CHAPTER 4

Examples

In this chapter, we give four computational examples for the measure of complete dependences φ and two examples for the measure of complete dependence ζ_p.

4.1 Example Calculations of Measure of Complete Dependence

First, we find $\varphi(Y|Y)$ where Y is a geometric distribution function.
Second, we compute $\varphi(Y|X)$ and $\varphi(Y|Y)$ in the case X and Y are continuous random vectors with FGM-copula.
Next, we show that $\varphi(H_\alpha|X_\alpha) = \frac{1}{\alpha}\varphi(F|X_1) + \frac{\alpha-1}{\alpha}\varphi(G|X_2)$ whenever F and G are joint distribution functions with the same marginals such that $F(1,\vec{v}) = G(1,\vec{v})$ and

$$H_\alpha(u,\vec{v}) = \begin{cases} \frac{1}{\alpha}F(\alpha u,\vec{v}) & \text{if } \; u \leq \frac{1}{\alpha}, \\ \\ \frac{1}{\alpha}F(1,\vec{v}) + \frac{\alpha-1}{\alpha}G(\frac{\alpha u-1}{\alpha-1},\vec{v}) & \text{otherwise.} \end{cases}$$

Finally, we compute $\varphi(C_{X,Y})$ where X and Y are continuous random vectors and $C_{X,Y}$ is the joint Archimedean copula.

For another measure of complete dependence ζ_p, there are two examples to be considered in this part. We start with the computation of $\zeta_p(Y|X)$ when X and Y are random vectors with FGM-copula.
Furthermore, we determine $\zeta_p(Y|X)$ where (X,Y) is normally distributed with mean zero and covariance matrix $\begin{pmatrix} I & P \\ P^t & I \end{pmatrix}$.

Example 4.1.1. If Y is a geometric distribution function, then

$$\varphi(Y|Y) = \frac{2(p-1)(p^2-2p+2)}{(p-2)p(p^2-3p+3)}.$$

Proof. Since Y is a geometric distribution function, we get $\mathbb{P}(Y=y)=(1-p)^y p$. Then

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \sum_{i=0}^{y} \mathbb{P}(Y=i) \\ &= p\sum_{i=0}^{y}(1-p)^i \\ &= \frac{p\left(1-(1-p)^{y+1}\right)}{1-(1-p)} \\ &= 1-(1-p)^{y+1}. \end{aligned}$$

Since $\varphi(Y|Y) = 2\int F_Y(y)\,(1-F_Y(y))\,\mathrm{d}F_Y(y)$, we can conclude that

$$\begin{aligned} \varphi(Y|Y) &= 2\int (1-(1-p)^{y+1})(1-p)^{y+1}\mathrm{d}F_Y(y) \\ &= 2\sum_{y=0}^{\infty}\left[(1-(1-p)^{y+1})(1-p)^{y+1}\right]\left(1-(1-p)^{y+1}\right) \\ &= 2\sum_{y=0}^{\infty}\left(1-2(1-p)^{y+1}+(1-p)^{2y+2}\right)(1-p)^{y+1} \\ &= 2\left\{\sum_{y=0}^{\infty}(1-p)^{y+1}-2\sum_{y=0}^{\infty}(1-p)^{2y+2}+\sum_{y=0}^{\infty}(1-p)^{3y+3}\right\} \\ &= 2\left\{\frac{1-p}{1-(1-p)}-2\left(\frac{(1-p)^2}{1-(1-p)^2}\right)+\left(\frac{(1-p)^3}{1-(1-p)^3}\right)\right\} \\ &= \frac{2(p-1)(p^2-2p+2)}{(p-2)p(p^2-3p+3)}. \end{aligned}$$

□

Example 4.1.2. Let X and Y be continuous random vectors with the FGM-copula

$$C_\theta(u,v) = \left(\prod_{i=1}^{m} u_i\right)\left(\prod_{j=1}^{n} v_j\right) + \theta\left(\prod_{i=1}^{m} u_i(1-u_i)\right)\left(\prod_{j=1}^{n} v_j(1-v_j)\right)$$

which is their joint distribution.

