# CHAPTER 4

# Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau$

In 1980, H.D. Alarcao showed that: A monoid $S$ is factorisable if and only if it is unit-regular [2]. In 2000, S. Leeratanavalee and K. Denecke defined a binary operation on the set of all generalized hypersubstitutions and then proved that this set together with the binary operation forms a monoid [16]. By Section 3.2 and Section 3.3 we characterize all unit-regular elements of the monoid of all generalized hypersubstitutions of type $\tau$. It is clear that the set of all unit-regular elements of the monoid of all generalized hypersubstitutions of type $\tau$ is a proper subset of the monoid of all generalized hypersubstitutions of type $\tau$. So the monoid of all generalized hypersubstitutions of type $\tau$ is not unit-regular, i.e. it is not factorisable. By Example 3.2.3 and Example 3.3.10, we show that the set of all unit-regular elements of the monoid of all generalized hypersubstitutions of type $\tau$ is not a submonoid of this monoid, that means the set of all unit-regular elements of this monoid is not factorisable. In this chapter we determine the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau$. The tool for determining the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau$ was difined in Chapter 3.



## 4.1 Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (2)$

To determine the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (2)$, at first we introduce some notations which will be used throughout of this section.

For a type $\tau = (n)$ with $n$-ary operation symbol $f$ and $t \in W_{(n)}(X)$, we denote
$leftmost(t) :=$ the first variable (from the left) that occurs in $t$,
$rightmost(t) :=$ the last variable (from the left) that occurs in $t$.

**Example 4.1.1.** Let $\tau = (3)$ and $t \in W_{(3)}(X)$ where

$$t = f(f(f(x_5, x_6, x_1), x_1, x_2), f(x_4, x_3, x_2), f(x_7, x_8, f(x_4, x_4, x_2))).$$

Then $leftmost(t) = x_5$ and $rightmost(t) = x_2$.

Denote $R_{(Hyp_G(2))_i}$ as in Section 3.2 where $i \in \{1, 2, 3, 4, 5, 6\}$. In Section 3.2, we show that $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is a set of all unit-regular elements in $Hyp_G(2)$ but it is not a submonoid of $Hyp_G(2)$, i.e. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not a factorisable submonoid of the monoid $Hyp_G(2)$.

By Example 3.2.3, we have:

(1) For each $\sigma_t \in R_1$ such that $t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ and $rigthmost(t') = x_2$, we have $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

(2) For each $\sigma_t \in R_2$ such that $t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ and $leftmost(t') = x_1$, we have $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

(3) For each $\sigma_t \in R_3$ and $\sigma_s \in R_4$ such that $t = f(x_1, t')$ and $s = f(s', x_2)$ where $t', s' \in W_{(2)}(X)$ and $rightmost(t') = x_1$, $leftmost(s') = x_2$, we have $\sigma_t \circ_G \sigma_s$, $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$.

Next, we find the maximal factorisable submonoid of the monoid $Hyp_G(2)$.

Let $\sigma_t \in Hyp_G(2)$, we denote

$R^*_{(Hyp_G(2))_1} := \{\sigma_t | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') \neq x_2\}$,

$R^*_{(Hyp_G(2))_2} := \{\sigma_t | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $leftmost(t') \neq x_1\}$,

$R^*_{(Hyp_G(2))_3} := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$,

$R^*_{(Hyp_G(2))_4} := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$,

$R_{(Hyp_G(2))_5} := \{\sigma_t | t = x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$ and

$R_{(Hyp_G(2))_6} := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

Denote $(UR)_{Hyp_G(2)} = \bigcup_{i=1}^{4} R^*_{(Hyp_G(2))_i} \cup R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}$.

**Theorem 4.1.2.** *$(UR)_{Hyp_G(2)}$ is submonoid of $Hyp_G(2)$.*

*Proof.* It is clear that $(UR)_{Hyp_G(2)} \subset Hyp_G(2)$. So we will show that $(UR)_{Hyp_G(2)}$ is closed under $\circ_G$.

