_G(n)$ where $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ such that $\pi \in S_n$. Since $(S_n, \circ)$ is a group, there exists $\pi' \in S_n$ such that $\pi \circ \pi' = (1) = \pi' \circ \pi$. Let $\sigma_s \in Hyp_G(n)$ where $s = f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})$. Then

$$\begin{aligned}
(\sigma_t \circ \sigma_s)(f) &= \widehat{\sigma}_t[f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})] \\
&= f(x_{(\pi'\circ\pi)(1)}, x_{(\pi'\circ\pi)(2)}, ..., x_{(\pi'\circ\pi)(n)}) \\
&= f(x_1, x_2, ..., x_n) \\
&= \sigma_{id}(f).
\end{aligned}$$

Similarly, we have $\sigma_s \circ \sigma_t = \sigma_{id}$. So $\sigma_t \in U(Hyp_G(n))$. □

Example 3.1.6. Let $\tau = (5)$ and $u \in W_{(5)}(X) \setminus X$ where $u = f(x_4, x_1, x_5, x_2, x_3)$. Let $\pi \in S_5$ such that $\pi(1) = 4$, $\pi(2) = 1$, $\pi(3) = 5$, $\pi(4) = 2$ and $\pi(5) = 3$. Then

$$u = f(x_4, x_1, x_5, x_2, x_3) = f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}, x_{\pi(5)}).$$

There exists $\pi^{-1} \in S_5$ such that $\pi^{-1}(1) = 2$, $\pi^{-1}(2) = 4$, $\pi^{-1}(3) = 5$, $\pi^{-1}(4) = 1$ and $\pi^{-1}(5) = 3$. Let

$$u^{-1} = f(x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, x_{\pi^{-1}(3)}, x_{\pi^{-1}(4)}, x_{\pi^{-1}(5)}) = f(x_2, x_4, x_5, x_1, x_3).$$

Consider

$$\begin{aligned}
(\sigma_u \circ_G \sigma_{u^{-1}})(f) &= \widehat{\sigma}_u[f(x_2, x_4, x_5, x_1, x_3)] \\
&= S^5(u, \widehat{\sigma}_u[x_2], \widehat{\sigma}_u[x_4], \widehat{\sigma}_u[x_5], \widehat{\sigma}_u[x_1], \widehat{\sigma}_u[x_3]) \\
&= S^5(f(x_4, x_1, x_5, x_2, x_3), x_2, x_4, x_5, x_1, x_3) \\
&= f(x_1, x_2, x_3, x_4, x_5) \\
&= \sigma_{id}(f)
\end{aligned}$$

and

$$\begin{aligned}
(\sigma_{u^{-1}} \circ_G \sigma_u)(f) &= \widehat{\sigma}_{u^{-1}}[f(x_4, x_1, x_5, x_2, x_3)] \\
&= S^5(u^{-1}, \widehat{\sigma}_{u^{-1}}[x_4], \widehat{\sigma}_{u^{-1}}[x_1], \widehat{\sigma}_{u^{-1}}[x_5], \widehat{\sigma}_{u^{-1}}[x_2], \widehat{\sigma}_{u^{-1}}[x_3]) \\
&= S^5(f(x_2, x_4, x_5, x_1, x_3), x_4, x_1, x_5, x_2, x_3) \\
&= f(x_1, x_2, x_3, x_4, x_5) \\
&= \sigma_{id}(f).
\end{aligned}$$

Hence $\sigma_{u^{-1}}$ is an inverse of σ_u. Therefore $\sigma_u, \sigma_{u^{-1}} \in U(Hyp_G(5))$.

By Theorem 3.1.5, we get

$$U(Hyp_G(n)) := \{\sigma_t \in Hyp_G(n) | t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) \text{ where } \pi \in S_n\}$$

is the set of all unit elements in $Hyp_G(n)$.

Corollary 3.1.7. $|U(Hyp_G(n))| = n!$.

Corollary 3.1.8. $U(Hyp_G(2)) = \{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

3.2 All Unit-regular Elements in $Hyp_G(2)$

In this section, we used the concepts of unit element, idempotent element and regular element as tools to determine the set of all unit-regular of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

First, we fix a type $\tau = (2)$ with the binary operation symbol f. Let $\sigma_t \in Hyp_G(2)$, we denote

$R_{(Hyp_G(2))_1} := \{\sigma_t | t = f(x_2, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin var(t')\}$,
$R_{(Hyp_G(2))_2} := \{\sigma_t | t = f(t', x_1) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin var(t')\}$,
$R_{(Hyp_G(2))_3} := \{\sigma_t | t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin var(t')\}$,
$R_{(Hyp_G(2))_4} := \{\sigma_t | t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin var(t')\}$,
$R_{(Hyp_G(2))_5} := \{\sigma_t | t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}\}$ and
$R_{(Hyp_G(2))_6} := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2011, W. Puninagool and S. Leeratanavalee showed that: $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is the set of all regular elements in $Hyp_G(2)$ [20]. In 2008, W. Puninagool and S. Leeratanavalee showed that: $\bigcup_{i=3}^{6} R_{(Hyp_G(2))_i} \setminus \{\sigma_{f(x_2,x_1)}\} = E(Hyp_G(2))$ [22]. By Corollary 3.1.8 we get $U(Hyp_G(2)) = \{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Since $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is a set of all regular elements in $Hyp_G(2)$, a set of all unit-regular elements in $Hyp_G(2)$ is a subset of $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$. Next, we will determine the set of all unit-regular elements in $Hyp_G(2)$.

Theorem 3.2.1. *$\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is a set of all unit-regular elements in $Hyp_G(2)$.*

Proof. Let $\sigma_t \in \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$, then $\sigma_t \in R_{(Hyp_G(2))_1}$ or $\sigma_t \in R_{(Hyp_G(2))_2}$ or $\sigma_t \in \bigcup_{i=3}^{6} R_{(Hyp_G(2))_i} \setminus \{\sigma_{f(x_2,x_1)}\}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$.

Case 1: $\sigma_t \in R_{(Hyp_G(2))_1}$. Then $t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[f(x_2, t')]] \\
&= \widehat{\sigma}_t[S^2(f(x_2, x_1), x_2, \widehat{\sigma}_{f(x_2,x_1)}[t'])] \\
&= \widehat{\sigma}_t[f(\widehat{\sigma}_{f(x_2,x_1)}[t'], x_2)] \\
&= S^2(f(x_2, t'), \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[t']], x_2) \\
&= f(x_2, t') \quad \text{since } x_1 \notin var(t') \\
&= \sigma_t(f).
\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_t$.

Case 2: $\sigma_t \in R_{(Hyp_G(2))_2}$. Then $t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[f(t', x_1)]] \\
&= \widehat{\sigma}_t[S^2(f(x_2, x_1), \widehat{\sigma}_{f(x_2,x_1)}[t'], x_1)] \\
&= \widehat{\sigma}_t[f(x_1, \widehat{\sigma}_{f(x_2,x_1)}[t'])] \\
&= S^2(f(t', x_1), x_1, \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[t']]) \\
&= f(t', x_1) \quad \text{since } x_2 \notin var(t') \\
&= \sigma_t(f).
\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_t$.

Case 3: $\sigma_t \in \bigcup_{i=3}^{6} R_{(Hyp_G(2))_i} \setminus \{\sigma_{f(x_2,x_1)}\} = E(Hyp_G(2))$. Then

$$\sigma_t \circ_G \sigma_{id} \circ_G \sigma_t = \sigma_t \circ_G \sigma_t = \sigma_t.$$

Case 4: $\sigma_t = \sigma_{f(x_2,x_1)}$. Then

$$\sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{id} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{f(x_2,x_1)}.$$

Therefore, for every $\sigma_t \in \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$, there exists $\sigma_u \in U(Hyp_G(2))$ such that $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$. Hence $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is a set of all unit-regular elements in $Hyp_G(2)$. $\square$

Then we get, for every element in $Hyp_G(2)$ is a regular element if and only if it is a unit-regular element.

Remark 3.2.2. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not closed under $\circ_G$, i.e. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not a subsemigroup of $Hyp_G(2)$.

Example 3.2.3. (1) Let $\sigma_t \in R_{(Hyp_G(2))_1}$ such that $t = f(x_2, t')$ where $t' = f(x_3, x_2)$. Then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_2, f(x_3, x_2))] \\
&= S^2(f(x_2, f(x_3, x_2)), \widehat{\sigma}_t[x_2], \widehat{\sigma}_t[f(x_3, x_2)]) \\
&= S^2(f(x_2, f(x_3, x_2)), x_2, f(x_2, f(x_3, x_2))) \\
&= f(f(x_2, f(x_3, x_2)), f(x_3, f(x_2, f(x_3, x_2)))).
\end{aligned}$$

So, $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

(2) Let $\sigma_t \in R_{(Hyp_G(2))_2}$ such that $t = f(t', x_1)$ where $t' = f(x_1, x_5)$. Then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(f(x_1, x_5), x_1)] \\
&= S^2(f(f(x_1, x_5), x_1), \widehat{\sigma}_t[f(x_1, x_5)], \widehat{\sigma}_t[x_1]) \\
&= S^2(f(f(x_1, x_5), x_1), f(f(x_1, x_5), x_1), x_1) \\
&= f(f(f(f(x_1, x_5), x_1), x_5), f(f(x_1, x_5), x_1)).
\end{aligned}$$

So, $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

(3) Let $\sigma_t \in R_{(Hyp_G(2))_3}$ and $\sigma_s \in R_{(Hyp_G(2))_4}$ such that $t = f(x_1, t')$ and $s = f(s', x_2)$ where $t' = f(x_5, x_1)$ and $s' = f(x_2, x_3)$.

Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(f(x_2, x_3), x_2)] \\
&= S^2(f(x_1, f(x_5, x_1)), \widehat{\sigma}_t[f(x_2, x_3)], \widehat{\sigma}_t[x_2]) \\
&= S^2(f(x_1, f(x_5, x_1), f(x_2, f(x_5, x_2)), x_2) \\
&= f(f(x_2, f(x_5, x_2)), f(x_5, f(x_2, f(x_5, x_2)))).
\end{aligned}$$

So $\sigma_t \circ_G \sigma_s \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

Consider

$$\begin{aligned}
(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, f(x_5, x_1))] \\
&= S^2(f(f(x_2, x_3), x_2), \widehat{\sigma}_s[x_1], \widehat{\sigma}_s[f(x_5, x_1)]) \\
&= S^2(f(f(x_2, x_3), x_2), x_1, f(f(x_1, x_3), x_1)) \\
&= f(f(f(f(x_1, x_3), x_1), x_3), f(f(x_1, x_3), x_1)).
\end{aligned}$$

So $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

By (1), (2) or (3), we have $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not a subsemigroup of $Hyp_G(2)$.

3.3 All Unit-regular Elements in $Hyp_G(n)$

In this section, we determine the set of all unit-regular of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Moreover, we will show that it is not a submonoid of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

For a type $\tau = (n)$ with n-ary operation f, we define:

Definition 3.3.1. Let $t \in W_{(n)}(X)$, a *subterm* of t is defined inductively by the following.

(i) Every variable $x \in var(t)$ is a subterm of t.

(ii) If $t = f(t_1, ..., t_n)$, then t itself, $t_1, ..., t_n$ are subterm of t.

(iii) If $t', t'' \in W_{(n)}(X)$ which t'' is a subterm of t' and t' is a subterm of t, then t'' is a subterm of t.

We denote the set of all subterms of t by $sub(t)$.

Example 3.3.2. Let $\tau = (2)$ and $t \in W_{(2)}(X)$ where $t = f(t_1, t_2)$ such that $t_1 = f(x_3, f(x_1, x_4))$ and $t_2 = f(f(x_7, x_1), f(x_2, x_1))$. Then

$$sub(t_1) = \{t_1, f(x_1, x_4), x_1, x_3, x_4\},$$
$$sub(t_2) = \{t_2, f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_7\},$$
$$sub(t) = \{t, t_1, t_2, f(x_1, x_4), f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_3, x_4, x_7\}.$$

Lemma 3.3.3. *For each $\sigma_s, \sigma_t \in Hyp_G(n)$ where $t = f(t_1, ..., t_n)$ such that $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. Let $h_1, ..., h_p \in \{j_1, \ldots, j_m\}$ and $h_l \neq h_r$ if $l \neq r$. Then $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ if and only if $s = f(s_1, ..., s_n)$ where $s_{h_q} = s_{j_l} = x_{i_l}$ for all $q \in \{1, ..., p\}$ and for some $l \in \{1, ..., m\}$.*

Proof. Assume that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ and let $s = f(s_1, ..., s_n)$. Suppose that, there exists $s_{h_q} = s_{j_l}$ for some $q \in \{1, ..., p\}$ and for some $l \in \{1, ..., m\}$ such that $s_{j_l} \in W_n(X) \setminus \{x_{i_l}\}$ for some $l \in \{1, ..., m\}$. Then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_s[t]] \\
&= \widehat{\sigma}_t[S^n(f(s_1, ..., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])] \\
&= \widehat{\sigma}_t[f(w_1, ..., w_n)] \quad \text{where } w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) \\
&\quad\ \text{for all } i \in \{1, ..., n\} \\
&= S^n(f(t_1, ..., t_n), \widehat{\sigma}_t[w_1], ..., \widehat{\sigma}_t[w_n]) \\
&= f(u_1, ..., u_n) \quad \text{where } u_i = S^n(t_i, \widehat{\sigma}_t[w_1], ..., \widehat{\sigma}_t[w_n]) \\
&\quad\ \text{for all } i \in \{1, ..., n\}.
\end{aligned}$$

Since $t_{i_l} = x_{j_l}$ for all $l \in \{1, ..., m\}$, thus $u_{i_l} = S^n(t_{i_l}, \widehat{\sigma}_t[w_1], ..., \widehat{\sigma}_t[w_n]) = \hat{\sigma}_t[w_{j_l}]$. Since $w_{j_l} = S^n(s_{j_l}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ and $s_{j_l} \neq x_{i_l}$, $w_{j_l} \neq \widehat{\sigma}_s[t_{i_l}] = x_{j_l}$, we get $u_{i_l} = \widehat{\sigma}_t[w_{j_l}] \neq x_{j_l}$, and then $f(u_1, ..., u_n) \neq t$. This is a contradiction. Hence $s_{h_q} = s_{j_l} = x_{i_l}$ for all $l \in \{1, ..., m\}$.

Conversely, let $s = f(s_1, ..., s_n)$ where $s_{h_q} = s_{j_l} = x_{i_l}$ for all $q \in \{1, ..., p\}$ and for some $l \in \{1, ..., m\}$. Then $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_t[f(w_1, ..., w_n)]$ where $w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ for all $i \in \{1, ..., n\}$. Since $s_{h_q} = s_{j_l} = x_{i_l}$ for all $q \in \{1, ..., p\}$ and for some $l \in \{1, ..., m\}$, $w_{j_l} = S^n(s_{j_l}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_{i_l}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{i_l}] = x_{j_l}$, we get

$$\widehat{\sigma}_t[f(w_1, ..., w_n)] = S^n(f(t_1, ..., t_n), \widehat{\sigma}_t[w_1], ..., \widehat{\sigma}_t[w_n]) = f(t_1, ..., t_n) = t.$$

Hence $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. □

Example 3.3.4. Let $\tau = (5)$ and let $\sigma_t \in Hyp_G(5)$ such that $t = f(t', x_1, x_4, t', x_2)$ where $t' \in W_{(5)}(X)$ and $var(t') \cap X_5 = \{x_1, x_2, x_4\}$. Choose $\sigma_s \in Hyp_G(5)$ such that $s = f(x_2, x_5, s', x_3, s'')$ where $s', s'' \in W_{(5)}(X) \setminus X_5$. Then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_s[t]] \\
&= \widehat{\sigma}_t[S^5(f(x_2, x_5, s', x_3, s''), \widehat{\sigma}_s[t'], \widehat{\sigma}_s[x_1], \widehat{\sigma}_s[x_4], \widehat{\sigma}_s[t'], \widehat{\sigma}_s[x_2])] \\
&= \widehat{\sigma}_t[S^5(f(x_2, x_5, s', x_3, s''), \widehat{\sigma}_s[t'], x_1, x_4, \widehat{\sigma}_s[t'], x_2)] \\
&= \widehat{\sigma}_t[f(x_1, x_2, s', x_4, s')] \\
&= S^5(f(t', x_1, x_4, t', x_2), \widehat{\sigma}_s[x_1], \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[s'], \widehat{\sigma}_s[x_4], \widehat{\sigma}_s[s'']) \\
&= S^5(f(t', x_1, x_4, t', x_2), x_1, x_2, \widehat{\sigma}_s[s'], x_4, \widehat{\sigma}_s[s'']) \\
&= f(t', x_1, x_4, t', x_2) = \sigma_t(f).
\end{aligned}$$

We see that σ_t is a regular element of $Hyp_G(5)$. If $\{s', s''\} = \{x_1, x_5\}$ then $\sigma_s \in U(Hyp_G(5))$ and so $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$, i.e. σ_t is a unit-regular element of $Hyp_G(5)$.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$,

$R_2 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \emptyset\}$,

$R_3 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ such that $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}\}$.

Example 3.3.5. Let $\tau = (3)$ and let $t = f(f(x_4, x_4, x_4), x_5, x_6)$, $s = f(x_3, f(x_4, x_3, x_4), x_2)$ and $w = f(x_3, f(x_1, x_3, x_4), x_2)$. Then $\sigma_t \in R_2$, $\sigma_s \in R_3$ but $\sigma_w \notin \bigcup_{i=1}^{3} R_i$, so $\bigcup_{i=1}^{3} R_i \subsetneq Hyp_G(3)$. It is clear that σ_t is a regular element in $Hyp_G(3)$. By Lemma 3.3.3, we get σ_s is a regular element but σ_w is not a regular element in $Hyp_G(3)$.

By the definition of R_1 and R_2 it is easy to check that for every element in $R_1 \cup R_2$ is a regular element in $Hyp_G(n)$. In 2010, W. Puninagool and S. Leeratanavalee [21] characterized the regular generalized hypersubstitutions of type $\tau = (n)$.

Theorem 3.3.6 ([21])**.** *Let $t = f(t_1, t_2, ..., t_n) \in W_{(n)}(X)$ and $var(t) \cap X_n = \{x_{j_1}, x_{j_2}, ..., x_{j_m}\}$. Then σ_t is regular if and only if there exist $i_1, i_2, ..., i_m \in \{1, 2, ..., n\}$ such that $t_{i_1} = x_{j_1}, t_{i_2} = x_{j_2}, ..., t_{i_m} = x_{j_m}$.*

By Theorem 3.3.6, we have every element in R_3 is regular. Then $\bigcup\limits_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$.

For each $\sigma_t \in Hyp_G(n)$, we denote

$E := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}\}$. Clearly, $E \subset R_3$.

