CHAPTER 2

Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Distribution Functions

Definition 2.1.1. Let X be a nonempty set and $A \subseteq X$. The *indicator function* of a set A defined on X will be denoted by $\mathbf{1}_A$, defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 2.1.2. Let $S \subseteq \mathbb{R}$. A function $f : S \to \mathbb{R}$ is said to be *nondecreasing* if

$$f(x) \leq f(y)$$

whenever $x \leq y$.

Definition 2.1.3. Let $S \subseteq \mathbb{R}$. A function $f : S \to \mathbb{R}$ is said to be *right-continuous* if for every $x \in S$ and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $x \leq y < x + \delta$.

Definition 2.1.4. A function $F : \mathbb{R} \to [0, 1]$ is called a *(cumulative) distribution function* if F satisfies the following properties:

i) F is right-continuous,

ii) F is nondecreasing, and

iii) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Example 2.1.1. A function $F : \mathbb{R} \to [0, 1]$ given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{x}{1 + x} & \text{if } x \geq 0 \end{cases}$$

is a distribution function.

Proof. We first show that F is right-continuous.

Since the function $x \mapsto 0$ is continuous on $(-\infty, 0)$ and the function $x \mapsto \frac{x}{1+x}$ is continuous on $(0, \infty)$, we focus only at 0.

Let $x = 0$ and $\epsilon > 0$. Choose $\delta = \epsilon > 0$.

For any $y \in [x, x + \delta) = [0, \delta)$, we get

$$
\begin{aligned}
|F(y) - F(x)| &= \left| \frac{y}{1+y} - 0 \right| \\
&= \left| \frac{y}{(1+y)} \right| \\
&\leq y \\
&< \delta \\
&= \epsilon.
\end{aligned}
$$

Then F is right-continuous.

Next, we show that F is nondecreasing.

It is clear that if $x = y$, then $F(x) = F(y)$. Suppose $x < y$.

Case i) $x < y \leq 0$.

In this case, we get $F(x) = F(y) = 0$.

Case ii) $x \leq 0 < y$.

In this case, we get

$$
\begin{aligned}
F(x) &= 0 \\
&< \frac{y}{1+y} \\
&= F(y).
\end{aligned}
$$

Case iii) $0 < x < y$.

In this case, we get $x + xy < y + xy$.

Then $x(1 + y) < y(1 + x)$.

This implies that $\dfrac{x}{1+x} < \dfrac{y}{1+y}$, that is, $F(x) < F(y)$.

Therefore, $F(x) \leq F(y)$ whenever $x \leq y$.

It is obvious that $\lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} 0 = 0$ and $\lim\limits_{x \to \infty} F(x) = \lim\limits_{x \to \infty} \dfrac{x}{1+x} = 1$.

Hence, F is a distribution function. □

Definition 2.1.5. Let $A \subseteq \mathbb{R}$ and $\Theta \subseteq \mathbb{R}^n$.

For $S : A \to \mathbb{R}$, define $\Delta_a^b S = S(b) - S(a)$ for $a, b \in A$.

For any $H : \Theta \times A \to \mathbb{R}$ the *difference*, denoted by $\Delta_a^b H$, is defined by

$$\Delta_a^b H(\theta) = H(\theta, b) - H(\theta, a)$$

for all $\theta \in \Theta$ and $a, b \in \mathbb{R}$ in which $a \leq b$.

Definition 2.1.6. Let $A \subseteq \mathbb{R}$ and H be a function from A^n to $\mathbb{R}$.
The *volume* of H, denoted by V_H, is defined by

$$V_H\left(\prod_{i=1}^{n}(a_i, b_i]\right) = \Delta_{a_1}^{b_1} \cdots \Delta_{a_n}^{b_n} H$$

where $a_i, b_i \in A$ such that $a_i < b_i$ for all $i = 1, ..., n$.

Let $A \subseteq \mathbb{R}$ and H be a function from A^2 to $\mathbb{R}$.
Given $a_1, a_2, b_1, b_2 \in A$ in which $a_1 \leq a_2$ and $b_1 \leq b_2$.
Consider

$$\begin{aligned}
V_H\left((a_1, b_1] \times (a_2, b_2]\right) &= \triangle_{a_1}^{b_1} \triangle_{a_2}^{b_2} H \\
&= \triangle_{a_1}^{b_1}\left(\triangle_{a_2}^{b_2} H\right) \\
&= \triangle_{a_2}^{b_2} H\left(b_1\right) - \triangle_{a_2}^{b_2} H\left(a_1\right) \\
&= \left(H\left(b_1, b_2\right) - H\left(b_1, a_2\right)\right) - \left(H\left(a_1, b_2\right) - H\left(a_1, a_2\right)\right) \\
&= H\left(b_1, b_2\right) - H\left(b_1, a_2\right) - H\left(a_1, b_2\right) + H\left(a_1, a_2\right).
\end{aligned}$$