**Case 1**: $\sigma_t \in R^*_{(Hyp_G(2))_1}$. Then $t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$. Let $\sigma_s \in (UR)_{Hyp_G(2)}$.

**Case 1.1**: $\sigma_s \in R^*_{(Hyp_G(2))_1}$. Then $s = f(x_2, s')$ where $s' \in W_{(2)}(X)$ such that $x_1 \notin var(s')$ and $rightmost(s') \neq x_2$.
Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_2, s')] \\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[x_2], \widehat{\sigma}_t[s']) \\ &= S^2(f(x_2, t'), x_2, \widehat{\sigma}_t[s']) \\ &= f(S^2(x_2, x_2, \widehat{\sigma}_t[s']), S^2(t', x_2, \widehat{\sigma}_t[s'])) \\ &= f(\widehat{\sigma}_t[s'], S^2(t', x_2, \widehat{\sigma}_t[s'])).\end{aligned}$$

Since $x_1 \notin var(s')$ and $rightmost(s') \neq x_2$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_1 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', x_2, \widehat{\sigma}_t[s'])$. Hence $\sigma_t \circ_G \sigma_s \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

**Case 1.2**: $\sigma_s \in R^*_{(Hyp_G(2))_2}$. Then $s = f(s', x_1)$ where $s' \in W_{(2)}(X)$ such that $x_2 \notin var(s')$ and $leftmost(s') \neq x_1$.
Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s', x_1)] \\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[s'], \widehat{\sigma}_t[x_1]) \\ &= f(S^2(x_2, \widehat{\sigma}_t[s'], x_1), S^2(t', \widehat{\sigma}_t[s'], x_1)) \\ &= f(x_1, S^2(t', \widehat{\sigma}_t[s'], x_1)).\end{aligned}$$

We have $x_2 \notin var(S^2(t', \widehat{\sigma}_t[s'], x_1))$. Since $rightmost(t') \neq x_2$, so $rightmost(S^2(t', \widehat{\sigma}_t[s'], x_1)) \neq x_1$. Then we have $\sigma_t \circ_G \sigma_s \in R^*_{(Hyp_G(2))_3} \subseteq (UR)_{Hyp_G(2)}$.
Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(s', x_1), \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[t']) \\ &= f(S^2(s', x_2, \widehat{\sigma}_s[t']), S^2(x_1, x_2, \widehat{\sigma}_s[t'])) \\ &= f(S^2(s', x_2, \widehat{\sigma}_s[t']), x_2).\end{aligned}$$

We have $x_1 \notin var(S^2(s', x_2, \widehat{\sigma}_s[t']))$. Since $leftmost(t') \neq x_1$, so $leftmost(S^2(s', x_2, \widehat{\sigma}_s[t'])) \neq x_2$. Then we have $\sigma_s \circ_G \sigma_t \in R^*_{(Hyp_G(2))_4} \subseteq (UR)_{Hyp_G(2)}$.

**Case 1.3**: $\sigma_s \in R^*_{(Hyp_G(2))_3}$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ such that $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')] \\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[s']) \\ &= S^2(f(x_2, t'), x_1, \widehat{\sigma}_t[s']) \\ &= f(S^2(x_2, x_1, \widehat{\sigma}_t[s']), S^2(t', x_1, \widehat{\sigma}_t[s'])) \\ &= f(\widehat{\sigma}_t[s'], S^2(t', x_1, \widehat{\sigma}_t[s'])).\end{aligned}$$

Since $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_1 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', x_1, \widehat{\sigma}_t[s']))$. Hence $\sigma_t \circ_G \sigma_s \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.

Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(x_1, s'), \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[t']) \\ &= f(S^2(x_1, x_2, \widehat{\sigma}_s[t']), S^2(s', x_2, \widehat{\sigma}_s[t'])) \\ &= f(x_2, S^2(s', x_2, \widehat{\sigma}_s[t'])).\end{aligned}$$

We have $x_1 \notin var(S^2(s', x_2, \widehat{\sigma}_s[t']))$. Since $rightmost(s') \neq x_1$, so $rightmost(S^2(s', x_2, \widehat{\sigma}_s[t'])) \neq x_2$. Then we have $\sigma_s \circ_G \sigma_t \in R^*_{(Hyp_G(2))_1} \subseteq (UR)_{Hyp_G(2)}$.

In case of $\sigma_s \in R^*_{(Hyp_G(2))_4}$, we can prove in the same manner as in Case 1.3 that $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 1.4**: $\sigma_s \in R_{(Hyp_G(2))_5}$, $s = x_1$ or $s = x_2$ or $s = f(x_1, x_2)$ or $s = f(x_2, x_1)$.

If $s = x_1$, then

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_1] = x_1 \quad \text{and}$$
$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_1}[f(x_2, t')] = S^2(x_1, x_2, \widehat{\sigma}_{x_1}[t']) = x_2.$$

If $s = x_2$, then

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_2] = x_2 \quad \text{and}$$
$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_2}[f(x_2, t')] = S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']).$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']) = x_i \notin \{x_1, x_2\}$.

If $s = f(x_1, x_2)$, then $\sigma_s = \sigma_{id}$ such that $\sigma_t \circ_G \sigma_{id} = \sigma_t = \sigma_{id} \circ_G \sigma_t$.

If $s = f(x_2, x_1)$, consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_s[f(x_2, x_1)] \\ &= S^2(f(x_2, t'), x_2, x_1) \\ &= f(S^2(x_2, x_2, x_1), S^2(t', x_2, x_1)) \\ &= f(x_1, S^2(t', x_2, x_1)).\end{aligned}$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $x_2 \notin var(S^2(t', x_2, x_1))$ and $rightmost(S^2(t', x_2, x_1)) \neq x_1$, we have $\sigma_t \circ_G \sigma_s \in R^*_{(Hyp_G(2))_1}$.
Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(x_2, x_1), x_2, \widehat{\sigma}_s[t']) \\ &= f(\widehat{\sigma}_s[t'], x_2).\end{aligned}$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $x_1 \notin var(\widehat{\sigma}_s[t'])$ and $leftmost(\widehat{\sigma}_s[t']) \neq x_2$, we have $\sigma_s \circ_G \sigma_t \in R^*_{(Hyp_G(2))_4}$.

Therefore $\sigma_s \circ_G \sigma_t, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 1.5**: $\sigma_s \in R_{(Hyp_G(2))_6}$. Then $s = f(s_1, s_2)$ where $x_1, x_2 \notin var(s)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2)] \\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \\ &= f(S^2(x_2, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]), S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2])) \\ &= f(\widehat{\sigma}_t[s_2], S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2])).\end{aligned}$$



Since $x_1, x_2 \notin var(s)$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s_1]) \cup var(\widehat{\sigma}_t[s_2])$ and then $x_1, x_2 \notin var(S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]))$. So that $\sigma_t \circ_G \sigma_s \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.
Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(s_1, s_2), x_2, \widehat{\sigma}_s[t']) \\ &= f(s_1, s_2) \qquad \text{since } x_1, x_2 \notin var(s).\end{aligned}$$

So that $\sigma_s \circ_G \sigma_t \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.

**Case 2**: $\sigma_t \in R^*_{(Hyp_G(2))_2}$ and $\sigma_s \in \bigcup_{i=2}^{4} R^*_{(Hyp_G(2))_i} \cup R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}$. We can prove similar to Case 1. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 3**: $\sigma_t \in R^*_{(Hyp_G(2))_3}$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$. Let $\sigma_s \in R^*_{(Hyp_G(2))_3} \cup R^*_{(Hyp_G(2))_4} \cup R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}$.