Example 3.3.7. Let $\tau = (3)$ and $\sigma_t \in Hyp_G(3)$ where $t = f(x_1, f(x_4, x_1, x_5), x_3)$. Then $\sigma_t \in E \subset R_3$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_1, f(x_4, x_1, x_5), x_3)] \\
&= S^3(t, \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[f(x_4, x_1, x_5)], \widehat{\sigma}_t[x_3]) \\
&= S^3(t, x_1, S^3(t, \widehat{\sigma}_t[x_4], \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[x_5]), x_3) \\
&= S^3(t, x_1, S^3(f(x_1, f(x_4, x_1, x_5), x_3), x_4, x_1, x_5), x_3) \\
&= S^3(t, x_1, f(x_4, f(x_4, x_4, x_5), x_5), x_3) \\
&= S^3(f(x_1, f(x_4, x_1, x_5), x_3), x_1, f(x_4, f(x_4, x_4, x_5), x_5), x_3) \\
&= f(x_1, f(x_4, x_1, x_5), x_3) \\
&= \sigma_t(f).
\end{aligned}$$

Hence $\sigma_t \in E(Hyp_G(3))$.

Let $s = f(x_3, f(x_4, x_1, x_5), x_1)$. Then $\sigma_s \in R_3 \setminus E$. Consider

$$\begin{aligned}
(\sigma_s \circ_G \sigma_s)(f) &= \widehat{\sigma}_s[f(x_3, f(x_4, x_1, x_5), x_1)] \\
&= S^3(s, \widehat{\sigma}_s[x_3], \widehat{\sigma}_s[f(x_4, x_1, x_5)], \widehat{\sigma}_s[x_1]) \\
&= S^3(s, x_3, S^3(s, \widehat{\sigma}_s[x_4], \widehat{\sigma}_s[x_1], \widehat{\sigma}_s[x_5]), x_1) \\
&= S^3(s, x_3, S^3(f(x_3, f(x_4, x_1, x_5), x_1), x_4, x_1, x_5), x_1) \\
&= S^3(s, x_3, f(x_5, f(x_4, x_4, x_5), x_4), x_1)
\end{aligned}$$

$$
\begin{aligned}
&= S^3(f(x_3, f(x_4, x_1, x_5), x_1), x_3, f(x_5, f(x_4, x_4, x_5), x_4), x_1)\\
&= f(x_1, f(x_4, x_3, x_5), x_3)\\
&\neq \sigma_s(f).
\end{aligned}
$$

Hence $\sigma_s \notin E(Hyp_G(3))$.

By the definition of R_1 and R_2 it is easy to check that for all elements in $R_1 \cup R_2$ are idempotent elements in $Hyp_G(n)$. In 2010, W. Puninagool and S. Leeratanavalee [21] characterized the idempotent generalized hypersubstitutions of type $\tau = (n)$.

Theorem 3.3.8 ([21]). *Let $t = f(t_1, t_2, ..., t_n) \in W_{(n)}(X)$ and $var(t) \cap X_n = \{x_{i_1}, x_{i_2}, ..., x_{i_m}\}$. Then σ_t is idempotent if and only if $t_{i_k} = x_{i_k}$ for all $k \in \{1, 2, ..., m\}$.*

By Theorem 3.3.8, we have that for every element in E is idempotent. It is clear that $E(Hyp_G(n)) = R_1 \cup R_2 \cup E$. By Example 3.3.7, $E(Hyp_G(n)) \subsetneq \bigcup_{i=1}^{3} R_i$.

Remark 3.3.9. $E(Hyp_G(n))$ is not subsemigroup of $Hyp_G(n)$.

Example 3.3.10. Let $\sigma_t, \sigma_s \in E(Hyp_G(3))$ where $t = f(x_5, x_2, x_4)$ and $s = f(x_1, f(x_1, x_1, x_1), x_5)$.
Consider

$$
\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, f(x_1, x_1, x_1), x_5)]\\
&= S^3(t, \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[f(x_1, x_1, x_1)], \widehat{\sigma}_t[x_5])\\
&= S^3(t, x_1, S^3(t, \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[x_1]), x_5)\\
&= S^3(t, x_1, S^3(f(x_5, x_2, x_4), x_1, x_1, x_1), x_5)\\
&= S^3(t, x_1, f(x_5, x_1, x_4), x_5)\\
&= S^3(f(x_5, x_2, x_4), x_1, f(x_5, x_1, x_4), x_5)\\
&= f(x_5, f(x_5, x_1, x_4), x_4).
\end{aligned}
$$

Then $\sigma_t \circ_G \sigma_s \notin E(Hyp_G(3))$. So $E(Hyp_G(3))$ is not closed under $\circ_G$, i.e. $E(Hyp_G(3))$ is not a subsemigroup of $Hyp_G(3)$.

By the definition of a regular element and a unit-regular element, we get the set of all unit-regular elements is a subset of the set of all regular elements. From now on, we show that the set of all unit-regular elements and the set of all regular elements in $Hyp_G(n)$ are the same.

Theorem 3.3.11. $\bigcup_{i=1}^{3} R_i$ *is a set of all unit-regular elements in* $Hyp_G(n)$.

Proof. Let $\sigma_t \in \bigcup_{i=1}^{3} R_i$. If $\sigma_t \in R_1 \cup R_2$, then $\sigma_t \in E(Hyp_G(n))$. So $\sigma_t \circ_G \sigma_{id} \circ_G \sigma_t = \sigma_t \circ_G \sigma_t = \sigma_t$. If $\sigma_t \in R_3$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. Choose $\sigma_u \in U(Hyp_G(n))$ where $u = f(u_1, ..., u_n) = f(x_{\pi(1)}, ..., x_{\pi(n)})$ for some $\pi \in S_n$ such that $\pi(j_1) = i_1, ..., \pi(j_m) = i_m$. Then $u_{j_l} = x_{\pi(j_l)} = x_{i_l}$ for all $l \in \{1, ..., m\}$. By Lemma 3.3.3, $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$. Hence σ_t is a unit-regular element in $Hyp_G(n)$. Since $\bigcup_{i=1}^{3} R_i$ is a set of all regular elements and all its elements are unit-regular, so $\bigcup_{i=1}^{3} R_i$ is a set of all unit-regular elements in $Hyp_G(n)$. □

Therefore, for every element in $Hyp_G(n)$ is a regular element if and only if it is a unit-regular element.

We have $\bigcup_{i=1}^{3} R_i$ is a proper subset of $Hyp_G(n)$, i.e. $Hyp_G(n)$ is not a regular semigroup. Next, we will prove that $\bigcup_{i=1}^{3} R_i$ is not closed under $\circ_G$. Firstly, we construct some tools used for this proof. We define:

Definition 3.3.12. Let $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, ..., t_n)$ for some $t_1, ..., t_n \in W_{(n)}(X)$. For each $s \in sub(t)$, $s \neq t$, sequences of s in t, denoted by $seq^t(s)$, is defined by

$$seq^t(s) = \{(i_1, ..., i_m) | m \in \mathbb{N} \text{ and } s = \pi_{i_m} \circ ... \circ \pi_{i_1}(t)\}$$

where $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ with $\pi_{i_l}(f(t_1, ..., t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, ..., n$.

Example 3.3.13. Let $t \in W_{(4)}(X)$ where $t = f(t_1, t_2, t_3, t_4)$ such that $t_1 = f(x_3, x_1, s, x_4)$, $t_2 = x_4$, $t_3 = f((x_7, s, x_1, x_4), x_4, f(x_8, f(x_3, x_1, s, x_4), x_2, f(x_3, x_1, s, x_4)), s)$ and $t_4 = s$ for some $s \in W_{(4)}(X)$. Then

$$seq^t(s) = \{(1,3), (3,1,2), (3,3,2,3), (3,3,4,3), (3,4), (4)\},$$
$$seq^{t_3}(s) = \{(1,2), (3,2,3), (3,4,3), (4)\},$$
$$seq^t(t_1) = \{(1), (3,3,2), (3,3,4)\},$$
$$seq^t(x_4) = \{(1,4), (2), (3,1,3)\}.$$

Lemma 3.3.14. *Let* $t, s \in W_{(n)}(X) \setminus X$, $x \in var(t)$ *and* $var(s) \cap X_n = \{x_{z_1}, ..., x_{z_k}\}$. *If* $(i_1, ..., i_m) \in seq^t(x)$ *where* $i_1, ..., i_m \in \{z_1, ..., z_k\}$ *then* $x \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ *and* *there is* $(a_{i_1},...,a_{i_m}) \in seq^{\widehat{\sigma}_s[t]}(x)$ *where* a_{i_j} *is a sequence of natural numbers* $j_1,...,j_h$ *such that* $(j_1,...,j_h) \in seq^s(x_{i_j})$ *for all* $j \in \{1,...,m\}$.