In general, let $A_i \subseteq \mathbb{R}$ for all $i = 1, \ldots, n$. For any $H : \prod_{i=1}^{n} A_i \to [0, 1]$ and all $a_i, b_i \in A_i$ in which $a_i \leq b_i$,

$$V_H\left(\prod_{i=1}^{n}\left(a_i, b_i\right]\right) = \sum_{\vec{v} \in \prod_{i=1}^{n}\{a_i, b_i\}} (-1)^{N(\vec{v})} H\left(\vec{v}\right),$$

where $N\left(\vec{v}\right) = N\left((v_1, ..., v_n)\right)$ is the number of i such that $v_i = a_i$.

Definition 2.1.7. Let $A \subseteq \mathbb{R}$.
A function F from A^n to $\mathbb{R}$ is called $n-$*increasing function* if the volume of F is non-negative, that is,

$$V_F\left(\prod_{i=1}^{n}(a_i, b_i]\right) \geq 0$$

for all $a_i, b_i \in A$ in which $a_i \leq b_i$.

Definition 2.1.8. Let $A_i \subseteq \mathbb{R}$ for all $i = 1, \ldots, n$. A function $H : \prod_{i=1}^{n} A_i \to [0,1]$ is said to be *continuous from above* if, for each $k = 1, \ldots, n$ and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|H(x_1, \ldots, x_k, \ldots, x_n) - H(x_1, x_2, \ldots, y_k, \ldots, x_n)| < \epsilon$$

for all $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} A_i$ and all $y_k \in A_k \cap [x_k, x_k + \delta)$.

Definition 2.1.9. Let $F : \mathbb{R}^n \to [0,1]$. Then F is called an *n-dimensional distribution function* if it satisfies the following properties:

i) $\lim_{v_i \to -\infty, \exists i} F((v_1, ..., v_n)) = 0$,

ii) $\lim_{v_i \to \infty, \forall i} F((v_1, ..., v_n)) = 1$,

iii) F is n-increasing, and

iv) F is continuous from above.

Let F be an n-dimensional distribution function. For each $i = 1, ..., n$, the function $F_i : \mathbb{R} \to [0,1]$ defined by $F_i(x_i) = \lim_{v_j \to \infty, \forall j \neq i} F((v_1, ..., v_n))$ is called the (i-th) *marginal distribution function* of F.

Remark 2.1.2. *Every marginal distribution function of an n-dimensional distribution function is a distribution function.*

Next, we will given an example of a 2-dimensional distribution function and its marginals.

Example 2.1.3. A function $F : \mathbb{R}^2 \to [0,1]$ given by

$$F(x,y) = \begin{cases} 1 & \text{if } x \geq 0, y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is a 2-dimensional distribution function.

Proof. It is easy to see that

$$\lim_{x \to -\infty} F(x,y) = 0,$$

$$\lim_{y \to -\infty} F(x,y) = 0,$$

$$\lim_{x \to \infty} \lim_{y \to \infty} F(x,y) = 1 \quad \text{and}$$

$$\lim_{y \to \infty} \lim_{x \to \infty} F(x,y) = 1.$$

We next show that F is a 2-increasing function.

Since $V_F((a,b] \times (c,d]) = F(b,d) + F(a,c) - F(b,c) - F(a,d)$ and range of the function F is $\{0,1\}$, we have three cases to consider, when $F(b,c) = 1, F(a,d) = 1$ and $F(b,c) = 0 = F(a,d)$.

If $F(b,c) = 0 = F(a,d)$, then the volume of F is not less than 0, that is, we have only two cases to consider, when $F(b,c) = 1$ or $F(a,d) = 1$.

Case i) $F(b,c) = 1$. By the definition of F, we obtain that $b, c \geq 0$.

Since $c \leq d$, we have $b, c, d \geq 0$.

If $a < 0$, then

$$\begin{aligned} V_F((a,b] \times (c,d]) &= F(b,d) + F(a,c) - F(b,c) - F(a,d) \\ &= 1 + 0 - 1 - 0 \\ &= 0. \end{aligned}$$

If $a \geq 0$, then

$$\begin{aligned} V_F((a,b] \times (c,d]) &= F(b,d) + F(a,c) - F(b,c) - F(a,d) \\ &= 1 + 1 - 1 - 1 \\ &= 0. \end{aligned}$$

Case ii) $F(a,d) = 1$. By the definition of F, we obtain that $a, d \geq 0$.

Since $a \leq b$, we have $a, b, d \geq 0$.