Then

$$\begin{aligned}
\varphi(Y|X) &= \int \left|F_{V|U}(v|u) - F_Y(v)\right| \mathrm{d}F_X(u)\mathrm{d}F_Y(v) \\
&= \int \left|\theta\left(\prod_{i=1}^{m}(1-2u_i)\right)\left(\prod_{j=1}^{n} v_j(1-v_j)\right)\right| \mathrm{d}u\mathrm{d}v \\
&= |\theta| \int \prod_{i=1}^{m}(1-2u_i)\mathrm{d}u \int \prod_{j=1}^{n} v_j(1-v_j)\mathrm{d}v \\
&= |\theta| \left(\int |1-2u|\mathrm{d}u\right)^m (v(1-v)\mathrm{d}v)^n \\
&= |\theta| \left(\int_{u\le\frac{1}{2}} (1-2u)\mathrm{d}u + \int_{u>\frac{1}{2}} (2u-1)\mathrm{d}u\right)^m \left(\int (v-v^2)\mathrm{d}v\right)^n \\
&= |\theta| \left(\frac{1}{2}\right)^m \left(\frac{1}{6}\right)^n \\
&= \frac{|\theta|}{2^m 6^n}.
\end{aligned}$$

Since $F_Y(v) = \prod_{i=1}^{n} v_i$, we obtain that

$$\begin{aligned}
\varphi(Y|Y) &= 2\int \prod_{i=1}^{n} v_i \left(\prod_{i=1}^{n} v_i\right) \mathrm{d}v \\
&= 2\int \left(\prod_{i=1}^{n} v_i - (\prod_{i=1}^{n} v_i)^2\right) \mathrm{d}v \\
&= 2\left[(\int v\mathrm{d}v)^n - (\int v^2\mathrm{d}v)^n\right] \\
&= 2\left(\frac{1}{2^n} - \frac{1}{3^n}\right) \\
&= \frac{2(3^n - 2^n)}{6^n}.
\end{aligned}$$

Therefore,

$$\frac{\varphi(Y|X)}{\varphi(Y|Y)} = \frac{|\theta|}{2^{m+1}(3^n - 2^n)}.$$

Example 4.1.3. Let X and Y be absolutely continuous random vectors with dimensions m and n, respectively, with the FGM-copula

$$C_\theta(\vec{u},\vec{v}) = \Pi(\vec{u})\Pi(\vec{v}) + \theta\Pi(\vec{u})\Pi(\vec{1}-\vec{u})\Pi(\vec{v})\Pi(\vec{1}-\vec{v})$$

as its linkage. Consider

$$\begin{aligned}\frac{\partial}{\partial\vec{u}}C_\theta(\vec{u},\vec{v}) &= \frac{\partial}{\partial\vec{u}}\left[\Pi(\vec{u})\Pi(\vec{v}) + \theta\Pi(\vec{u})\Pi(\vec{1}-\vec{u})\Pi(\vec{v})\Pi(\vec{1}-\vec{v})\right] \\ &= \Pi(\vec{v}) + \theta\Pi(\vec{v})\Pi(\vec{1}-\vec{v})\frac{\partial}{\partial\vec{u}}\Pi(\vec{u})\Pi(\vec{1}-\vec{u}) \\ &= \Pi(\vec{v}) + \theta\left(\Pi_{i=1}^m|1-2u_i|\right)\left(\Pi_{j=1}^n v_j(1-v_j)\right).\end{aligned}$$

Then

$$\begin{aligned}\zeta_p(Y|X) &= \left[\int\left|\frac{\partial}{\partial\vec{u}}C_\theta(\vec{u},\vec{v}) - \Pi(\vec{v})\right|^p \mathrm{d}\vec{u}\mathrm{d}\vec{v}\right]^{\frac{1}{p}} \\ &= \left[\int\left|\theta\left(\Pi_{i=1}^m|1-2u_i|\right)\left(\Pi_{j=1}^n v_j(1-v_j)\right)\right|^p \mathrm{d}u_1...\mathrm{d}u_m\mathrm{d}v_1...\mathrm{d}v_n\right]^{\frac{1}{p}} \\ &= |\theta|\left[\int|\vec{1}-2\vec{u}|^p\mathrm{d}\vec{u}\right]^{\frac{m}{p}}\left[\int\vec{v}^p(\vec{1}-\vec{v})^p\mathrm{d}\vec{v}\right]^{\frac{n}{p}} \\ &= |\theta|\left(\frac{1}{p+1}\right)^{\frac{m}{p}}\beta(1+p,1+p)^{\frac{n}{p}}\end{aligned}$$

where $\beta(1+p,1+p) = \int_0^1 \vec{v}^p(\vec{1}-\vec{v})^p\mathrm{d}\vec{v}$ is the beta function.
Particularly, $\frac{|\theta|}{2^{m+n}3^n} \le \zeta_p(Y|X) \le \frac{|\theta|}{6^n}$.