Proof. Let $t = f(t_1,...,t_n)$ for some $t_1,...,t_n \in W_{(n)}(X)$ and $(i_1,...,i_m) \in seq^t(x)$ where $i_1,...,i_m \in \{z_1,...,z_k\}$. Let us proceed by mathematical induction on m. If $(i_1) \in seq^t(x)$ where $i_1 \in \{z_1,...,z_k\}$, then $x = \pi_{i_1}(t) = t_{i_1}$ where $t_{i_1} \in \{t_1,...,t_n\}$. Hence $\widehat{\sigma}_s[t_{i_1}] = \widehat{\sigma}_s[x] = x$. Consider

$$\sigma_s \circ_G \sigma_t(f) = \widehat{\sigma}_s[t] = S^n(s, \widehat{\sigma}_s[t_1],...,\widehat{\sigma}_s[t_n])$$

Since $x_{i_1} \in var(s) \cap X_n$, $x = \widehat{\sigma}_s[t_{i_1}] \in var(\widehat{\sigma}_s[t])$ and there is $(a_{i_1}) \in seq^{\widehat{\sigma}_s[t]}(x)$ where a_{i_1} is a sequence of natural numbers $j_1,...,j_h$ such that $(j_1,...,j_h) \in seq^s(x_{i_1})$. Let m be a natural number and assume that, for each $u \in W_{(n)}(X) \setminus X$, $x \in var(u)$ and $(l_1,...,l_p) \in seq^u(x)$ where $l_1,...,l_p \in \{z_1,...,z_k\}$, then $x \in var(\widehat{\sigma}_s[u]) = var(\sigma_s \circ_G \sigma_u)$ and there is $(a_{l_1},...,a_{l_p}) \in seq^{\widehat{\sigma}_s[u]}(x)$ where a_{l_q} is a sequence of natural numbers $r_1,...,r_{h^*}$ such that $(r_1,...,r_{h^*}) \in seq^s(x_{l_q})$ for all $q \in \{1,...,p\}$ is true for all natural numbers $p < m$. If $(i_1,...,i_m) \in seq^t(x)$ where $i_1,...,i_m \in \{z_1,...,z_k\}$, then $x = \pi_{i_m} \circ ... \circ \pi_{i_1}(t) = \pi_{i_m} \circ ... \circ \pi_{i_2}(t_{i_1})$, i.e. $x \in var(t_{i_1})$ and $(i_2,...,i_m) \in seq^{t_{i_1}}(x)$. By our assumption, we get $x \in var(\widehat{\sigma}_s[t_{i_1}])$ and there is $(a_{i_2},...,a_{i_m}) \in seq^{\widehat{\sigma}_s[t_{i_1}]}(x)$ where a_{i_j} is a sequence of natural numbers $j_1,...,j_h$ such that $(j_1,...,j_h) \in seq^s(x_{i_j})$ for all $j \in \{2,...,m\}$. Since $x_{i_1} \in var(s) \cap X_n$, $\widehat{\sigma}_s[t_{i_1}] \in sub(S^n(s, \widehat{\sigma}_s[t_1],...,\widehat{\sigma}_s[t_n])) = sub(\widehat{\sigma}_s[t])$ and $seq^{\widehat{\sigma}_s[t]}(\widehat{\sigma}_s[t_{i_1}]) = seq^s(x_{i_1})$. Hence $x \in var(\widehat{\sigma}_s[t])$ and there is $(a_{i_1}, a_{i_2},...,a_{i_m}) \in seq^{\widehat{\sigma}_s[t]}(x)$ where a_{i_j} is a sequence of natural numbers $j_1,...,j_h$ such that $(j_1,...,j_h) \in seq^s(x_{i_j})$ for all $j \in \{1,2,...,m\}$. □

Theorem 3.3.15. *Let* $t = f(t_1,...,t_n)$ *where* $t_{i_1} = x_{j_1},...,t_{i_m} = x_{j_m}$ *for some* $i_1,...,i_m$, $j_1,...,j_m \in \{1,...,n\}$ *and* $var(t) \cap X_n = \{x_{j_1},...,x_{j_m}\}$. *If* $x_{j_l} \in var(t_k)$ *for some* $l \in \{1,...,m\}$ *and* $k \in \{1,...,n\} \setminus \{i_1,...,i_m\}$ *where* $(k_1,...,k_p) \in seq^{t_k}(x_{j_l})$ *for some* $k_1,...,k_p \in \{1,...,n\} \setminus \{i_l\}$ *then there exists* $\sigma_s \in Hyp_G(n)$ *such that* $\sigma_s \circ_G \sigma_t$ *is not a unit-regular element in* $Hyp_G(n)$.

Proof. Assume that the condition holds. Since $(k_1,...,k_p) \in seq^{t_k}(x_{j_l})$, we get $(k, k_1,..., k_p) \in seq^t(x_{j_l})$. Let $h_1,...,h_q \in \{k, k_1,...,k_p\}$ and $h_l \neq h_r$ if $l \neq r$. Then $q \leq n$. Choose $\sigma_s \in Hyp_G(n)$ where $s = f(s_1,...,s_n)$ such that $s_1 = x_{h_1},...,s_q = x_{h_q}$ and $s_{q+1},...,s_n \in W_{(n)}(X)$ and $var(s_r) \cap X_n = \emptyset$ for all $r \in \{q+1,...,n\}$. Then $s_i \neq x_{i_l}$ for all $i \in \{1,...,n\}$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1,...,t_n)] = S^n(f(s_1,..,s_n), \widehat{\sigma}_s[t_1],...,\widehat{\sigma}_s[t_n]) = f(u_1,...,u_n)$$

where $u_i = S^n(s_i, \widehat{\sigma_s}[t_1], ..., \widehat{\sigma_s}[t_n])$ for all $i \in \{1, ..., n\}$. Since $s_i \neq x_{i_l}$, $u_i \neq x_{j_l}$ for all $i \in \{1, ..., n\}$. By Lemma 3.3.14, we get $x_{j_l} \in var(\sigma_s \circ_G \sigma_t)$ such that $x_{j_l} \in var(u_j)$ where $u_j \in W_{(n)}(X) \setminus X$ for some $j \in \{1, ..., n\}$. Hence $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{3} R_i$, so $\sigma_s \circ_G \sigma_t$ is not a unit-regular element in $Hyp_G(n)$. □

Example 3.3.16. Let $\tau = (3)$ and $\sigma_t \in \bigcup_{i=1}^{3} R_i$ where $t = f(x_2, f(f(x_4, x_4, x_5), x_2, x_5), f(x_5, x_2, x_5))$. Choose $\sigma_s \in R_3$ where $s = f(x_2, x_3, x_4)$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, f(f(x_4, x_4, x_5), x_2, x_5), f(x_5, x_2, x_5))] \\ &= S^3(s, x_2, f(x_2, x_5, x_4), f(x_2, x_5, x_4)) \\ &= f(f(x_2, x_5, x_4)), f(x_2, x_5, x_4), x_4).\end{aligned}$$

We see that $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{3} R_i$. So $\sigma_s \circ_G \sigma_t$ is not a unit-regular element in $Hyp_G(3)$. Hence $\bigcup_{i=1}^{3} R_i$ is not closed under $\circ_G$.

Therefore $\bigcup_{i=1}^{3} R_i$ is not unit-regular submonoid and it is not regular submonoid of $Hyp_G(n)$.

3.4 All Completely Regular Elements in $Hyp_G(n)$

In semigroup theory, the principle special study of a regular element are inverse of an element and a completely regular element with a great diversity of their various generalization.

In the monoid of all generalized hypersubstitutions, a regular element was studied by W. Puninagool and S. Leeratanavalee in 2010 [21]. The main tool used to study a regular element of the monoid of all generalized hypersubstitutions is the concept of a regular element of the monoid of all hypersubstitutions. The concept of a regular element of the monoid of all hypersubstitutions originated by Th. Changphas and K. Denecke [7].

In this section, we used the concepts of regular element as tools to determine the set of all completely regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ and we have that a completely regular element is both left regular and right regular element of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

Denote R_1, R_2, R_3 and E as in Section 3.3. Then $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$. By the definition of completely regular we get the set of all completely regular elements is a subset of $\bigcup_{i=1}^{3} R_i$.

In 2010, W. Puninagool and S. Leeratanavalee showed that $E(Hyp_G(n)) = R_1 \cup R_2 \cup E$ is the set of all idempotent elements in $Hyp_G(n)$ such that $E(Hyp_G(n)) \subset \bigcup_{i=1}^{3} R_i$ [21].

Theorem 3.4.1. *For each $\sigma_t \in E(Hyp_G(n))$, σ_t is a completely regular element in $Hyp_G(n)$.*

Proof. The proof is obvious. □

Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$ and let $\sigma_t \in Hyp_G(n)$. By Section 3.1, we have

$$U(Hyp_G(n)) := \{\sigma_t \in Hyp_G(n) | t = f(x_{\pi(1)}, ..., x_{\pi(n)}) \text{ where } \pi \in S_n\}$$

is the set of all unit elements in $Hyp_G(n)$. We see that $U(Hyp_G(n)) \subset R_3 \subset \bigcup_{i=1}^{3} R_i$.

Theorem 3.4.2. *For each $\sigma_t \in U(Hyp_G(n))$, σ_t is a completely regular element in $Hyp_G(n)$.*

Proof. Let $\sigma_t \in U(Hyp_G(n))$. Then there exists $\sigma_{t^{-1}} \in U(Hyp_G(n)) \subseteq Hyp_G(n)$ such that $\sigma_t \circ_G \sigma_{t^{-1}} = \sigma_{id} = \sigma_{t^{-1}} \circ_G \sigma_t$ and $\sigma_t \circ_G \sigma_{t^{-1}} \circ_G \sigma_t = \sigma_t$. □

Let $\sigma_t \in Hyp_G(n)$, we denote

$CR(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ and $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ where π is a bijective map on $\{i_1, ..., i_m\}$ for some $i_1, \ldots, i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, ..., x_{\pi(i_m)}\}\}$. Then we have $(E \cup U(Hyp_G(n))) \subseteq CR(R_3) \subset R_3$.

Example 3.4.3. Let $\tau = (5)$ and $t = f(t_1, t_2, t_3, t_4, t_5)$ where $t_1 = x_3, t_2 = f(x_6, x_6, x_3, x_6, x_6)$, $t_3 = x_4$, $t_4 = x_1$ and $t_5 = x_3$. Let π be a bijective map on $\{1, 3, 4\}$ where $\pi(1) = 3, \pi(3) = 4$ and $\pi(4) = 1$. Then $t_1 = x_{\pi(1)}$, $t_3 = x_{\pi(3)}$ and $t_4 = x_{\pi(4)}$. So $\sigma_t \in CR(R_3)$.