If $c < 0$, then

$$\begin{aligned} V_F((a,b] \times (c,d]) &= F(b,d) + F(a,c) - F(b,c) - F(a,d) \\ &= 1 + 0 - 0 - 1 \\ &= 0. \end{aligned}$$

If $c \geq 0$, then

$$\begin{aligned} V_F((a,b] \times (c,d]) &= F(b,d) + F(a,c) - F(b,c) - F(a,d) \\ &= 1 + 1 - 1 - 1 \\ &= 0. \end{aligned}$$

Therefore, $V_F((a,b] \times (c,d]) \geq 0$.

Finally, we show that F is continuous from above.

Let $x, y \in \mathbb{R}$ and $\epsilon > 0$. Then we have four cases to consider.

Case i) $x \geq 0, y \geq 0$.

Choose $\delta = \epsilon > 0$.

For any $x^+ \in [x, x+\delta)$, $|F(x,y) - F(x^+, y)| = |1-1| = 0 < \epsilon$.

Similarly, $|F(x, y^+) - F(x,y)| < \epsilon$ for all $y^+ \in [y, y+\delta)$.

Case ii) $x < 0, y \geq 0$.

Choose $\delta_1 = -\frac{x}{2} > 0$ and $\delta_2 = \epsilon > 0$.

For any $x^+ \in [x, x+\delta_1)$, $|F(x,y) - F(x^+, y)| = |0-0| = 0 < \epsilon$

and for any $y^+ \in [y, y+\delta_2)$, $|F(x, y^+) - F(x,y)| = |0-0| = 0 < \epsilon$.

Case iii) $x \geq 0, y < 0$.

Choose $\delta_1 = \epsilon > 0$ and $\delta_2 = -\frac{y}{2} > 0$.

For any $x^+ \in [x, x+\delta_1)$, $|F(x,y) - F(x^+, y)| = |0-0| = 0 < \epsilon$

and for any $y^+ \in [y, y+\delta_2)$, $|F(x, y^+) - F(x,y)| = |0-0| = 0 < \epsilon$.

Case iv) $x < 0, y < 0$.

Choose $\delta_1 = -\frac{x}{2} > 0$ and $\delta_2 = -\frac{y}{2} > 0$.

For any $x^+ \in [x, x+\delta_1)$, $|F(x,y) - F(x^+, y)| = |0-0| = 0 < \epsilon$

and for any $y^+ \in [y, y+\delta_2)$, $|F(x, y^+) - F(x,y)| = |0-0| = 0 < \epsilon$.

Hence, F is a continuous from above function.

Therefore, F is a 2-dimensional distribution function.

If $F_1 : \mathbb{R} \to [0,1]$ is defined by

$$F_1(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

and $F_2 : \mathbb{R} \to [0,1]$ is defined by

$$F_2(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ 0 & \text{if } y < 0, \end{cases}$$

then F_1 and F_2 are marginal distribution functions of F. □

2.2 Probability Measures

Definition 2.2.1. Let Ω be a nonempty set and 2^Ω denote the power set of Ω. A class $\Sigma \subseteq 2^\Omega$ is called a *σ-algebra* on Ω if it satisfies the following properties:

i) $\emptyset \in \Sigma$,

ii) if $E \in \Sigma$, then $E^C = \Omega \setminus E \in \Sigma$, and

iii) if $E_1, E_2, E_3, \ldots \in \Sigma$, then $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

The ordered pair (Ω, Σ) is called a *measurable space* and the elements of Σ are called *measurable sets.*

Let Ω be a nonempty set. For any $\Lambda \subseteq 2^{\Omega}$, denote the intersection of all σ-algebras containing Λ by $\sigma(\Lambda)$. Note that $\sigma(\Lambda)$ is the smallest σ-algebra containing Λ.

Definition 2.2.2. Let (X, Σ_X) and (Y, Σ_Y) be two measurable spaces.
A function $f : X \to Y$ is said to be *measurable function* if $f^{-1}(E) \in \Sigma_X$ for all $E \in \Sigma_Y$.

Definition 2.2.3. Let $\Omega \subseteq \mathbb{R}^n$ where $n \in \mathbb{N}$ and $\mathscr{O}$ be the set of all open subsets of Ω. Then $\sigma(\mathscr{O})$ is called *the Borel σ-algebra on* Ω which specifically is denoted by $\mathscr{B}(\Omega)$. The elements of $\mathscr{B}(\Omega)$ are called *Borel sets.*

Definition 2.2.4. Let Ω be a nonempty set, Σ be a σ-algebra on Ω and $\mathscr{I} \subseteq \mathbb{N}$. A function $\mu : \Sigma \to [0,1]$ is called a *probability measure* if it satisfies the following properties:

i) $\mu(\Omega) = 1$, and

ii) for any countable collection $\{E_i\}_{i \in \mathscr{I}}$ of elements in Σ such that $E_j \cap E_k = \emptyset$ when $j, k \in \mathscr{I}$ and $j \neq k$, $\mu\left(\bigcup_{i \in \mathscr{I}} E_i\right) = \sum_{i \in \mathscr{I}} \mu(E_i)$.