Let F and G be joint distribution functions with the same marginals and $\alpha \in (1,\infty)$. Then the function H_α defined by

$$H_\alpha(u,\vec{v}) = \begin{cases}\frac{1}{\alpha}F(\alpha u,\vec{v}) & \text{if } u \le \frac{1}{\alpha}, \\ \frac{1}{\alpha}F(1,\vec{v}) + \frac{\alpha-1}{\alpha}G(\frac{\alpha u-1}{\alpha-1},\vec{v}) & \text{otherwise}\end{cases}$$

also has the same marginals as F and G.

Example 4.1.4. Let F and G be joint distribution functions with the same marginals such that $F(1,\vec{v}) = G(1,\vec{v})$. Let $(X_1,Y_1),(X_2,Y_2)$ and (X_α,Y_α) have the distribution functions F,G and H_α, respectively. Then

$$\begin{aligned}
\varphi(H_\alpha|X_\alpha) &= \int \left|\frac{\partial}{\partial u}H_\alpha(u,\vec{v}) - H_\alpha(1,\vec{v})\right| \mathrm{d}u\mathrm{d}H_\alpha(1,\vec{v}) \\
&= \int_{u\le\frac{1}{\alpha}} \left|\frac{1}{\alpha}\frac{\partial}{\partial u}F(\alpha u,\vec{v}) - \frac{1}{\alpha}F(1,\vec{v}) - \frac{\alpha-1}{\alpha}G(1,\vec{v})\right| \mathrm{d}u\mathrm{d}F(1,\vec{v}) \\
&\quad + \int_{u>\frac{1}{\alpha}} \left|\frac{\alpha-1}{\alpha}\frac{\partial}{\partial u}G(\frac{\alpha u-1}{\alpha-1},\vec{v}) - \frac{1}{\alpha}F(1,\vec{v}) - \frac{\alpha-1}{\alpha}G(1,\vec{v})\right| \mathrm{d}u\mathrm{d}G(1,\vec{v}) \\
&= \frac{1}{\alpha}\int \left|\frac{\partial}{\partial w}F(w,\vec{v}) - F(1,\vec{v})\right| \mathrm{d}w\mathrm{d}F(1,\vec{v}) \\
&\quad + \frac{\alpha-1}{\alpha}\int \left|\frac{\partial}{\partial x}G(x,\vec{v}) - G(1,\vec{v})\right| \mathrm{d}x\mathrm{d}G(1,\vec{v}) \\
&= \frac{1}{\alpha}\varphi(F|X_1) + \frac{\alpha-1}{\alpha}\varphi(G|X_2).
\end{aligned}$$

Example 4.1.5. Let X and Y be continuous random vectors with the joint Archimedean copula $C_{X,Y}$ of the form

$$C_{X,Y}(u_1,...,u_m,v_1,...,v_n) = \phi(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m) + \phi^{-1}(v_1) + ... + \phi^{-1}(v_n))$$

for some function ϕ.

In the case of $C_{X,Y}$ is Clayton copula we have $\phi_\theta(x) = (1+\theta x)^{-\frac{1}{\theta}}$ and
$\phi_\theta^{(m)}(x) = \theta^m \left(1 - m - \frac{1}{\theta}_m (\phi_\theta(x))^{m\theta+1}\right)$ where $(a)_m = a(a+1)...(a+m-1)$.
Consider

$$\begin{aligned}
C_{Y|X}(v|u) &= \frac{\phi^{(m)}(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m) + \phi^{-1}(v_1) + ... + \phi^{-1}(v_n))}{\phi^{(m)}(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m))} \\
&= \frac{\theta^m \left(1 - m - \frac{1}{\theta}\right)_m [\phi_\theta(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m) + \phi^{-1}(v_1) + ... + \phi^{-1}(v_n))]^{m\theta+1}}{\theta^m \left(1 - m - \frac{1}{\theta}\right)_m [\phi_\theta(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m))]^{m\theta+1}} \\
&= \left(\frac{C_\theta(u,v)}{C_\theta(u)}\right)^{m\theta+1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C_X}{\partial u}(u_1,...,u_m) &= \frac{\phi^{(m)}(\phi^{-1}(u_1) + ... + \phi^{-1}(u_m))}{\phi'(\phi^{-1}(u_1))...\phi(\phi^{-1}(u_m)))} \\
&= \frac{\theta^m \left(1 - m - \frac{1}{\theta}\right)_m (C_\theta(u))^{m\theta+1}}{(-u_1^{\theta+1})...(-u_m^{\theta+1})},
\end{aligned}$$

and

$$\frac{\partial C_Y}{\partial v}(v_1, ..., v_n) = \frac{\phi^{(n)}(\phi^{-1}(v_1) + ... + \phi^{-1}(v_n))}{\phi'(\phi^{-1}(v_1) + ... + \phi^{-1}(v_n))}$$
$$= \frac{\theta^n \left(1 - n - \frac{1}{\theta}\right)_n (C_\theta(v))^{n\theta+1}}{(-v_1^{\theta+1})...(-v_n^{\theta+1})}.$$

