CHAPTER 2

Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Metric spaces

Definition 2.1.1. Let X be a set and d be a function from $X \times X$ to $[0, \infty)$ such that for all $x, y, z \in X$ we have

(M1) $d(x, y) = 0$ if and only if $x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

A function d satisfying the above conditions is said to be a *distance function* or a *metric* and the pair (X, d) a *metric space*. We sometimes write X for a metric space (X, d).

Example 2.1.1. The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric d is called the *usual metric* for $\mathbb{R}$.

Example 2.1.2. Let X be a nonempty set. Define a metric d on X by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a metric space, called a *discrete space*.

Example 2.1.3. Let X be the set of all continuous functions from a closed interval $[a, b]$ to $\mathbb{R}$. We define a metric d by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \text{ for all } f, g \in X.$$

Then (X, d) is a metric space and usually denoted by $C[a, b]$.

Definition 2.1.2. A sequence $\{x_n\}$ in (X, d) is said to *converge* to a point $x \in X$ if for each $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. In this case we write either $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.1.3. A sequence $\{x_n\}$ in a metric space (X, d) is said to be a *Cauchy sequence* if for each $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Theorem 2.1.4. (cf. [25]) *Every convergent sequence in a metric space is a Cauchy sequence.*

Definition 2.1.5. A metric space X is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 2.1.6. Given a point $x_0 \in X$ and a real number $r > 0$, we define

(i) $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$, the *open ball* with center x_0 and redius r;

(ii) $\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$, the *closed ball* with center x_0 and redius r;

(iii) $S(x_0, r) = \{x \in X \mid d(x, x_0) = r\}$, the *sphere* with center x_0 and redius r.

Definition 2.1.7. Let (X, d) be a metric space. A set $G \subseteq X$ is called an *open set* if for every $x \in G$, there is $r > 0$ such that $B(x, r) \subseteq G$. A set $F \subseteq X$ is called a *closed set* if its complement is open. A set $C \subseteq X$ is called a *compact set* if any sequence $\{x_n\}$ in C has a subsequence $\{x_{n_k}\}$ which converges to a point in C.

Theorem 2.1.8. (cf. [25]) *A subset C of a metric space X is closed if and only if*

$$\{x_n\} \subset C \text{ and } \lim_{n\to\infty} x_n = x \text{ imply } x \in C.$$

Lemma 2.1.9. (cf. [25]) *A compact subset C of a metric space is closed and bounded.*

Definition 2.1.10. (cf. [53]) Let X and Y be metric spaces and let f be a mapping from X into Y. Then f is said to be *continuous* at x_0 in X if

$$x_n \to x_0 \implies f(x_0) \to f(x_0).$$

Moreover, f is said to be *continuous on* X if it is continuous at every point of X.

Theorem 2.1.11. (cf. [24]) *Let X be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous mapping. Then there is a point $x_0 \in X$ such that*

$$f(x_0) = \inf\{f(x) : x \in X\}.$$

2.2 Banach spaces

A *linear space* or *vector space* X over field $\mathbb{F}$ (the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation $+$ called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{F}\times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$:

1. $x + y = y + x$;
2. $x + (y + z) = (x + y) + z$;
3. there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$;
4. for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse* or the *negative* of x such that $x + (-x) = 0$;
5. $\alpha(x + y) = \alpha x + \alpha y$;
6. $(\alpha + \beta)x = \alpha x + \beta x$;
7. $(\alpha\beta)x = \alpha(\beta x)$;
8. $1x = x$.

The elements of a vector space X are called *vectors*, and the elements of $\mathbb{F}$ are called *scalars*.

A finite subset $\{x_1, \ldots, x_n\}$ of a linear space X is said to be *linearly independent* if for any $\alpha_1 \ldots, \alpha_n \in \mathbb{R}$ with $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$. If, in addition, every $x \in X$ is a linear combination of $x_1, \ldots, x_n$, that is $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ for some $\alpha_1 \ldots, \alpha_n \in \mathbb{R}$, then we say that X has the *dimension* n.

A function $\|\cdot\|$ from a (real) linear space X into $\mathbb{R}$ is called a *norm* if it satisfies the following properties for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$;
2. $\|x\| = 0$ if and only if $x = 0$;
3. $\|\alpha x\| = |\alpha|\|x\|$;
4. $\|x + y\| \leq \|x\| + \|y\|$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x,y) = \|x-y\|, \quad x,y \in X.$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space*. A normed linear space $(X, \|\cdot\|)$ which is complete is called a *Banach space*.

Definition 2.2.1. A subset C of a Banach space X is said to be *convex* if $\alpha x + (1-\alpha)y \in C$ for each $x, y \in C$ and $\alpha \in [0,1]$.

Definition 2.2.2. Let C be a subset of a linear space X. The *closed convex hull* of C in X is the intersection of all closed convex subsets of X containing C which is denoted by $\overline{co}(C)$. Thus,

$$\overline{co}(C) = \cap \{D \subseteq X \ : \ C \subseteq D, \ D \text{ is closed and convex}\}.$$

Definition 2.2.3. Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges strongly* to x written by $x_n \to x$, if $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 2.2.4. A Banach space X is said to be *strictly convex* if

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \text{ imply } \left\|\frac{x+y}{2}\right\| < 1.$$

Example 2.2.1.