A *probability space* is a triplet (Ω, Σ, μ).

Definition 2.2.5. Let $\mathscr{B}([0,1])$ be the Borel σ-algebra on $[0,1]$,

$$\Gamma = \{(a,b] \subseteq [0,1] \,|0 \leq a < b \leq 1 \} \cup \{\emptyset\},$$

and $\tau : \Gamma \to [0,1]$ be defined by $\tau(A) = \begin{cases} b-a & \text{if } A = (a,b], \\ 0 & \text{if } A = \emptyset. \end{cases}$

A function $\lambda : \mathscr{B}([0,1]) \to [0,1]$ defined by

$$\lambda(A) = \inf\left\{ \sum_{n=1}^{\infty} \tau(D_n) \,| A \subseteq \bigcup_{n=1}^{\infty} D_n, (D_n)_{n=1}^{\infty} \subset \Gamma \right\}$$

is called the *Lebesgue measure on* $[0,1]$.

Remark 2.2.1. *λ is a probability measure on* $[0,1]$.

Theorem 2.2.2. *Let Ω be a nonempty set and $\Lambda \subseteq 2^{\Omega}$ be nonempty and closed under finite intersections. If P_1 and P_2 are probability measures on $\sigma(\Lambda)$ such that $P_1 = P_2$ on Λ, then $P_1 = P_2$ on $\sigma(\Lambda)$.*

Definition 2.2.6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A *random variable* is a Borel measurable function from Ω to $\mathbb{R}$. A *random vector* is a Borel measurable function from Ω to $\mathbb{R}^n$.

Definition 2.2.7. For any random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, its *distribution function* is a function F_X defined by

$$F_X(x) = \mathbb{P}(X \leq x)$$

for all $x \in \mathbb{R}$.

Definition 2.2.8. For any random vector $(X_1, \ldots, X_n)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, its *(joint) distribution function* is a function H defined by

$$H(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

for all $x_i \in \mathbb{R}$ where $i = 1, \ldots, n$.

Here and henceforth, we define $(x_1, ..., x_n) < (y_1, ..., y_n)$ if $x_i < y_i$ for all $i = 1, ..., n$ and $(x_1, ..., x_n) \leq (y_1, ..., y_n)$ if $x_i \leq y_i$ for all $i = 1, ..., n$.
Denote also $(\vec{a}, \vec{b}] = (a_1, b_1] \times ... \times (a_n, b_n]$ where $\vec{a} = (a_1, ..., a_n)$ and $\vec{b} = (b_1, ..., b_n)$.

Definition 2.2.9. Let X be a random vector with the F_X.
A random vector X is said to be *(absolutely) continuous* if there is a function f_X such that $F_X(x) = \int_{(-\infty,x]} f_X \mathrm{d}\lambda$ for all $x \in \mathbb{R}$. A function f_X is called the density function of F_X.

Definition 2.2.10. An n-dimensional random vector U is said to be *uniform* if its distribution function F_U is the product function Π, that is,

$$F_U(\vec{u}) = \Pi(\vec{u}) = \Pi_{i=1}^n u_i$$

for all $\vec{u} = (u_1, ..., u_n) \in [0, 1]^n$.

Proposition 2.2.3. *Let X and Y be random vectors with distribution functions F_X and F_Y, respectively. Then the three following properties are equivalent:*

i) X and Y are independent,

ii) $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for all measurable sets A and B, and

iii) $F_{X,Y}(\vec{x}, \vec{y}) = F_X(\vec{x})F_Y(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Definition 2.2.11. A random vector Y is said to be *completely dependent* on a random vector X if Y takes only one value for each value of X with probability one, that is, there is a measurable function f such that $Y = f(X)$ almost everywhere.