**Case 3.1**: $\sigma_s \in R^*_{(Hyp_G(2))_3}$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')]\\ &= S^2(f(x_1, t'), x_1, \widehat{\sigma}_t[s'])\\ &= f(S^2(x_1, x_1, \widehat{\sigma}_t[s']), S^2(t', x_1, \widehat{\sigma}_t[s']))\\ &= f(x_1, t') \qquad \text{since } x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R^*_{(Hyp_G(2))_3} \subseteq (UR)_{Hyp_G(2)}$.

**Case 3.2**: $\sigma_s \in R^*_{(Hyp_G(2))_4}$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$.

Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, t')]\\ &= S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])\\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), S^2(x_2, x_1, \widehat{\sigma}_s[t']))\\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), \widehat{\sigma}_s[t']).\end{aligned}$$

Since $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$, so $x_1, x_2 \notin var(\widehat{\sigma}_s[t'])$. Since $x_1 \notin var(s')$ and $x_1, x_2 \notin var(\widehat{\sigma}_s[t'])$, so $x_1, x_2 \notin var(S^2(s', x_1, \widehat{\sigma}_s[t']))$. Then $\sigma_s \circ_G \sigma_t \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.



Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s', x_2)]\\ &= S^2(f(x_1, t'), \widehat{\sigma}_t[s'], x_2)\\ &= f(S^2(x_1, \widehat{\sigma}_t[s'], x_2), S^2(t', \widehat{\sigma}_t[s'], x_2))\\ &= f(\widehat{\sigma}_t[s'], S^2(t', \widehat{\sigma}_t[s'], x_2)).\end{aligned}$$

Since $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_2 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', \widehat{\sigma}_t[s'], x_2))$. Then $\sigma_t \circ_G \sigma_s \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.

If $\sigma_s \in R_{(Hyp_G(2))_5}$ and $\sigma_s \in R_{(Hyp_G(2))_6}$, we can prove similar to Case 1.4 and Case 1.5. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 4**: $\sigma_t \in R^*_{(Hyp_G(2))_4}$ and $\sigma_s \in R^*_{(Hyp_G(2))_4} \cup R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}$. We can prove similar to Case 3. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 5**: $\sigma_t \in R_{(Hyp_G(2))_5}$ and $\sigma_s \in R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}$. We can prove similar to Case 1.4. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

**Case 6**: $\sigma_t \in R_{(Hyp_G(2))_6}$ and $\sigma_s \in R_{(Hyp_G(2))_6}$. Then $\sigma_t \circ_G \sigma_s = \sigma_t \in R_{(Hyp_G(2))_6} \subseteq (UR)_{Hyp_G(2)}$.

Therefore $((UR)_{Hyp_G(2)}, \circ_G, \sigma_{id})$ is a submonoid of $Hyp_G(2)$. □

**Theorem 4.1.3.** *$(UR)_{Hyp_G(2)}$ is a unit-regular submonoid of $Hyp_G(2)$.*

*Proof.* By Corollary 3.1.8, $U(Hyp_G(2)) = \{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\}$ and $\{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\} \subset (UR)_{Hyp_G(2)}$, so $U(Hyp_G(2)) = U((UR)_{Hyp_G(2)})$ is the set of all unit elements of the monoid $(UR)_{Hyp_G(2)}$. By Theorem 3.2.1, we get $(UR)_{Hyp_G(2)}$ is a unit-regular submonoid of $Hyp_G(2)$. □

**Theorem 4.1.4.** *$(UR)_{Hyp_G(2)}$ is a maximal unit-regular submonoid of $Hyp_G(2)$.*

*Proof.* Let $H$ be a proper unit-regular submonoid of $Hyp_G(2)$ such that $(UR)_{Hyp_G(2)} \subseteq H \subset Hyp_G(2)$. Let $\sigma_t \in H$, then $\sigma_t$ is a unit-regular element.

