CHAPTER 5

Conclusion

In this study, we found that

5.1 All Unit Elements in $Hyp_G(n)$

1. Let $\sigma_t \in Hyp_G(n)$ where $t = f(t_1, t_2, ..., t_n) \in W_{(n)}(X)$. If $t_i \in W_{(n)}(X) \backslash X$ for some $i \in \{1, 2, ..., n\}$, then σ_t is not unit.

2. Let $\sigma_t \in Hyp_G(n)$ where $t = f(x_{m_1}, x_{m_2}, ..., x_{m_n}) \in W_{(n)}(X)$. If $m_i > n$ for some $i \in \{1, 2, ..., n\}$, then σ_t is not unit in $Hyp_G(n)$.

3. An element $\sigma_t \in U(Hyp_G(n))$ if and only if $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ where $\pi \in S_n$ and S_n is the set of all permutations of $\{1, 2, ..., n\}$.

4. $U(Hyp_G(n)) = \{\sigma_t \in Hyp_G(n) | t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where $\pi \in S_n\}$.

5. $|U(Hyp_G(n))| = n!$.

6. $U(Hyp_G(2)) = \{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

5.2 All Unit-regular Elements in $Hyp_G(2)$

Let $\sigma_t \in Hyp_G(2)$. We denote

$R_{(Hyp_G(2))_1} := \{\sigma_t | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')\}$,
$R_{(Hyp_G(2))_2} := \{\sigma_t | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')\}$,
$R_{(Hyp_G(2))_3} := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')\}$,
$R_{(Hyp_G(2))_4} := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')\}$,
$R_{(Hyp_G(2))_5} := \{\sigma_t | t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}\}$ and
$R_{(Hyp_G(2))_6} := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

Then we have:

1. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is a set of all unit-regular elements in $Hyp_G(2)$.

2. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not closed under $\circ_G$, i.e. $\bigcup_{i=1}^{6} R_{(Hyp_G(2))_i}$ is not a subsemigroup of $Hyp_G(2)$.

5.3 All Unit-regular Elements in $Hyp_G(n)$

1. For each $\sigma_s, \sigma_t \in Hyp_G(n)$ where $t = f(t_1, ..., t_n)$ such that $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. Let $h_1, ..., h_p \in \{j_1, \ldots, j_m\}$ and $h_l \neq h_r$ if $l \neq r$. Then $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ if and only if $s = f(s_1, ..., s_n)$ where $s_{h_q} = s_{j_l} = x_{i_l}$ for all $q \in \{1, ..., p\}$ and for some $l \in \{1, ..., m\}$.

2. $E(Hyp_G(n))$ is not a subsemigroup of $Hyp_G(n)$.

3. Let $t, s \in W_{(n)}(X) \setminus X$, $x \in var(t)$ and $var(s) \cap X_n = \{x_{z_1}, ..., x_{z_k}\}$. If $(i_1, ..., i_m) \in seq^t(x)$ where $i_1, ..., i_m \in \{z_1, ..., z_k\}$ then $x \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ and there is $(a_{i_1}, ..., a_{i_m}) \in seq^{\widehat{\sigma}_s[t]}(x)$ where a_{i_j} is a sequence of natural numbers $j_1, ..., j_h$ such that $(j_1, ..., j_h) \in seq^s(x_{i_j})$ for all $j \in \{1, ..., m\}$.

4. Let $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m$, $j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. If $x_{j_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ where $(k_1, ..., k_p) \in seq^{t_k}(x_{j_l})$ for some $k_1, ..., k_p \in \{1, ..., n\} \setminus \{i_l\}$ then there exists $\sigma_s \in Hyp_G(n)$ such that $\sigma_s \circ_G \sigma_t$ is not a unit-regular element in $Hyp_G(n)$.

Let $\sigma_t \in Hyp_G(n)$. We denote
$R_1 := \{\sigma_{x_i} | x_i \in X\}$,
$R_2 := \{\sigma_t | t \in W_{(n)}(X) \setminus X \text{ and } var(t) \cap X_n = \emptyset\}$,
$R_3 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ such that $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}\}$.
Then we have :

5. $\bigcup_{i=1}^{3} R_i$ is the set of all unit-regular elements in $Hyp_G(n)$.

6. $\bigcup_{i=1}^{3} R_i$ is not a unit-regular submonoid and it is not a regular submonoid of $Hyp_G(n)$.

5.4 All Completely Regular Elements in $Hyp_G(n)$

1. For each $\sigma_t \in E(Hyp_G(n))$, σ_t is a completely regular element in $Hyp_G(n)$.

2. For each $\sigma_t \in U(Hyp_G(n))$, σ_t is a completely regular element in $Hyp_G(n)$.

3. Let $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m,\ j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$. If there exists $l \in \{1, ..., m\}$ such that $t_{i_l} = x_{j_l}$ where $i_l \notin \{j_1, ..., j_m\}$, then $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$.

   Let $\sigma_t \in Hyp_G(n)$. We denote R_1, R_2, R_3 as in Section 5.3 and denote

   $CR(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ there is $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, ..., i_m\}$ for some $i_1, \ldots, i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, ..., x_{\pi(i_m)}\}\}$.

   Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$.

   Then we have:

4. For each $\sigma_t \in CR(R_3)$, σ_t is a completely regular element in $Hyp_G(n)$.

5. Let $CR(Hyp_G(n))$. Then $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$.

6. Let $\sigma_t \in CR(Hyp_G(n))$. Then σ_t is both left regular and right regular element in $Hyp_G(n)$, and σ_t is an intra-regular element in $Hyp_G(n)$.