Definition 2.2.12. Let X and Y be random vectors.
The *conditional distribution function* $F_{Y|X}$ of Y given X is defined by

$$\begin{aligned} F_{Y|X}(y|x) &= \lim_{h \searrow 0} \frac{\mathbb{P}(x-h < X \le x+h, Y \le y)}{\mathbb{P}(x-h < X \le x+h)} \\ &= \lim_{h \searrow 0} \frac{V_{F_{X,Y}}\left((x-h, x+h] \times (-\infty, y]\right)}{V_{F_X}((x-h, x+h])}. \end{aligned}$$

Lemma 2.2.4. *Let X and Y be random vectors. Given the distribution function F_Y of Y and the conditional distribution function $F_{Y|X}$ of Y given X. Then*

$$\int F_{Y|X}(y|x) \mathrm{d}F_X(x) = F_Y(y).$$

Proof. Denote $\mu_y(A) = \mathbb{P}(Y \le y | X \in A)$ and $\mu(A) = \mathbb{P}(X \in A)$.
Then μ_y is absolutely continuous with respective to μ.
By Lebesgue point theorem, $\lim\limits_{h \searrow 0} \frac{1}{\mu(\mathcal{B}(x,\epsilon))} \int_{\mathcal{B}(x,\epsilon)} \frac{d\mu_y}{d\mu} d\mu = \frac{\mu_y(\mathcal{B}(x,\epsilon))}{\mu(\mathcal{B}(x,\epsilon))}$ converges to $\frac{d\mu_y}{d\mu}(x)$.
Thus, $F_{Y|X}(y|x) = \frac{d\mu_y}{\dagger\mu}(x.)$ Therefore,

$$\begin{aligned} \int F_{Y|X}(y|t) dF_X(t) &= \int \frac{d\mu_y}{d\mu}(t) d\mu(t) \\ &= \mu(\mathbb{R}^n) \\ &= F_Y(y). \end{aligned}$$

□

Lemma 2.2.5. *Let X and Y be random vectors. Given the distribution function F_Y of Y and the conditional distribution function $F_{Y|X}$ of Y given X. Then X and Y are independent if and only if $F_{Y|X} = F_Y$ a.e. with respect to the product of the distribution of X and Y.*

Proof. Let X and Y be independent random vectors, we have

$$\begin{aligned} F_{Y|X}(y|x) &= \lim_{h \searrow 0} \frac{V_{F_{X,Y}}\left((x-h, x+h] \times (-\infty, y]\right)}{V_{F_X}\left((x-h, x+h]\right)} \\ &= \lim_{h \searrow 0} \frac{V_{F_X}(x-h, x+h] V_{F_Y}(-\infty, y]}{V_{F_X}\left((x-h, x+h]\right)} \\ &= \lim_{h \searrow 0} V_{F_Y}(-\infty, y] \\ &= F_Y(y). \end{aligned}$$

Conversely, we show that X and Y are independent when $F_{Y|X} = F_Y$.
Let $F_{Y|X} = F_Y$ and $t \in \mathbb{R}$. Consider

$$\begin{aligned} F_{X,Y}(x,t) &= \mathbb{P}(X \leq x, Y \leq t) \\ &= \int_{(-\infty,t]} \mathbb{P}(Y \leq y | X = x) \mathrm{d}F_X(x) \\ &= \int_{(-\infty,t]} F_{Y|X}(y|x) \mathrm{d}F_X(x) \\ &= \int_{(-\infty,t]} F_Y(y) \mathrm{d}F_X(x) \\ &= F_X(x)F_Y(t) \quad \text{a.e.} \end{aligned}$$

By Proposition 2.2.3, we obtain that X and Y are independent. □

2.3 Copulas

Definition 2.3.1. An *n-dimensional copula* or *n-copula* is a continuous function $C : [0,1]^n \to [0,1]$ satisfying the following properties:

i) C is an n-increasing,

ii) C is grounded, that is, $C(u_1, ..., u_n) = 0$ whenever $u_i = 0$ for some $i = 1, ..., n$, and

iii) C has uniform marginals, that is, $C(u_1, ..., u_n) = u_i$ whenever $u_j = 1$ for all $j \neq i$.

Example 2.3.1. Let $\vec{u} = (u_1, ..., u_n) \in [0,1]^n$. The function $\Pi^n : [0,1]^n \to [0,1]$ defined by $\Pi^n(u_1, ..., u_n) = u_1 \cdots u_n$ is an n-copula.

Proof. Let $\vec{u} = (u_1, \ldots, u_n) \in [0,1]^n$.

i) If $\vec{u}$ has at least one coordinate which is equal to 0, then $\Pi^n(\vec{u}) = 0$.

ii) If all coordinates of $\vec{u}$ are equal to 1 except possibly u_k, then $\Pi^n(\vec{u}) = u_k$.

iii) Next, we show that Π^n is n-increasing.
Let $\left(\vec{a}, \vec{b}\right] = (a_1, b_1] \times \cdots \times (a_n, b_n]$.