**Case 1**: $\sigma_t \in R_{(Hyp_G(2))_1} \setminus R^*_{(Hyp_G(2))_1}$. Then $t = f(x_2, t')$ where $x_1 \notin var(t')$ and $rightmost(t') = x_2$. Since $x_2 \in var(t)$ and $rightmost(t') = x_2$, so $t \in sub(\widehat{\sigma}_t[t'])$. Then $\widehat{\sigma}_t[t'] \in W_{(2)}(X) \setminus X$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_2, t')] \\ &= S^2(f(x_2, t'), x_2, \widehat{\sigma}_t[t']) \\ &= f(S^2(x_2, x_2, \widehat{\sigma}_t[t']), S^2(t', x_2, \widehat{\sigma}_t[t'])) \\ &= f(\widehat{\sigma}_t[t'], S^2(t', x_2, \widehat{\sigma}_t[t'])).\end{aligned}$$

Since $x_2 \in var(t')$ and $t \in sub(\widehat{\sigma}_t[t'])$, so $t \in sub(S^2(t', x_2, \widehat{\sigma}_t[t']))$, we have $x_2 \in var(S^2(t', x_2, \widehat{\sigma}_t[t']))$. Since $x_2 \in var(S^2(t', x_2, \widehat{\sigma}_t[t'])) \cup var(\widehat{\sigma}_t[t'])$ and $S^2(t', x_2, \widehat{\sigma}_t[t']), \widehat{\sigma}_t[t'] \in W_{(2)}(X) \setminus X$, so $\sigma_t \circ_G \sigma_t$ is not unit-regular. Then $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') = x_2\} \nsubseteq H$.

**Case 2**: $\sigma_t \in R_{(Hyp_G(2))_2} \setminus R^*_{(Hyp_G(2))_2}$. Then $t = f(t', x_1)$ where $x_2 \notin var(t')$ and $leftmost(t') = x_1$. Since $x_1 \in var(t')$ and $leftmost(t') = x_1$, so $t \in sub(\widehat{\sigma}_t[t'])$. Then $\widehat{\sigma}_t[t'] \in W_{(2)}(X) \setminus X$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(t', x_1)] \\ &= S^2(f(t', x_1), \widehat{\sigma}_t[t'], x_1) \\ &= f(S^2(t', \widehat{\sigma}_t[t'], x_1), S^2(x_1, \widehat{\sigma}_t[t'], x_1)) \\ &= f(S^2(t', \widehat{\sigma}_t[t'], x_1), \widehat{\sigma}_t[t']).\end{aligned}$$

Since $x_1 \in var(t')$ and $t \in sub(\widehat{\sigma}_t[t'])$, so $t \in sub(S^2(t', \widehat{\sigma}_t[t'], x_1))$, we have $x_1 \in var(S^2(t', \widehat{\sigma}_t[t'], x_1))$. Since $x_1 \in var(S^2(t', \widehat{\sigma}_t[t'], x_1)) \cup var(\widehat{\sigma}_t[t'])$ and $S^2(t', \widehat{\sigma}_t[t'], x_1), \widehat{\sigma}_t[t'] \in W_{(2)}(X) \backslash X$, so $\sigma_t \circ_G \sigma_t$ is not unit-regular. Then $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $leftmost(t') = x_1\} \nsubseteq H$.

**Case 3**: $\sigma_t \in R_{(Hyp_G(2))_3} \setminus R^*_{(Hyp_G(2))_3}$. Then $t = f(x_1, t')$ where $x_2 \notin var(t')$ and $rightmost(t') = x_1$. Choose $\sigma_s \in R^*_{(Hyp_G(2))_4} \subseteq H$. Then $s = f(s', x_2)$ such that $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$.

Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, t')] \\ &= S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t']) \\ &= S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t']) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), S^2(x_2, x_1, \widehat{\sigma}_s[t'])) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), \widehat{\sigma}_s[t']).\end{aligned}$$

Since $x_2 \in var(s)$ and $rightmost(t') = x_1$, so $x_1 \in var(\widehat{\sigma}_s[t'])$. Since $x_1 \in var(\widehat{\sigma}_s[t'])$, so $\sigma_s \circ_G \sigma_t$ is not unit-regular. Then $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') = x_1\} \nsubseteq H$.

**Case 4**: $\sigma_t \in R_{(Hyp_G(2))_4} \setminus R^*_{(Hyp_G(2))_4}$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$ and $leftmost(t') = x_2$. Choose $\sigma_s \in R'_{(Hyp_G(2))_3} \subseteq H$. Then $s = f(x_1, s')$ such that $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$.

Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(t', x_2)] \\ &= S^2(f(x_1, s'), \widehat{\sigma}_s[t'], x_2) \\ &= f(S^2(x_1, \widehat{\sigma}_s[t'], x_2), S^2(s', \widehat{\sigma}_s[t'], x_2)) \\ &= f(\widehat{\sigma}_s[t'], S^2(s', \widehat{\sigma}_s[t'], x_2)).\end{aligned}$$

Since $x_1 \in var(s)$ and $leftmost(t') = x_2$, so $x_2 \in var(\widehat{\sigma}_s[t'])$. Since $x_2 \in var(\widehat{\sigma}_s[t'])$, so $\sigma_s \circ_G \sigma_t$ is not unit-regular. Then $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(t', x_2)$ where

$t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $leftmost(t') = x_2\} \nsubseteq H$.

Therefore $H = (UR)_{Hyp_G(2)}$. □

**Theorem 4.1.5.** *$(UR)_{Hyp_G(2)}$ is the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (2)$.*

*Proof.* By Theorem 2.1.6. □

## 4.2 Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (n)$

In this section, we find the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (n)$.

Let $\tau = (n)$ be a type of operation symbol $f$. Denote $R_1, R_2$ and $R_3$ as in Section 3.3. We have $\bigcup\limits_{i=1}^{3} R_i$ is a set of all unit-regular elements in $Hyp_G(n)$ but it is not a submonoid of $Hyp_G(n)$, i.e. $\bigcup\limits_{i=1}^{3} R_i$ is not factorisable submonoid of the monoid $Hyp_G(n)$. Next, we find the maximal factorisable submonoid of the monoid $Hyp_G(n)$.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_3^* := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$ and if $x_{j_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ then there exists $(k_1, ..., k_p) \in seq^{t_k}(x_{j_l})$ such that $k_q = i_l$ for some $q \in \{1, ..., p\}\}$.

We denote $(UR)_{Hyp_G(n)} = R_1 \cup R_2 \cup R_3^*$.



**Example 4.2.1.** Let $\tau = (4)$ and let $t \in W_{(4)}(X)$ such that

$$t = f(f(x_6, x_6, x_6, f(x_7, x_3, x_7, x_3)), x_8, x_8, x_8).$$

Then $seq^t(x_3) = \{(1, 4, 2), (1, 4, 4)\}$. If $\sigma_s, \sigma_w, \sigma_u \in Hyp_G(4)$ where $s = f(x_3, t, x_1, x_2)$, $w = f(x_1, t, x_3, x_5)$ and $u = f(x_5, x_2, t, x_4)$. Then $\sigma_s \in R_3^*$, $\sigma_w \in R_3 \setminus R_3^*$ and $\sigma_u \notin \bigcup\limits_{i=1}^{3} R_i$.

**Theorem 4.2.2.** *$(UR)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$.*

*Proof.* We have $(UR)_{Hyp_G(n)} \subset Hyp_G(n)$, so we will show that $(UR)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$, i.e. $\sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(n)}$ for all $\sigma_t, \sigma_s \in (UR)_{Hyp_G(n)}$.

If $\sigma_t \in R_1$ then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in (UR)_{Hyp_G(n)}$.