Theorem 3.4.4. *For each $\sigma_t \in CR(R_3)$, σ_t is a completely regular element in $Hyp_G(n)$.*

Proof. Let $\sigma_t \in CR(R_3)$. Then $t = f(t_1, ..., t_n)$ and $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ where π is a bijective map on $\{i_1, ..., i_m\}$ for some $i_1, \ldots, i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, ..., x_{\pi(i_m)}\}$. Let $s \in W_{(n)}(X)$ where $s = f(s_1, ..., s_n)$ such that $s_{\pi(i_1)} = x_{i_1}, ..., s_{\pi(i_m)} = x_{i_m}$. Let $t_k \in sub(t_j)$ and $s_k \in sub(s_j)$ for all $j \in \{1, .., n\} \setminus \{i_1, ..., i_m\}$ and

$k \in \{1, .., n\}$. If $var(t_k) \cap X_n = \emptyset$ then we choose $s_k = t_k$. And, if $t_k = x_{\pi(i_l)}$ and $\pi(i_p) = i_l$ for some $i_p, i_l \in \{i_1, .., i_m\}$ we choose $s_k = x_{i_p}$. By Lemma 3.3.3, we have $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. Next, we will show that $\sigma_t \circ_G \sigma_s = \sigma_s \circ_G \sigma_t$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = S^n(f(t_1, ..., t_n), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = f(w_1, ..., w_n)$$

where $w_i = S^n(t_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n])$ for all $i \in \{1, .., n\}$. And consider

$$(\sigma_s \circ_G \sigma_t)(f) = S^n(f(s_1, ..., s_n), \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(s_i, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n])$ for all $i \in \{1, .., n\}$.

Case 1: $i_l \in \{i_1, ..., i_m\}$.

Since π is a bijective map on $\{i_1, ..., i_m\}$, there exists $i_p \in \{i_1, .., i_m\}$ such that $\pi(i_p) = i_l$. Then

$$u_{i_l} = S^n(s_{i_l}, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n]) = S^n(x_{i_p}, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n]) = \hat{\sigma_s}[t_{i_p}] = x_{\pi(i_p)} = x_{i_l}$$

and

$$w_{i_l} = S^n(t_{i_l}, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = S^n(x_{\pi(i_l)}, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = \hat{\sigma_t}[s_{\pi(i_l)}] = x_{i_l}.$$

So $u_{i_l} = w_{i_l}$ for all $l \in \{1, ..., m\}$.

Case 2: $j \in \{1, .., n\} \setminus \{i_1, ..., i_m\}$.

Let $t_k \in sub(t_j)$ and $s_k \in sub(s_j)$ for all $k \in \{1, ..., n\}$. Then $w_j = S^n(t_j, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n])$ and $u_j = S^n(s_j, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n])$. We put $w'_k = S^n(t_k, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n])$ and $u'_k = S^n(s_k, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n])$ for all $k \in \{1, ..., n\}$. If $var(t_k) \cap X_n = \emptyset$, then $w'_k = t_k$ and $u'_k = s_k = t_k$. If $t_k = x_{\pi(i_l)}$ and $\pi(i_p) = i_l$, then

$$w'_k = S^n(t_k, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = S^n(x_{\pi(i_l)}, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) = \hat{\sigma_t}[s_{\pi(i_l)}] = x_{i_l}$$

and

$$u'_k = S^n(s_k, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n]) = S^n(x_{i_p}, \hat{\sigma_s}[t_1], ..., \hat{\sigma_s}[t_n]) = \hat{\sigma_s}[t_{i_p}] = x_{\pi(i_p)} = x_{i_l}.$$

So $w_j = u_j$ for all $j \in \{1, .., n\} \setminus \{i_1, ..., i_m\}$.

Hence $f(w_1, ..., w_n) = f(u_1, ..., u_n)$, so $\sigma_t \circ_G \sigma_s = \sigma_s \circ_G \sigma_t$. Therefore σ_t is a completely regular element in $Hyp_G(n)$. □

Lemma 3.4.5. *Let $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$, $j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. If there exists $l \in \{1, ..., m\}$ such that $t_{i_l} = x_{j_l}$ where $i_l \notin \{j_1, ..., j_m\}$, then $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$.*

Proof. Assume that the condition holds. Consider

$$(\sigma_t \circ_G \sigma_t)(f) = \hat{\sigma_t}[t] = S^n(f(t_1, ..., t_n), \hat{\sigma_t}[t_1], ..., \hat{\sigma_t}[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(t_i, \hat{\sigma_t}[t_1], ..., \hat{\sigma_t}[t_n])$ for all $i \in \{1, ..., n\}$. We have $u_i = S^n(t_i, \hat{\sigma_t}[t_1], ..., \hat{\sigma_t}[t_n]) \in \{x_{j_1}, ..., x_{j_m}\}$ if and only if $t_i = x_{i_k}$ for some $k \in \{1, ..., m\}$. Since $i_l \notin \{j_1, ..., j_m\}$, $t_i \neq x_{i_l}$ for all $i \in \{1, ..., n\}$. So $u_i \neq x_{j_l}$. Hence $\sigma_t^2(f) = f(u_1, ..., u_n)$ where $u_i \neq x_{j_l}$ for all $i \in \{1, ..., n\}$. Let $\sigma_s \in Hyp_G(n)$. Next, we will show that $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$. If $s = x_i$ where $x_i \in X$, then $(\sigma_s \circ_G \sigma_t^2)(f) = x_j \neq \sigma_t(f)$ for some $x_j \in X$. If $s = f(s_1, ..., s_n)$ where $s_1, ..., s_n \in W_{(n)}(X)$, then

$$\begin{aligned}(\sigma_s \circ_G \sigma_t^2)(f) &= \hat{\sigma_s}[f(u_1, ..., u_n)]\\ &= S^n(f(s_1, ..., s_n), \hat{\sigma_s}[u_1], ..., \hat{\sigma_s}[u_n])\\ &= f(w_1, ..., w_n)\end{aligned}$$

where $w_i = S^n(s_i, \hat{\sigma_s}[u_1], ..., \hat{\sigma_s}[u_n])$ for all $i \in \{1, ..., n\}$. Since $u_i \neq x_{j_l}$ for all $i \in \{1, ..., n\}$, $\hat{\sigma_s}[u_i] \neq x_{j_l}$. So $w_i \neq x_{j_l}$ for all $i \in \{1, ..., n\}$. Hence $f(w_1, ..., w_n) \neq f(t_1, ..., t_n)$, so $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$. □

Theorem 3.4.6. *Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$. Then $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$.*

Proof. By Theorem 3.4.1 and Theorem 3.4.4, every element in $CR(Hyp_G(n))$ is completely regular. Let σ_t be a regular element where $\sigma_t \notin CR(Hyp_G(n))$. Then $\sigma_t \in R_3 \setminus CR(R_3)$. By Lemma 3.4.5, $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$. Then $\sigma_t \neq (\sigma_t^2 \circ_G \sigma_u) \circ_G \sigma_t^2$ where $\sigma_t^2 \circ_G \sigma_u \in Hyp_G(n)$. By Theorem 2.1.3, σ_t is not a completely regular element in $Hyp_G(n)$. Therefore $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$. □

Corollary 3.4.7. *Let $\sigma_t \in CR(Hyp_G(n))$. Then σ_t is both left regular and right regular element in $Hyp_G(n)$, and σ_t is an intra-regular element in $Hyp_G(n)$.*

Corollary 3.4.8. *If $\sigma_t \in R_3 \setminus CR(R_3)$, then σ_t is not a left regular element in $Hyp_G(n)$.*

Example 3.4.9. Let $\tau = (3)$ and let $\sigma_t \in Hyp_G(3)$ where $t = f(x_3, f(x_4, x_4, x_4), x_5)$ then $\sigma_t \in R_3 \setminus CR(Hyp_G(3))$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_3, f(x_4, x_4, x_4), x_5)]\\ &= S^3(t, \widehat{\sigma}_t[x_3], \widehat{\sigma}_t[f(x_3, f(x_4, x_4, x_4), x_5)], \widehat{\sigma}_t[x_5])\\ &= S^3(t, x_3, S^3(t, \widehat{\sigma}_s[x_4], \widehat{\sigma}_t[x_4], \widehat{\sigma}_t[x_4]), x_5)\\ &= S^3(t, x_3, S^3(f(x_3, f(x_4, x_4, x_4), x_5), x_4, x_4, x_4), x_5)\end{aligned}$$

$$
\begin{aligned}
&= S^3(t, x_3, f(x_4, f(x_4, x_4, x_4), x_5), x_5) \\
&= S^3(f(x_3, f(x_4, x_4, x_4), x_5), x_3, f(x_4, f(x_4, x_4, x_4), x_5), x_5) \\
&= f(x_5, f(x_4, x_4, x_4), x_5).
\end{aligned}
$$

Let $\sigma_s \in Hyp_G(3)$, if $s \in X$ then $\sigma_t^2 \circ_G \sigma_s \in X$ and $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_s \in X$ for all $\sigma_u \in Hyp_G(3)$. If $s \in W_{(3)}(X) \setminus X$ then $\sigma_t^2 \circ_G \sigma_s = \sigma_t^2 \neq \sigma_t$ and $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_s = \sigma_u \circ_G \sigma_t^2 \neq \sigma_t$ for all $\sigma_u \in Hyp_G(3)$. So σ_t is not a right regular element and it is not an intra-regular element in $Hyp_G(3)$.