Since $b_i - a_i \geq 0$ for all $i = 1, ..., n$ and

$$\begin{aligned}
(b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) &= \sum_{\forall i=1,\dots,n; v_i \in \{b_i, -a_i\}} v_1 \cdots v_n \\
&= \sum_{\forall i=1,\dots,n; v_i \in \{b_i, -a_i\}} \Pi^n(v_1, \cdots, v_n) \\
&= \sum_{\vec{v} \in \prod_{i=1}^{n}\{b_i, -a_i\}} \Pi^n(\vec{v}) \\
&= \sum_{\vec{v} \in \prod_{i=1}^{n}\{b_i, a_i\}} (-1)^{N(\vec{v})}\, \Pi^n(\vec{v}) \\
&= V_{\Pi^n}\left(\left(\vec{a}, \vec{b}\right]\right),
\end{aligned}$$

it follows that $V_{\Pi^n}\left(\left(\vec{a}, \vec{b}\right]\right) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) \geq 0$.
Hence, Π^n is n-increasing.

By i)−iii), we obtain that Π^n is an n-copula. □

Theorem 2.3.2. *[6] Let H be an n-dimensional distribution function of a random vector $(X_1, ..., X_n)$ with marginals $F_1, ..., F_n$. Then there exists an n-copula C such that*

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \tag{2.1}$$

for all $x_1, ..., x_n \in \mathbb{R}$.
Any copula C satisfying equation (2.1) is said to be associated with $(X_1, ..., X_n)$.

If $F_1, ..., F_n$ are all continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F_1) \times ... \times Ran(F_n)$ where $Ran(F)$ is the range of F.

Conversely, if C is an n-copula and $F_1, ..., F_n$ are distribution functions, then the function H defined by (2.1) is an n-dimensional distribution.

A joint distribution function of uniform random vectors is called a *linkage*.

Note that a linkage is always a copula.

For each distribution function F_X of an absolutely continuous random vector $X = (X_1, ..., X_n)$, define a *transformation* $\Psi_{F_X} : \mathbb{R}^n \to [0,1]^n$ by letting

$$\Psi_{F_X}(x_1, ..., x_n) = \left(F_{X_1}(x_1), F_{X_2|X_1}(x_2|x_1), ..., F_{X_n|(X_1,...,X_{n-1})}(x_n|(x_1, ..., x_{n-1}))\right)$$

for all $(x_1, ..., x_n) \in \mathbb{R}^n$. It is known that $U = \Psi_{F_X}(X)$ has uniform distribution (see in [5]). H. Li, M. Scarsini, and M. Shaked [5] showed that for each random variables $X_1, ..., X_n$ with an absolutely continuous joint distribution F, these random variables $U_1, ..., U_n$ are independent uniform $[0,1]$ whenever $(U_1, ..., U_n) = \Psi_F(X_1, ..., X_n)$. Similarly, if the univariate random variable X has the continuous distribution function F, then $F(X)$ is a uniform $[0,1]$ random variable.

By inverting, Ψ_F can express the X's as functions of the independent uniform random variables $U_1, ..., U_n$. Denote

$$x_1 = F^{-1}(u_1) \tag{2.2}$$

and, by induction,

$$x_i = F^{-1}_{i|1,...,i-1}(u_i|x_1, ..., x_{i-1}), \qquad i = 2, ..., n. \tag{2.3}$$

Consider the transformation $\Psi^* : [0,1]^n \to \mathbb{R}^n$ defined by (here the x_i's are functions of the u_i' s as given in (2.2) and (2.3))

$$\Psi^*_F(u_1, ..., u_n) = (x_1, ..., x_n), \qquad (u_1, ..., u_n) \in [0,1]^n.$$

Let

$$(\hat{X}_1, ..., \hat{X}_n) = \Psi^*_F(U_1, ..., U_n).$$

Then $(\hat{X}_1, ..., \hat{X}_n)$ has the same distribution as $(X_1, ..., X_n)$.

In fact, it is well known, if F is absolutely continuous, then

$$\Psi^*_F \Psi_F(X_1, X_2, ..., X_n) =_{\text{a.s.}} (X_1, X_2, ..., X_n),$$

where $=_{\text{a.s.}}$ denotes an equality almost surely under the probability measure associated with F.

Since $U_i = \Psi_i(X_i)$ is uniform, the joint distribution of $U_1, ..., U_k$ is a linkage. We called this linkage, the *linkage associated with* $X_1, ..., X_k$ and will be denoted by $C_{X_1,...,X_k}$.

We will also denote $C_{X_k|X_1,...,X_{k-1}}$ the conditional distribution function of U_k given $(U_1, ..., U_{k-1})$.

For further information on linkages, see [5].

Example 2.3.3. Let X and Y be random vectors with dimensions m and n, respectively, with FGM-copula

$$C_\theta(\vec{u}, \vec{v}) = \Pi(\vec{u})\Pi(\vec{v}) + \theta\Pi(\vec{u})\Pi(\vec{v})\Pi(\vec{1} - \vec{u})\Pi(\vec{1} - \vec{v})$$

where $\theta \in [-1, 1]$ as their joint distribution.