If $\sigma_t \in R_2$ then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in R_1$ and $\sigma_s \circ_G \sigma_t \in R_2$ for all $\sigma_s \in (R_2 \cup R_3^*)$.

If $\sigma_t \in R_3^*$ then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in R_1$ and $\sigma_s \circ_G \sigma_t \in R_2$ for all $\sigma_s \in R_2$. Denote $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. Let $\sigma_s \in R_3^*$. We denote $s = f(s_1, ..., s_n)$ where $s_{r_1} = x_{h_1}, ..., s_{r_{m^*}} = x_{h_{m^*}}$ for some $r_1, ..., r_{m^*}, h_1, ..., h_{m^*} \in \{1, ..., n\}$ and $var(s) \cap X_n = \{x_{h_1}, ..., x_{h_{m^*}}\}$. Hence

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, ..., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ for all $i \in \{1, ..., n\}$.

**Case 1**: $i \in \{r_1, ..., r_{m^*}\}$. Then $i = r_\alpha$ for some $\alpha \in \{1, ..., m^*\}$. So

$$u_i = S^n(s_{r_\alpha}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_{h_\alpha}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{h_\alpha}].$$

**Case 1.1**: $h_\alpha \in \{i_1, ..., i_m\}$. Then $h_\alpha = i_\beta$ for some $\beta \in \{1, ..., m\}$ and $t_{i_\beta} = x_{j_\beta}$, so $u_i = \widehat{\sigma}_s[t_{i_\beta}] = x_{j_\beta}$.

**Case 1.2**: $h_\alpha = k$ where $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$. Then $u_i = \widehat{\sigma}_s[t_k]$.

**Case 1.2.1**: $var(t_k) \cap X_n = \emptyset$. Then $var(u_i) \cap X_n = \emptyset$.

**Case 1.2.2**: $var(t_k) \cap X_n \neq \emptyset$. Then there exists $x_{j_\beta} \in var(t_k)$ where $t_{i_\beta} = x_{j_\beta}$ for some $\beta \in \{1, ..., m\}$ and there exists $(k_1, ..., k_p) \in seq^{t_k}(x_{j_\beta})$ such that $k_q = i_\beta$ for some $q \in \{1, ..., p\}$. If $x_{k_1}, ..., x_{k_p} \in var(s)$ then $x_{i_\beta} = x_{k_q} \in var(s)$ where $k_q \neq k$, so $x_{k_q} = x_{i_\beta} = s_{r_\varepsilon}$ for some $\varepsilon \in \{1, ..., m^*\}$ and there exists $(r_\varepsilon) \in seq^s(x_{k_q})$ such that $r_\varepsilon \neq r_\alpha = i$. Hence

$$u_{r_\varepsilon} = S^n(s_{r_\varepsilon}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_{i_\beta}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{i_\beta}]) = x_{j_\beta}.$$

By Theorem 3.5.3, we get $x_{j_\beta} \in var(\widehat{\sigma}_s[t_k]) = var(u_i)$ and there exists $(a_{k_1}, ..., a_{k_P}) \in seq^{u_i}(x_{j_\beta})$ where $a_{k_q} = r_\varepsilon$ and $a_{k_j}$ is a sequence $j_1, ..., j_h$ such that $(j_1, ..., j_h) \in seq^s(x_{k_j})$ for all $j \in \{1, ..., p\} \setminus \{q\}$. If $x_{k_\gamma} \notin var(s)$ for some $1 \leq \gamma \leq p$ then $x_{j_\beta} \notin var(u_i)$. So $var(u_i) \cap X_n = \emptyset$.