Therefore, $\zeta_1(C_{X,Y}) = \int \left| \frac{(C_\theta(u,v))^{1+m\theta}(C_\theta(v))^{1+n\theta} - (C_\theta(u))^{1+m\theta}(C_\theta(v))^{2+n\theta}}{(u_1...u_m v_1...v_n)^{\theta+1}} \right| \mathrm{d}u\mathrm{d}v.$

Example 4.1.6. Let X and Y be absolutely continuous random vectors with dimensions m and n, respectively, such that (X, Y) is normally distributed with mean zero and covariance matrix $\begin{pmatrix} I & P \\ P^t & I \end{pmatrix}$. Since components of X and Y are independent,

$$\zeta_p(Y|X) = \left[\int\int \left| F_{Y|X}(y|x) - F_Y(y) \right|^p \mathrm{d}F_X(x)\mathrm{d}F_Y(y) \right]^{\frac{1}{p}}$$

by the change of variable formula.

Given $X = x, Y$ is normally distributed with mean $P^t x$ and covariance matrix $I - P^t P$. Therefore,

$$F_{Y|X}(y|x) = \Phi_{I-P^tP}(y - P^t x)$$

where Φ_Σ is the normal distribution function with mean zero and covariance matrix Σ. Thus,

$$\zeta_p(Y|X) = \left[\int\int \left| \Phi_{I-P^tP}(y - P^t x) - \Phi_I(y) \right|^p \mathrm{d}\Phi_I(x)\mathrm{d}\Phi_I(y) \right]^{\frac{1}{p}}.$$

Particularly, $\zeta_2(Y|X) = \sqrt{\int\int \left| \Phi^2_{I-P^tP}(y - P^t x) \right| \mathrm{d}\Phi_I(x)\mathrm{d}\Phi_I(y) - \left(\frac{1}{3}\right)^n}$.

Since $\int\int \Phi^2_{I-P^tP}(y - P^t x)\mathrm{d}\Phi_I(x)\mathrm{d}\Phi_I(y)$ is the expectation of $\Phi^2_{I-P^tP}(Z - P^t W)$ where (Z, W) has standard normal distribution which directly implies $Z - P^t W$ is normally distributed with mean zero and covariance matrix $I + P^t P$,

$$\zeta_2(Y|X) = \sqrt{\int \Phi^2_{I-P^tP}(z)\mathrm{d}\Phi_{I+P^tP}(z) - \left(\frac{1}{3}\right)^n}.$$

Note that the same idea can also be extended to the case of X and Y with dependent components. For example, let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be such that (X, Y) is normally distributed with mean zero and covariance matrix

$$\begin{pmatrix} 1 & \rho_X & \rho & \rho \\ \rho_X & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho_Y \\ \rho & \rho & \rho_Y & 1 \end{pmatrix}.$$

Then the linkage associated with X and Y is the same as the linkage associated with a normal random vector with mean zero and covariance matrix

$$\begin{pmatrix} 1 & 0 & \rho & \sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho \\ 0 & 1 & \sqrt{\frac{1-\rho_X}{1+\rho_X}}\rho & \sqrt{\frac{1-\rho_X}{1+\rho_X}}\sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho \\ \rho & \sqrt{\frac{1-\rho_X}{1+\rho_X}}\rho & 1 & 0 \\ \sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho & \sqrt{\frac{1-\rho_X}{1+\rho_X}}\sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho & 0 & 1 \end{pmatrix}$$

(see Example 3.3 in [5]). Therefore, we can apply the above result to

$$P = \begin{pmatrix} \rho & \sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho \\ \sqrt{\frac{1-\rho_X}{1+\rho_X}}\rho & \sqrt{\frac{1-\rho_X}{1+\rho_X}}\sqrt{\frac{1-\rho_Y}{1+\rho_Y}}\rho \end{pmatrix}$$

to yield $\zeta_p(Y|X)$ in this case also.