Then the marginals F_X and F_Y of C_θ are defined by $F_X(\vec{u}) = \Pi(\vec{u})$ and $F_Y(\vec{v}) = \Pi(\vec{v})$. Therefore, X and Y are uniform random vectors, that is, C_θ is the linkage.

Measure of Complete Dependence

Let X and Y be two random variables. The random variable Y is *completely dependent* on X if there is a Borel measurable function f such that $\mathbb{P}(Y = f(X)) = 1$. If Y is completely dependent on X, and X is completely dependent on Y, then X and Y are called *mutually completely dependent* (m.c.d.).

In 2010, Siburg and Stoimenov [1] defined a measure of dependence

$$\begin{aligned}\omega(X,Y) &= \sqrt{3\int_{[0,1]^2}\left[\left(\frac{\partial}{\partial x}C_{X,Y}(x,y)-y\right)^2+\left(\frac{\partial}{\partial y}C_{X,Y}(x,y)-x\right)^2\right]\mathrm{d}x\mathrm{d}y} \\ &= \sqrt{3\int_{[0,1]^2}\left[\left(\frac{\partial}{\partial x}C_{X,Y}(x,y)\right)^2+\left(\frac{\partial}{\partial y}C_{X,Y}(x,y)\right)^2\right]\mathrm{d}x\mathrm{d}y-2}\end{aligned}$$

for any continuous random variables X and Y. In their works, it was shown that, for any random variables X and Y with continuous distribution functions, $\omega(X,Y)$ has the following properties:

i) $\omega(X,Y) = \omega(Y,X)$,

ii) $0 \le \omega(X,Y) \le 1$,

iii) $\omega(X,Y) = 0$ if and only if X and Y are independent,

iv) $\omega(X,Y) = 1$ if and only if X and Y are m.c.d.,

v) $\omega(X,Y) \in (\sqrt{2}/2, 1]$ if Y is completely dependent on X (or vice versa),

vi) If $f, g : \mathbb{R} \to \mathbb{R}$ are strictly monotone functions, then $\omega(f(X), g(Y)) = \omega(X,Y)$, and

vii) If $(X_n, Y_n)_{n\in\mathbb{N}}$ is a sequence of pairs of random variables with continuous marginal distribution functions and copulas $(C_n)_{n\in\mathbb{N}}$ and if $\lim_{n\to\infty} ||C_n - C|| = 0$, then $\lim_{n\to\infty} \omega(X_n, Y_n) = \omega(X,Y)$ where $||.||$ is a modified Sobolev norm and C is a copula associated with random variables X and Y.

Let $\mathfrak{F}$ be the set of all bivariate distribution functions with continuous marginal distribution functions, as well as $\mathfrak{X}$ be the set of all bivariate random vectors with distribution functions in $\mathfrak{F}$. Dette et al. [2] defined a measure $r : \mathfrak{X} \to [0,1]$ via the following formula

$$r(X,Y) = 6\int_0^1\int_0^1 F_{V|U}(v|u)^2\mathrm{d}v\mathrm{d}u - 2.$$

Since $F_{V|U}(v|u) = \frac{\partial}{\partial u} C_{X,Y}(u, v)$, we obtain that

$$r(X, Y) = 6 \left\| \frac{\partial}{\partial u} C_{X,Y} \right\|_2^2 - 2$$

where $|| \cdot ||$ denotes the L^2-norm.

In their works, it was shown that, if $f, g : \mathbb{R} \to \mathbb{R}$ are strictly monotone functions, then $r(f(X), g(Y)) = r(X, Y)$.

Moreover, r can be viewed as a functional on the set of copulas $\mathfrak{C}$, and we write $r(C_{X,Y}) = r(X, Y)$ which can be considered as an asymmetric version of $\omega(X, Y)$, that is, $\omega(X, Y) = \sqrt{\frac{r(X, Y) + r(Y, X)}{2}}$.

Let $\mathcal{C}$ be the set of all two-dimensional copulas and $\mathcal{C}_d$ be the class of all completely dependent copulas. During the same period, Trutschnig [3] also defined a measure of complete dependence ζ_1 based on the Markov kernel associated with X and Y.
When written using the copula $C_{X,Y}$ associated with X and Y,

$$\zeta_1(Y|X) = 3 \int \int \left| \frac{\partial}{\partial x} C_{X,Y}(x, y) - y \right| \mathrm{d}x\mathrm{d}y$$

and showed that for every $C \in \mathcal{C}$, $\zeta_1(C) \in [0, 1]$. Furthermore, $\zeta_1(C) = 1$ if and only if $C \in \mathcal{C}_d$, that is, all completely dependent copulas have a maximum dependence measure.

