CHAPTER 1

Introduction

1.1 The background of fixed point theory

Let X be a nonempty set and $T : X \to X$ be a mapping. A point $x \in X$ is said to be a *fixed point* of T if and only if $T(x) = x$. The set of all fixed points of T is denoted by $F(T)$. For example, if T is defined on the set of real numbers by $T(x) = x^2 - 2$ then $F(T) = \{-1, 2\}$. If the mapping T does not have a fixed point we often say that T is fixed point free.

The fixed point theory is the most important tool to solve a problem in many branches of science and new technologies. This is because many practical problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings.

One of the first and most celebrated results on this matter is the one proved by Brouwder [6] in 1912.

Theorem 1.1.1. (Brower's fixed point theorem) *Let B be a closed ball in $\mathbb{R}^n$, then each continuous mapping $T : B \to B$ has a fixed point.*

The underlying conditions behind Brouwer's theorem are the compactness and convexity of the unit ball of $\mathbb{R}^n$, Schauder [48] used them to proved an extension theorem in 1930.

Theorem 1.1.2. (Schauder's fixed point theorem) *Let X be a Banach space. If C is a nonempty compact convex subset of X, then each continuous mapping $T : C \to C$ has a fixed point.*

Let (X, d) be a metric space. Recall that a mapping $T : X \to X$ is said to be *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in X$, and a mapping $T : X \to X$ is called *nonexpansive* if $d(T(x), T(y)) \leq d(x, y)$ for all $x, y \in X$.

In 1992, Banach [3] obtained a fixed point theorem which has turned out to be a very powerful and useful tool in Mathematics, and probably be the most frequently cited in literature.

Theorem 1.1.3. (The Banach contraction principle) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed point x_0 in X. Moreover, for each $x \in X$, we have*

$$\lim_{n\to\infty} T^n(x) = x_0.$$

The existence of fixed points for nonexpansive mappings in Banach spaces was studied independently by three authors in 1965 (see Browder [7], Göhde [18], and Kirk [29]). They showed that every nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space always has a fixed point.

Theorem 1.1.4. (Browder's theorem) *Let C ba a convex bounded closed subset of a uniformly convex Banach space and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point.*

Many researchers have been tried to find conditions guaranteeing the existence of fixed points for nonexpansive mappings. Moreover, the fixed point theory for mappings which are more general than nonexpansive mappings is also studied.

Several authors have studied methods for the iterative approximation of fixed points of nonlinear mappings. The following classical iteration methods are often used to approximate a fixed point of a mapping T.

In 1953, Mann [40] introduced an iteration scheme by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in the interval $[0, 1]$. This iteration is called *the Mann iteration process*. He proved, under some appropriate conditions, that the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T.

In 1974, Ishikawa [23] generalized the Mann iteration process as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n), \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. This iteration is called *the Ishikawa iteration process*. He proved, under some appropriate conditions, that the sequence defined by (1.2) converges weakly to a fixed point of T.

In 2008, Kirk and Xu [33] introduced the concept of asymptotic pointwise nonexpansive mappings which generalizes the concept of nonexpansive mappings and proved the existence of fixed points for such mappings in uniformly convex Banach spaces.

In 2011, Kozlowski [35] defined an iterative sequence for an asymptotic pointwise nonexpansive mapping T on a convex subset C of a Banach space X by

$$\begin{cases} x_1 \in C, \\ x_{k+1} = (1 - t_k)x_k + t_k T^{n_k}(y_k), \\ y_k = (1 - s_k)x_k + s_k T^{n_k}(x_k), \text{ for } k \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{t_k\}$ and $\{s_k\}$ are sequences in $[0, 1]$ and $\{n_k\}$ is an increasing sequence of natural numbers. This iteration is called *the generalized Ishikawa iteration process*. He proved, under some appropriate conditions, that the sequence defined by (1.3) converges weakly to a fixed point of T.

In 2010, Kocourek et al. [34] introduced the concept of generalized hybrid mappings in Hilbert spaces. Later on, Lin et al. [39] defined a generalized hybrid mapping, which is more general than that of Kocourek et al. This class of mappings properly contains the class of nonexpansive mappings. In [39], the authors also obtained the demiclosed principle, fixed point theorems as well as convergence theorems for generalized hybrid mappings in CAT(0) spaces.

In 2013, Salahifard et al. [47] introduced the concept of fundamentally nonexpansive mappings which generalizes the concept of nonexpansive mappings and proved that every fundamentally nonexpansive mappings on a bounded closed convex subset of a complete CAT(0) space always has a fixed point.

Fixed point theory in CAT(κ) spaces was first studied by Kirk [30, 31]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see e.g., [1, 11, 13, 14, 15, 19, 26, 28, 32, 37, 42, 45, 46, 47, 50, 54, 55]). Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see Theorem 2.4.12), all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \leq 0$. However, there are only a few articles that contain fixed point results in the setting of CAT(κ) spaces with $\kappa > 0$.

The purpose of this thesis are three folds. Firstly, we study convergence theorems of the Ishikawa iteration for asymptotic pointwise nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Secondly, we study fixed point theorems and convergence theorems of the Ishikawa iteration for generalized hybrid mapping in CAT(κ) spaces with $\kappa > 0$. Finally, we study fixed point theorems and convergence theorems of the Ishikawa iteration for fundamentally nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Our results extend and improve several results in the literature.

This thesis is divided in to 5 chapters. Chapter 1 is an introduction to the research problems. In Chapter 2, we collect some basic definitions and results which are needed in later chapters. Chapter 3, we give some convergence theorems of the Ishikawa iteration for asymptotic pointwise nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Chapter 4, we prove some fixed point theorems and convergence theorems of the Ishikawa iteration for generalized hybrid mappings in CAT(κ) spaces with $\kappa > 0$. Chapter 5, we prove fixed point theorems and convergence theorems of the Ishikawa iteration for fundamentally nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$.