By Corollary 3.4.8 and Example 3.4.9, there exist regular elements in $Hyp_G(\tau)$ such that it is not left regular, right regular and intra-regular elements in $Hyp_G(\tau)$.

Example 3.4.10. Let $\tau = (3)$ and let $\sigma_t, \sigma_s \in Hyp_G(3)$ where $t = f(x_3, x_5, x_1)$, $s = f(x_4, x_3, x_2)$ then $\sigma_t, \sigma_s \in CR(Hyp_G(3))$. Consider

$$
\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_4, x_3, x_2)] \\
&= S^3(t, \widehat{\sigma}_t[x_4], \widehat{\sigma}_t[x_3], \widehat{\sigma}_t[x_2]) \\
&= S^3(f(x_3, x_5, x_1), x_4, x_3, x_2) \\
&= f(x_2, x_5, x_4).
\end{aligned}
$$

We see that $\sigma_t \circ_G \sigma_s \notin CR(Hyp_G(3))$. So $CR(Hyp_G(3))$ is not closed under $\circ_G$.

Therefore $CR(Hyp_G(\tau))$ is not a submonoid of $Hyp_G(\tau)$.

3.5 All Intra-regular Elements in $Hyp_G(2)$

By Theorem 2.1.4, we conclude that a completely regular element is an intra-regular element. In general, an intra-regular element need not be a completely regular element. In this section, we use the concept in Section 3.4 to show that an intra-regular element of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ is a completely regular element. Moreover, we have a relationship of completely regular, left regular, right regular and intra-regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

3.5.1 Sequence of Terms

At first, we construct some tools used to characterize all intra-regular elements in $Hyp_G(2)$. These tools are called the *sequence* of a term and the *depth* of a term, respectively.

Let $t \in W_{(n)}(X) \setminus X$, and $t_i \in sub(t)$. It can be possible that t_i occurs in the term t more than once, we denote

$t_i^{(j)}$:= subterm t_i occurring in the j^{th} order of t (from the left).

Definition 3.5.1. Let $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, ..., t_n)$ for some $t_1, ..., t_n \in W_{(n)}(X)$ and let $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ with $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, ..., t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, ..., n$. For each $s^{(j)} \in sub(t)$ for some $j \in \mathbb{N}$, we denote the *sequence* of $s^{(j)}$ in t by $seq^t(s^{(j)})$ and denote the *depth* of $s^{(j)}$ in t by $depth^t(s^{(j)})$. If $s^{(j)} = \pi_{i_m} \circ ... \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then

$$seq^t(s^{(j)}) = (i_1, ..., i_m) \quad \text{and} \quad depth^t(s^{(j)}) = m.$$

Example 3.5.2. Let $\tau = (3)$ and let $t \in W_{(3)}(X) \setminus X$ where $t = f(t_1, t_2, t_3)$ such that $t_1 = x_5$, $t_2 = f(x_3, f(x_4, f(x_2, x_7, x_{10}), x_5), x_5)$ and $t_3 = f(f(x_5, x_4, f(x_2, x_7, x_{10})), x_1, x_6)$. Then

$$\begin{aligned}
seq^t(x_5^{(1)}) &= (1) \quad \text{and} \quad depth^t(x_5^{(1)}) = 1;\\
seq^t(x_5^{(2)}) &= (2, 2, 3) \quad \text{and} \quad depth^t(x_5^{(2)}) = 3;\\
seq^t(x_5^{(3)}) &= (2, 3) \quad \text{and} \quad depth^t(x_5^{(3)}) = 2;\\
seq^t(x_5^{(4)}) &= (3, 1, 1) \quad \text{and} \quad depth^t(x_5^{(4)}) = 3;\\
seq^t(f(x_2, x_7, x_{10})^{(1)}) &= (2, 2, 2) \quad \text{and} \quad depth^t(f(x_2, x_7, x_{10})^{(1)}) = 3;\\
seq^t(f(x_2, x_7, x_{10})^{(2)}) &= (3, 1, 3) \quad \text{and} \quad depth^t(f(x_2, x_7, x_{10})^{(2)}) = 3;\\
seq^{t_3}(f(x_2, x_7, x_{10})^{(1)}) &= (1, 3) \quad \text{and} \quad depth^{t_3}(f(x_2, x_7, x_{10})^{(1)}) = 2;\\
seq^t(x_{10}^{(1)}) &= (2, 2, 2, 3) \quad \text{and} \quad depth^t(x_{10}^{(1)}) = (4);\\
seq^t(x_{10}^{(2)}) &= (3, 1, 3, 3) \quad \text{and} \quad depth^t(x_{10}^{(2)}) = 4;\\
seq^{t_3}(x_{10}^{(1)}) &= (1, 3, 3) \quad \text{and} \quad depth^{t_3}(x_{10}^{(1)}) = 3.
\end{aligned}$$

Let $t, s_1, s_2, ..., s_k \in W_{(n)}(X) \setminus X$ and $x_i \in var(t)$. We donote

$x_i^{(j)}$:= the variable x_i occurring in the j^{th} order of t (from the left);

$x_i^{(j,j_1)}$:= the variable $x_i^{(j)}$ occurring in the j_1^{th} order of $\widehat{\sigma}_{s_1}[t]$ (from the left);

$x_i^{(j,j_1,j_2)}$:= the variable $x_i^{(j,j_1)}$ occurring in the j_2^{th} order of $\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]$ (from the left).

Similarly,

$x_i^{(j,j_1,j_2,...,j_k)}$:= the variable $x_i^{(j,j_1,...,j_{k-1})}$ occurring in the j_k^{th} order of $\widehat{\sigma}_{s_k}[\widehat{\sigma}_{s_{k-1}}[...[\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]...]$ (from the left).

Theorem 3.5.3. *Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i^{(j)} \in var(t)$ for some $i, j \in \mathbb{N}$ and let $seq^t(x_i^{(j)}) = (i_1, ..., i_m)$. Then $x_{i_1}, ..., x_{i_m} \in var(s) \cap X_n$ if and only if $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) =$*

$var(\sigma_s \circ_G \sigma_t)$ for some $j_1 \in \mathbb{N}$ and $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, ..., a_{i_m})$ where a_{i_l} is a sequence of natural number $p_1, ..., p_q$ such that $(p_1, ..., p_q) = seq^s(x_{i_l}^{h_l})$ for some $h_l \in \mathbb{N}$ and for all $l \in \{1, ..., m\}$.

Proof. ($\Rightarrow$). The proof similar to Lemma 3.3.14.
($\Leftarrow$). Assume that $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ for some $j_1 \in \mathbb{N}$ and $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, ..., a_{i_m})$ where a_{i_l} is a sequence of natural number $p_1, ..., p_q$ such that $(p_1, ..., p_q) = seq^s(x_{i_l}^{h_l})$ for some $h_l \in \mathbb{N}$ and for all $l \in \{1, ..., m\}$. Then

$$vb^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = vb^s(x_{i_1}) \times vb^s(x_{i_2}) \times ... \times vb^s(x_{i_m}).$$

Suppose that $x_{i_k} \notin var(s) \cap X_n$ for some $1 \leq k \leq m$, so $vb^s(x_{i_z}) = 0$, i.e. $vb^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = 0$, which contradicts to our assumption. Hence $x_{i_1}, ..., x_{i_m} \in var(s) \cap X_n$. □

Example 3.5.4. Let $\tau = (3)$ and let $t = f(x_2, f(x_4, x_5, x_2), f(x_2, x_6, x_7))$ and $s = f(x_3, x_1, x_3)$. Then $seq^t(x_2^{(1)}) = (1)$, $seq^t(x_2^{(2)}) = (2, 3)$, $seq^t(x_2^{(3)}) = (3, 1)$ and $seq^t(x_7^{(1)}) = (3, 3)$. By Theorem 3.5.3, there exist $x_2^{(1,h)}, x_2^{(3,k_1)}, x_2^{(3,k_2)}, x_7^{(1,l_1)}, x_7^{(1,l_2)}, x_7^{(1,l_3)}, x_7^{(1,l_4)} \in var(\widehat{\sigma}_s[t])$ for some $h, k_1, k_2, l_1, l_{2,3}, l_4 \in \mathbb{N}$ and

$$seq^{\widehat{\sigma}_s[t]}(x_2^{(1,h)}) = (2) = seq^{\widehat{\sigma}_s[t]}(x_2^{(1,2)}) \text{ where } seq^s(x_1^{(1)}) = (2);$$
$$seq^{\widehat{\sigma}_s[t]}(x_2^{(3,k_1)}) = (1,2) = seq^{\widehat{\sigma}_s[t]}(x_2^{(3,1)}) \text{ where } seq^s(x_3^{(1)}) = (1) \text{ and } seq^s(x_1^{(1)}) = (2);$$
$$seq^{\widehat{\sigma}_s[t]}(x_2^{(3,k_2)}) = (3,2) = seq^{\widehat{\sigma}_s[t]}(x_2^{(3,3)}) \text{ where } seq^s(x_3^{(2)}) = (3) \text{ and } seq^s(x_1^{(1)}) = (2);$$
$$seq^{\widehat{\sigma}_s[t]}(x_7^{(1,l_1)}) = (1,1) = seq^{\widehat{\sigma}_s[t]}(x_7^{(1,1)}) \text{ where } seq^s(x_3^{(1)}) = (1) \text{ and } seq^s(x_3^{(1)}) = (1);$$
$$seq^{\widehat{\sigma}_s[t]}(x_7^{(1,l_2)}) = (1,3) = seq^{\widehat{\sigma}_s[t]}(x_7^{(1,2)}) \text{ where } seq^s(x_3^{(1)}) = (1) \text{ and } seq^s(x_3^{(2)}) = (3);$$
$$seq^{\widehat{\sigma}_s[t]}(x_7^{(1,l_3)}) = (3,1) = seq^{\widehat{\sigma}_s[t]}(x_7^{(1,3)}) \text{ where } seq^s(x_3^{(2)}) = (3) \text{ and } seq^s(x_3^{(1)}) = (1);$$
$$seq^{\widehat{\sigma}_s[t]}(x_7^{(1,l_4)}) = (3,3) = seq^{\widehat{\sigma}_s[t]}(x_7^{(1,4)}) \text{ where } seq^s(x_3^{(2)}) = (3) \text{ and } seq^s(x_3^{(2)}) = (3).$$