Both r and ζ_1 are similar in the sense that both measures reach the maximum value if and only if the random variable Y is a measurable function of the random variable X.

Recently, Tasena and Dhompongsa [4] also defined measures of complete dependence

$$\omega_k(Y|X) = \left[\int \int \left| F_{Y|X}(y|x) - \frac{1}{2} \right|^k \mathrm{d}F_X(x)\mathrm{d}F_Y(y) \right]^{\frac{1}{k}}$$

and

$$\overline{\omega}_k(Y|X) = \left[\frac{\omega_k^k(Y|X) - \hat{\omega}_k(Y)}{\omega_k^k(Y|Y) - \hat{\omega}_k(Y)} \right]^{\frac{1}{k}}$$

where $\hat{\omega}_k(Y) = \int \left| F_Y(y) - \frac{1}{2} \right|^k \mathrm{d}F_Y(y)$ for any random vectors X and Y.

In their works, it was shown that ω_k has the following properties:

i) $\omega_k(Y^\perp|X^\perp) \leq \omega_k(Y|X) \leq \omega_k(Y|Y)$ where $(Y^\perp, X^\perp)$ have the same marginals as (Y, X) but $X^\perp$ and $Y^\perp$ are independent,

ii) $\omega_k(Y^\perp|X^\perp) = 0$ and $\omega_k(Y|Y) = 1$,

iii) $\omega_k(Y^\perp|X^\perp) = \omega_k(Y|X)$ if and only if X and Y are independent,

iv) $\omega_k(Y|X) = \omega_k(Y|Y)$ if and only if Y is a function of X,

v) $\omega_k(Y, Y, Z|X) = \omega_k(Y, Z|X)$ for all random vectors X, Y, and Z,

vi) $\omega_k(Y|X, X, Z) = \omega_k(Y|X, Z)$ for all random vectors X, Y, and Z,

vii) $\omega_k(Y|X, Z) \geq \omega_k(Y|X)$ for all random vectors X, Y, and Z,

viii) $\omega_k(Y, f(X)|X) \geq (Y|X)$ for all measurable functions f and random vectors X and Y,

ix) $\omega_k(Y|f(X)) \leq \omega_k(Y|X)$ for all measurable functions f and random vectors X and Y,

while $\overline{\omega}_k(Y|X)$ satisfies the properties:

i) $\overline{\omega}_k(Y^{\perp}|X^{\perp}) \leq \overline{\omega}_k(Y|X) \leq \overline{\omega}_k(Y|Y)$ where $(Y^{\perp}, X^{\perp})$ have the same marginals as (Y, X) but $X^{\perp}$ and $Y^{\perp}$ are independent,

ii) $\overline{\omega}_k(Y^{\perp}|X^{\perp}) = 0$ and $\overline{\omega}_k(Y|Y) = 1$,

iii) $\overline{\omega}_k(Y^{\perp}|X^{\perp}) = \overline{\omega}_k(Y|X)$ if and only if X and Y are independent,

iv) $\overline{\omega}_k(Y|X) = \overline{\omega}_k(Y|Y)$ if and only if Y is a function of X,

v) $\overline{\omega}_k(Y, Y, Z|X) = \overline{\omega}_k(Y, Z|X)$ for all random vectors X, Y, and Z,

vi) $\overline{\omega}_k(Y|X, X, Z) = \overline{\omega}_k(Y|X, Z)$ for all random vectors X, Y, and Z,

vii) $\overline{\omega}_k(Y|X, Z) \geq \overline{\omega}_k(Y|X)$ for all random vectors X, Y, and Z, and

viii) $\overline{\omega}_k(Y|f(X)) \leq \overline{\omega}_k(Y|X)$ for all measurable functions f and random vectors X and Y. Moreover, if f is invertible, then $\overline{\omega}_k(Y|f(X)) = \overline{\omega}_k(Y|X)$.

Moreover,

$$\overline{\omega}_2(Y|X) = \sqrt{\frac{\int\int \left(F_{Y|X}(y|x) - F_Y(y)\right)^2 \mathrm{d}F_X(x)\mathrm{d}F_Y(y)}{\int F_Y(y)(1 - F_Y(y))\mathrm{d}F_Y(y)}}.$$

It can be easily seen that both ω_k and $\overline{\omega}_k$ depend on F_Y except only for the case Y is a continuous random variable. In this case, we can use the change of variable formula to nullity the effect of F_Y. Also,

$$\overline{\omega}_k(Y|X) \neq \left[\frac{\int\int \left|F_{Y|X}(y|x) - F_Y(y)\right|^k \mathrm{d}F_X(x)\mathrm{d}F_Y(y)}{\omega_k^k(Y|Y) - \hat{\omega}_k(Y)}\right]^{\frac{1}{k}}$$

in general.