**Case 2**: $i = k^*$ where $k^* \in \{1, ..., n\} \setminus \{r_1, ..., r_{m^*}\}$. Then

$$u_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(s_{k^*}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]).$$

**Case 2.1**: $var(s_{k^*}) \cap X_n = \emptyset$. Then $u_i = s_{k^*}$ and $var(u_i) \cap X_n = \emptyset$.

**Case 2.2**: $var(s_{k^*}) \cap X_n \neq \emptyset$. Then $x_{h_\alpha} \in var(s_{k^*})$ for some $\alpha \in \{1, ..., m^*\}$. So $s_{r_\alpha} = x_{h_\alpha}$ and there exists $(k_1^*, ..., k_p^*) \in seq^{s_{k^*}}(x_{h_\alpha})$ such that $k_q^* = r_\alpha$ for some $q \in \{1, ..., p\}$.

**Case 2.2.1**: $h_\alpha \in \{i_1, ..., i_m\}$. Then $h_\alpha = i_\beta$ for some $\beta \in \{1, ..., m\}$, so $x_{i_\beta} = x_{h_\alpha} \in var(s)$. By Theorem 3.5.3, we get $x_{j_\beta} \in var(\widehat{\sigma}_s[t])$ and $seq^s(x_{i_\beta}) \subseteq seq^{\widehat{\sigma}_s[t]}(x_{j_\beta})$. Since $(r_\alpha) \in seq^s(x_{i_\beta})$ and $(k^*, k_1^*, ..., k_p^*) \in seq^s(x_{i_\beta})$, so $(r_\alpha) \in seq^{\widehat{\sigma}_s[t]}(x_{j_\beta})$

and $(k^*, k_1^*, ..., k_p^*) \in seq^{\widehat{\sigma}_s[t]}(x_{j_\beta})$. Hence $u_{r_\alpha} = x_{j_\beta}$ and $x_{j_\beta} \in var(u_{k^*}) = var(u_i)$ and there exists $(k_1^*, ..., k_p^*) \in seq^{u_i}(x_{j_\beta})$ such that $k_q^* = r_\alpha$.

**Case 2.2.2**: $h_\alpha = k$ where $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$. We can prove similar to Case 1.2.

Therefore $\sigma_s \circ_G \sigma_t \in (R_2 \cup R_3^*) \subset (UR)_{Hyp_G(n)}$.

Hence $(UR)_{Hyp_G(n)}$ is closed under $\circ_G$ and we have $\sigma_{id} \in (UR)_{Hyp_G(n)}$, i.e. $(UR)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$. □

**Theorem 4.2.3.** *$(UR)_{Hyp_G(n)}$ is a unit-regular submonoid of $Hyp_G(n)$.*

*Proof.* By Theorem 3.1.5, we have

$$U(Hyp_G(n)) := \{\sigma_t \in Hyp_G(n) | t = f(x_{\pi(1)}, ..., x_{\pi(n)}) \text{ where } \pi \in S_n\}$$

such that $U(Hyp_G(n)) \subset (UR)_{Hyp_G(n)}$. So $U(Hyp_G(n)) = U((UR)_{Hyp_G(n)})$ is the set of all unit elements of the monoid $(UR)_{Hyp_G(n)}$. By Theorem 3.3.11, we get $(UR)_{Hyp_G(n)}$ is a unit-regular submonoid of $Hyp_G(n)$. □

**Theorem 4.2.4.** *$(UR)_{Hyp_G(n)}$ is a maximal unit-regular submonoid of $Hyp_G(n)$.*

*Proof.* Let $H$ be a proper unit-regular submonoid of $Hyp_G(n)$ such that $(UR)_{Hyp_G(n)} \subseteq H \subset Hyp_G(n)$. Let $\sigma_t \in H$ where $\sigma_t \in R_3 \setminus R_3^*$. By Theorem 3.3.15, we can choose $\sigma_s \in R_3^*$ such that $\sigma_s \circ_G \sigma_t$ is not unit-regular. So $\sigma_t \notin H$. Hence $H = (UR)_{Hyp_G(n)}$. □

**Theorem 4.2.5.** *$(UR)_{Hyp_G(n)}$ is the maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (n)$.*

*Proof.* By Theorem 2.1.6. □