Since $x_2 \notin var(s)$, so $x_2^{(2,i)} \notin var(\widehat{\sigma}_s[t])$ for all $i \in \mathbb{N}$. Consider,

$$\begin{aligned}
\widehat{\sigma}_s[t] &= \widehat{\sigma}_s[f(x_2^{(1)}, f(x_4, x_5, x_2^{(2)}), f(x_2^{(3)}, x_6, x_7^{(1)}))] \\
&= S^3(f(x_3, x_1, x_3), \widehat{\sigma}_s[x_2^{(1)}], \widehat{\sigma}_s[f(x_4, x_5, x_2^{(2)})], \widehat{\sigma}_s[f(x_2^{(3)}, x_6, x_7^{(1)})]) \\
&= f(f(x_7^{(1,1)}, x_2^{(3,1)}, x_7^{(1,2)}), x_2^{(1,2)}, f(x_7^{(1,3)}, x_2^{(3,3)}, x_7^{(1,4)})) \\
&= f(f(x_7, x_2, x_7), x_2, f(x_7, x_2, x_7)).
\end{aligned}$$

Corollary 3.5.5. *Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i^{(j)} \in var(t)$ for some $i, j \in \mathbb{N}$ such that $seq^t(x_i^{(j)}) = i_1, i_2, ..., i_m$ for some $i_1, i_2, ..., i_m \in \{1, ..., n\}$ and $x_{i_k} \in var(s)$ for all $1 \leq k \leq m$. Then there exists $j_1 \in \mathbb{N}$ such that*

$$depth^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = depth^s(x_{i_1}^{(l_1)}) + depth^s(x_{i_2}^{(l_2)}) + ... + depth^s(x_{i_m}^{(l_m)})$$

for some $l_1, l_2, ..., l_m \in \mathbb{N}$, *and*

$$vb^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = vb^s(x_{i_1}) \times vb^s(x_{i_2}) \times ... \times vb^s(x_{i_m}).$$

Let $vb^t(x_i) = d$.

If $x_i \in X_n$, *then* $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

If $x_i \in X \setminus X_n$ *where* $x_i \notin var(s)$, *then* $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

Example 3.5.6. For each $\tau = (3)$. Let $t, s \in W_{(2)}(X) \setminus X$ where

$$t = f(f(x_3, x_5, x_4), x_5, f(x_2, x_5, x_4)) \text{ and } s = f(x_2, f(x_2, x_3, x_3), x_3)$$

Then

$$\begin{aligned}
seq^t(x_3^1) = (1,1) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_3^1) = vb^s(x_1) \times vb^s(x_1) = 0 \times 0 = 0;\\
seq^t(x_5^1) = (1,2) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_5^1) = vb^s(x_1) \times vb^s(x_2) = 0 \times 2 = 0;\\
seq^t(x_5^2) = (2) \quad &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_5^2) = vb^s(x_2) = 2;\\
seq^t(x_5^3) = (3,2) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_5^3) = vb^s(x_3) \times vb^s(x_2) = 3 \times 2 = 6;\\
seq^t(x_4^1) = (1,3) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_4^1) = vb^s(x_1) \times vb^s(x_3) = 0 \times 3 = 0;\\
seq^t(x_4^2) = (3,3) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_4^2) = vb^s(x_3) \times vb^s(x_3) = 3 \times 3 = 9;\\
seq^t(x_2^1) = (3,1) &\Longrightarrow vb^{\widehat{\sigma}_s[t]}(x_2^1) = vb^s(x_3) \times vb^s(x_1) = 3 \times 0 = 0.
\end{aligned}$$

Consider

$$\begin{aligned}
\sigma_s \circ_G \sigma_t &= \widehat{\sigma}_s[f(f(x_3, x_5, x_4), x_5, f(x_2, x_5, x_4))]\\
&= S^3(s, \widehat{\sigma}_s[f(x_3, x_5, x_4)], \widehat{\sigma}_s[x_5], \widehat{\sigma}_s[f(x_2, x_5, x_4)])\\
&= S^3(s, S^3(s, \widehat{\sigma}_s[x_3], \widehat{\sigma}_s[x_5], \widehat{\sigma}_s[x_4])], x_5, S^3(s, \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[x_5], \widehat{\sigma}_s[x_4]]))\\
&= S^3(s, f(x_5, f(x_5, x_4, x_4), x_4), x_5, f(x_5, f(x_5, x_4, x_4), x_4))\\
&= f(x_5, f(x_5, f(x_5, f(x_5, x_4, x_4), x_4), f(x_5, f(x_5, x_4, x_4), x_4)),\\
&\quad\ f(x_5, f(x_5, x_4, x_4), x_4)).
\end{aligned}$$

3.5.2 All Intra-regular Elements in $Hyp_G(2)$

In this section, we characterize the set of all intra-regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$. Finally, we show that the set of all completely regular elements and the set of all intra-regular elements in $Hyp_G(2)$ are the same.

We recall first the characterization of all completely regular elements in $Hyp_G(2)$.

Let $\tau = (2)$ be a type with a binary operation symbol f. By the definition of R_1, R_2 and R_3 in Section 3.3 and the definition of $CR(R_3)$ in Section 3.4, we get

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_{(2)}(X) \setminus X \text{ and } var(t) \cap X_2 = \emptyset\}$;

$R_3 := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $t = f(t_1, t_2)$ where $t_i = x_j$ for some $i, j \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_j\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$;

$CR(R_3) := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

Then we have $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(2)$ [21]. By Theorem 3.4.6 and by Corollary 3.4.7, we have $CR(Hyp_G(2)) := CR(R_3) \cup R_1 \cup R_2 = E(Hyp_G(2)) \cup \{\sigma_{f(x_2,x_1)}\}$ is the set of all completely regular elements in $Hyp_G(2)$ and every element in $CR(Hyp_G(2))$ is intra-regular. In Lemma 3.5.7 - Lemma 3.5.11, we determine some elements in $Hyp_G(2) \setminus CR(Hyp_G(2))$ which are not intra-regular.

Lemma 3.5.7. *If* $t = f(t_1, x_1)$ *where* $t_1 \in W_{(2)}(X) \setminus X_2$ *then* σ_t *is not intra-regular in* $Hyp_G(2)$.

Proof. Let $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X_2$. For each $u \in X$, we get $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ and $\sigma_v \circ_G \sigma_t^2 \circ_G \sigma_u \neq \sigma_t$ for all $v \in W_{(2)}(X)$. Let $u, v \in W_{(2)}(X) \setminus X$ where $u = f(u_1, u_2)$ and $v = f(v_1, v_2)$ for some $u_1, u_2, v_1, v_2 \in W_{(2)}(X)$, we will show that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in X \setminus X_2$ then $x_2 \notin var(t)$. By Theorem 3.5.3, $x_1 \notin var(\widehat{\sigma}_t[t]) = var(\sigma_t^2)$, i.e. $var(\sigma_t^2) \cap X_2 = \emptyset$. Hence $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in W_{(2)}(X) \setminus X$, then

$$\sigma_t^2(f) = \widehat{\sigma}_t[t] = S^2(f(t_1, x_1), \widehat{\sigma}_t[t_1], x_1) = f(w_1, w_2)$$

where $w_1 = S^2(t_1, \widehat{\sigma}_t[t_1], x_1)$ and $w_2 = S^2(x_1, \widehat{\sigma}_t[t_1], x_1) = \widehat{\sigma}_t[t_1]$ and denote $w = f(w_1, w_2)$. Since $t_1 \notin X$, so $w_1 \notin X$ and $w_2 = \widehat{\sigma}_t[t_1] \notin X$. Consider

$$\sigma_t^2 \circ_G \sigma_v(f) = \widehat{\sigma}_w[v] = S^2(f(w_1, w_2), \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2]) = f(s_1, s_2)$$

where $s_i = S^2(w_i, \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2])$ for all $i \in \{1, 2\}$. Since $w_i \notin X$ for all $i \in \{1, 2\}$, $s_i \notin X$ for all $i \in \{1, 2\}$. Then $\widehat{\sigma}_u[s_i] \notin X$ for all $i \in \{1, 2\}$. Consider

$$\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v(f) = S^2(f(u_1, u_2), \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2]) = f(r_1, r_2)$$

where $r_i = S^2(u_i, \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2])$ for all $i \in \{1, 2\}$. If $u_2 \in W_{(2)}(X) \setminus X$ or $u_2 \in X_2$ then $r_2 \notin X$. If $u_2 \in X \setminus X_2$ then $u_2 = r_2$. So $r_2 \neq x_1$. Therefore $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. Hence σ_t is not intra-regular in $Hyp_G(2)$. □

Lemma 3.5.8. *If* $t = f(x_2, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$ *then* σ_t *is not intra-regular in* $Hyp_G(2)$.

Proof. The proof is similar to the proof of Lemma 3.5.7. □

Lemma 3.5.9. *If* $t = f(x_1, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$ *and* $x_2 \in var(t)$ *then* σ_t *is not intra-regular in* $Hyp_G(2)$.