7. If $\sigma_t \in R_3 \setminus CR(R_3)$, then σ_t is not a left regular element in $Hyp_G(n)$.

8. $CR(Hyp_G(\tau))$ is not a submonoid of $Hyp_G(\tau)$.

5.5 All Intra-regular Elements in $Hyp_G(2)$

5.5.1 Sequence of Terms

1. Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i^{(j)} \in var(t)$ for some $i, j \in \mathbb{N}$ and let $seq^t(x_i^{(j)}) = i_1, ..., i_m$. Then $x_{i_1}, ..., x_{i_m} \in var(s) \cap X_n$ if and only if $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ for some $j_1 \in \mathbb{N}$ and $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, ..., a_{i_m})$ where a_{i_l} is a sequence of natural numbers $p_1, ..., p_q$ such that $(p_1, ..., p_q) = seq^s(x_{i_l}^{h_l})$ for some $h_l \in \mathbb{N}$ and for all $l \in \{1, ..., m\}$.

2. Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i^{(j)} \in var(t)$ for some $i, j \in \mathbb{N}$ such that $seq^t(x_i^{(j)}) = i_1, i_2, ..., i_m$ for some $i_1, i_2, ..., i_m \in \{1, ..., n\}$ and $x_{i_k} \in var(s)$ for all $1 \leq k \leq m$. Then there is $j_1 \in \mathbb{N}$ such that

$$depth^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = depth^s(x_{i_1}^{(l_1)}) + depth^s(x_{i_2}^{(l_2)}) + ... + depth^s(x_{i_m}^{(l_m)})$$

for some $l_1, l_2, ..., l_m \in \mathbb{N}$, and

$$vb^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = vb^s(x_{i_1}) \times vb^s(x_{i_2}) \times ... \times vb^s(x_{i_m}).$$

Let $vb^t(x_i) = d$.

If $x_i \in X_n$, then $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

If $x_i \in X \setminus X_n$ where $x_i \notin var(s)$, then $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

5.5.2 All Intra-regular Elements in $Hyp_G(2)$

1. If $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X_2$ then σ_t is not intra-regular in $Hyp_G(2)$.

2. If $t = f(x_2, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X_2$ then σ_t is not intra-regular in $Hyp_G(2)$.

3. If $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X_2$ and $x_2 \in var(t)$ then σ_t is not intra-regular in $Hyp_G(2)$.

4. If $t = f(t_1, x_2)$ where $t_1 \in W_{(2)}(X) \setminus X_2$ and $x_1 \in var(t)$ then σ_t is not intra-regular in $Hyp_G(2)$.

5. If $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ and $var(t) \cap X_2 \neq \emptyset$ then σ_t is not intra-regular in $Hyp_G(2)$.

   Let $\sigma_t \in Hyp_G(2)$. By the definition of R_1, R_2, R_3 in Section 5.3 and $CR(R_3)$ in Section 5.4, we have

   $R_1 := \{\sigma_{x_i} | x_i \in X\}$;

   $R_2 := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \emptyset\}$;

   $R_3 := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $t = f(t_1, t_2)$ where $t_i = x_j$ for some $i, j \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_j\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$;

   $CR(R_3) := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

   Let $CR(Hyp_G(2)) := CR(R_3) \cup R_1 \cup R_2 = E(Hyp_G(2)) \cup \{\sigma_{f(x_2,x_1)}\}$.

   Then we have:

6. $CR(Hyp_G(2))$ is the set of all intra-regular elements in $Hyp_G(2)$.

7. Let $\sigma_t \in Hyp_G(2)$. The following statements are equivalent:

   (i) σ_t is completely regular;

(ii) σ_t is left regular;

(iii) σ_t is right regular;

(iv) σ_t is intra-regular.

5.6 Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (2)$

Let $\sigma_t \in Hyp_G(2)$. We denote

$R^*_{(Hyp_G(2))_1} := \{\sigma_t | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') \neq x_2\}$,

$R^*_{(Hyp_G(2))_2} := \{\sigma_t | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $leftmost(t') \neq x_1\}$,

$R^*_{(Hyp_G(2))_3} := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$,

$R^*_{(Hyp_G(2))_4} := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$,

$R_{(Hyp_G(2))_5} := \{\sigma_t | t = x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$,

$R_{(Hyp_G(2))_6} := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$ and

$$(UR)_{Hyp_G(2)} = \bigcup_{i=1}^{4} R^*_{(Hyp_G(2))_i} \cup R_{(Hyp_G(2))_5} \cup R_{(Hyp_G(2))_6}.$$

Then we have:

1. $(UR)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$.

2. $(UR)_{Hyp_G(2)}$ is a unit-regular submonoid of $Hyp_G(2)$.

3. $(UR)_{Hyp_G(2)}$ is a maximal unit-regular submonoid of $Hyp_G(2)$.

4. $(UR)_{Hyp_G(2)}$ is a maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (2)$.

5.7 Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (n)$

Let $\sigma_t \in Hyp_G(n)$. We denote R_1, R_2 as in Section 5.3 and denote

$R^*_3 := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m, j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}$ and if $x_{j_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$

and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ then there exists $(k_1, ..., k_p) \in seq^{t_k}(x_{j_l})$ such that $k_q = i_l$ for some $q \in \{1, ..., p\}\}$.

Let $(UR)_{Hyp_G(n)} := R_1 \cup R_2 \cup R_3^*$.

Then we have:

1. $(UR)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$.
2. $(UR)_{Hyp_G(n)}$ is a unit-regular submonoid of $Hyp_G(n)$.
3. $(UR)_{Hyp_G(n)}$ is a maximal unit-regular submonoid of $Hyp_G(n)$.
4. $(UR)_{Hyp_G(n)}$ is a maximal factorisable submonoid of the monoid generalized hypersubstitutions of type $\tau = (n)$.