Proof. Assume that $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X_2$ and $x_2 \in var(t)$. Let $m = max\{depth^t(x_2^{(i)}) | x_2^{(i)} \in var(t)$ for some $i \in \mathbb{N}\}$ $(*)$, then there exists $h \in \mathbb{N}$ such that $seq^t(x_2^{(h)}) = (i_1, i_2, ..., i_m)$ where $i_1, i_2, ..., i_m \in \{1, 2\}$. It means $x_2^{(h)} = \pi_{i_m} \circ \pi_{i_{m-1}} \circ ... \circ \pi_{i_1}(t)$ where maps $\pi_{i_1}, ..., \pi_{i_{m-1}}, \pi_{i_m}$ are defined on $W_{(2)}(X) \setminus X_2$ to $W_{(2)}(X)$. Since $x_2^{(h)} \in var(t_2)$, $\pi_{i_1}(t) = t_2$, i.e. $i_1 = 2$. So $seq^t(x_2^{(h)}) = (2, i_2, ..., i_m)$. By Theorem 3.5.3, there is $x_2^{(h,h_1)} \in var(\widehat{\sigma}_t[t]) = var(\sigma_t^2)$ for some $h_1 \in \mathbb{N}$ such that

$$seq^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, ..., i_m, a_{i_2}, ..., a_{i_m})$$

where $(2, i_2, ..., i_m) = seq^t(x_2^{(h)})$ and a_{i_z} is a sequence of natural numbers such that $(a_{i_z}) = seq^s(x_{i_z}^{(h_{i_z})})$ for some $h_{i_z} \in \mathbb{N}$ and for all $2 \leq z \leq m$. [Note: $x_2^{(h)}$ is a variable x_2 occurring in the h^{th} order of t (from the left) and $x_2^{(h,h_1)}$ is a variable $x_2^{(h)}$ occurring in the h_1^{th} order of σ_t^2 (from the left)]. Instead of a sequence $a_{i_2}, ..., a_{i_m}$, we write a sequence of natural numbers $w_1, ..., w_d$ for some $d \in \mathbb{N}$ and $w_1, ..., w_d \in \{1, 2\}$. Then

$$seq^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, ..., i_m, w_1, ..., w_d).$$

Suppose that there exist $u, v \in W_{(2)}(X)$ such that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v = \sigma_t$ $(**)$, i.e. $u = f(x_1, u_2)$ and $v = f(x_1, v_2)$ for some $u_2, v_2 \in W_2(X)$ where $x_2 \in var(u_2) \cap var(v_2)$. Choose $x_2^{(j)} \in var(v)$ for some $j \in \mathbb{N}$. Then $seq^v(x_2^{(j)}) = (2, p_1, ..., p_q)$ for some $p_1, ..., p_q \in \{1, 2\}$ and for some $q \in \mathbb{N}$. By Theorem 3.5.3, there is $x_2^{(j,j_1)} \in var(\sigma_t^2 \circ_G \sigma_v)$ for some $j_1 \in \mathbb{N}$ such that

$$seq^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, ..., i_m, w_1, ..., w_d, a_{p_1}, ..., a_{p_q})$$

where $(2, i_2, ..., i_m, w_1, ..., w_d) = seq^{\sigma_t^2}(x_2^{(h,h_1)})$ and a_{p_z} is a sequence of natural numbers such that $(a_{p_z}) = seq^s(x_{p_z}^{(l_z)})$ for some $l_z \in \mathbb{N}$ and for all $1 \leq z \leq q$. [Note: $x_2^{(j)}$ is a variable x_2 occurring in the j^{th} order of v (from the left) and $x_2^{(j,j_1)}$ is a variable $x_2^{(j)}$ occurring in the j_1^{th} order of $\sigma_t^2 \circ_G \sigma_v$ (from the left)]. Instead of a sequence $a_{p_1}, ..., a_{p_q}$ we write a sequence of natural numbers $w_{d+1}, ..., w_k$ for some $k \in \mathbb{N}$ and $w_{d+1}, ..., w_k \in \{1, 2\}$. Then

$$seq^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, ..., i_m, w_1, ..., w_d, w_{d+1}, ..., w_k).$$

By Theorem 3.5.3, we have $x_2^{(j,j_1,j_2)} \in var(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)$ for some $j_2 \in \mathbb{N}$. By Corollary 3.5.5, we have

$$\begin{aligned} depth^{\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1,j_2)}) &= depth^u(x_2^{(b_1)}) + depth^u(x_{i_2}^{(b_2)}) + ... + depth^u(x_{i_m}^{(b_m)}) \\ &\quad + depth^u(x_{w_1}^{(b_{m+1})}) + ... + depth^u(x_{w_d}^{(b_{m+d})}) \\ &\quad + depth^u(x_{w_{d+1}}^{(b_{m+d+1})}) + ... + depth^u(x_{w_k}^{(b_{m+k})}) \\ &> m \end{aligned}$$

for some $b_1, ..., b_m, b_{m+1}, ..., b_{m+d}, b_{m+d+1}, ..., b_{m+k} \in \mathbb{N}$, which contradicts to $(*)$ and $(**)$. Therefore σ_t is not intra-regular in $Hyp_G(2)$. □

Lemma 3.5.10. *If $t = f(t_1, x_2)$ where $t_1 \in W_{(2)}(X) \setminus X_2$ and $x_1 \in var(t)$ then σ_t is not intra-regular in $Hyp_G(2)$.*

Proof. The proof is similar to the proof of Lemma 3.5.9. □

Lemma 3.5.11. *If $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ and $var(t) \cap X_2 \neq \emptyset$ then σ_t is not intra-regular in $Hyp_G(2)$.*

Proof. Let $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ and $var(t) \cap X_2 \neq \emptyset$.
Case1: $var(t) \cap X_2 = \{x_i\}$ for some $i \in \{1, 2\}$. Let $j \in \{1, 2\}$ where $i \neq j$.

If j is occurring in $seq^t(x_i^{(h)})$ for all $x_i^{(h)} \in var(t)$ then $var(\sigma_t^2) \cap X_2 = \emptyset$, i.e. $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

If j is not occurring in $seq^t(x_i^{(h)})$ for some $x_i^{(h)} \in var(t)$ then $seq^t(x_i^{(h)}) = (i_1, i_2, ..., i_m)$ where $i_1, i_2, ..., i_m \in \{i\}$ for some $m \in \mathbb{N}$. We can prove similar to the proof of Lemma 3.5.9, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.
Case2: $var(t) \cap X_2 = X_2$. We can prove similar to the proof of Lemma 3.5.9, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

Therefore σ_t is not intra-regular in $Hyp_G(2)$. □

Theorem 3.5.12. *$CR(Hyp_G(2))$ is the set of all intra-regular elements in $Hyp_G(2)$.*

Proof. By Corollary 3.4.7 and by Lemma 3.5.7 - Lemma 3.5.11. □

In 2014, S. Sudsanit, S. Leeratanavalee and W. Puninagool [24] characterized left-right regular elements in the monoid generalized hypersustitutions of type $\tau = (2)$.

Proposition 3.5.13 ([24])**.** *If σ_t is idempotent, then σ_t is left(right) regular.*

Proposition 3.5.14 ([24])**.** *$\sigma_{f(x_2,x_1)}$ is left(right) regular in $Hyp_G(2)$.*

By Proposition 3.5.13 and Proposition 3.5.14, S. Sudsanit, S. Leeratanavalee and W. Puninagool showed that every element in $CR(Hyp_G(2))$ is left(right) regular.

Proposition 3.5.15 ([24]). *$\sigma_{f(x_2,x_m)}$ where $m \in \mathbb{N}$ with $m > 2$ is not left(right) regular in $Hyp_G(2)$.*

Proposition 3.5.16 ([24]). *$\sigma_{f(x_m,x_1)}$ where $m \in \mathbb{N}$ with $m > 2$ is not left(right) regular in $Hyp_G(2)$.*

Proposition 3.5.17 ([24]). *Let $t \in W_{(2)}(X) \setminus X$. Then the following statements hold:*
(i) If $x_2 \in var(t)$, then $\sigma_{f(x_1,t)}$ is not left(right) regular;
(ii) If $x_1 \in var(t)$, then $\sigma_{f(t,x2)}$ is not left(right) regular;
(iii) $\sigma_{f(t,x_1)}$ and $\sigma_{f(x_2,t)}$ are not left(right) regular;
(iv) If $x_1 \in var(t)$ or $x_2 \in var(t)$ then $\sigma_{f(x_m,t)}$ and $\sigma_{f(t,x_m)}$ are not left(right) regular where $m \in \mathbb{N}$ with $m > 2$.

Proposition 3.5.18 ([24]). *Let $t_1, t_2 \in W_{(2)}(X) \setminus X$. If $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$ then $\sigma_{f(t_1,t_2)}$ is not left(right) regular.*

By Proposition 3.5.15 - Proposition 3.5.18, S. Sudsanit, S. Leeratanavalee and W. Puninagool showed that every element in $Hyp_G(2) \setminus CR(Hyp_G(2))$ is not left(right) regular, i.e. $CR(Hyp_G(2))$ is the set of all left(right) regular elements in $Hyp_G(2)$.

By Section 3.4, we have the set of all completely regular elements, the set of all left regular and the set of all right regular elements in $Hyp_G(2)$ are the same. Then

Theorem 3.5.19. *Let $\sigma_t \in Hyp_G(2)$. The following statements are equivalent:*

(i) σ_t is completely regular;

(ii) σ_t is left regular;

(iii) σ_t is right regular;

(iv) σ_t is intra-regular.