(i) Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then X is not strictly convex.

(ii) Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_2$ defined by $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then X is strictly convex.

Definition 2.2.5. A Banach space X is called *uniformly convex* if for any $\varepsilon \in (0,2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1-\delta$.

Theorem 2.2.6. (cf. [53]) *Every uniformly convex Banach space is strictly convex.*

2.3 Hilbert spaces

Definition 2.3.1. Let X be a vector space over field $\mathbb{F} = \mathbb{R}$ (or $\mathbb{C}$). An *inner product* on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ such that for any $x, y, z \in X$ and $\alpha \in \mathbb{F}$, one has

(1) $\langle x, x \rangle \geq 0$;

(2) $\langle x, x\rangle = 0$ if and only if $x = 0$;

(3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$; and

(5) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

A vector space X together with an inner product $\langle\cdot,\cdot\rangle$ is called an *inner product space* or a *pre-Hilbert space* and it is denoted by $(X, \langle\cdot,\cdot\rangle)$ or simply by X.

Remark 2.3.2. An inner product on X defines a norm $||\cdot||$ on X which is given by

$$||x|| = \sqrt{\langle x, x\rangle}.$$

Definition 2.3.3. A complete inner product space is called a *Hilbert space*.

Example 2.3.1. The Euclidean space $\mathbb{R}^n$ is a Hilbert space with an inner product defined by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n,$$

where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$.

Theorem 2.3.4. (**The Schwarz inequality**) *If x and y are any two vectors in an inner product space X, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.*

Theorem 2.3.5. (**The parallelogram law**) *If x and y are any two vectors in an inner product space X, then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Remark 2.3.6. Any Hilbert space is a uniformly convex Banach space.

A function $f : X \to \mathbb{R}$ is said to be *linear* if $f(\alpha x + y) = \alpha f(x) + f(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{F}$. In addition, if there is $M > 0$ such that $|f(x)| \leq M\|x\|$ for all $x \in X$, we say that f is a *bounded linear functional.* Notice that the class of all bounded linear functionals on X, denoted by X^*, is a Banach space equipped with the norm defined by

$$\|f\| = \sup\left\{|f(x)| : x \in B_X\right\} = \sup\left\{|f(x)| : x \in S_X\right\},$$

where $B_X = \left\{x \in X : \|x\| \leq 1\right\}$ is the unit ball of X and $S_X = \left\{x \in X : \|x\| = 1\right\}$ is the unit sphere of X (see [25]). We also denote $X^{**} := (X^*)^*$ and call it the *second dual space* of X.

The most well known theorem in Banach space theory is the Hahn-Banach theorem.

Theorem 2.3.7. *Let x be a nonzero element of a normed space X. Then there exists $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$.*

The topology induced by a norm is too strong in the sense that it has many open sets. Indeed, in order that each bounded sequence in X has a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces which are related to the linear structure of the spaces to search for subsequential extraction principles. So it is worthwhile to define the weaker topology for a Banach space X. We say that a sequence $\{x_n\}$ in X *converges weakly* to x, denoted by

$$w - \lim_{n\to\infty} x_n = x,$$

if $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in X^*$. A subset K of X is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of each of its weakly convergent sequences. The weakly open sets are now taken as those sets whose complements are weakly closed. The resulting topology on X is called the *weak topology* on X. Sets which are compact in this topology are said to be weakly compact.

It is important to know that the weak topology on a Banach space is a Hausdorff topology, and that weak limits are unique. This is because the functionals in X^* separate points in X, that is, given any two points $x \neq y \in X$ there exists an $f \in X^*$ such that $f(x) \neq f(y)$. This is an another consequence of the Hahn-Banach theorem.

For $x \in X$ and $f \in X^*$ define $c(x)(f) = f(x)$. It is easily seen that $c(x) \in X^{**}$ and that, in fact, the mapping $c : X \to X^{**}$ is an isometric isomorphism, called the canonical embedding of X into X^{**}. If $c(X) = X^{**}$, then X is said to be *reflexive.*

Proposition 2.3.8. *For a Banach space X the following are equivalent:*

1. *X is reflexive.*
2. *X^* is reflexive.*
3. *B_X is weakly compact in X.*
4. *Any bounded sequence in X has a weakly convergent subsequence.*
5. *For any $f \in X^*$ there exists $x \in B_X$ such that $f(x) = \|f\|$.*
6. *For any bounded closed convex subset C of X and any $f \in X^*$ there exists $x \in C$ such that $f(x) = \sup\{f(y) : y \in C\}$.*

7. *If* (C_n) *is any descending sequence of nonempty bounded closed convex subsets of* X, *then* $\cap_{n=1}^{\infty} C_n \neq \emptyset$.

Note that property (7) above offers a quick way, which we will not prove here, to confirm the following fact.

Theorem 2.3.9. (cf. [24]) *If* X *is a uniformly convex Banach space, then* X *is reflexive.*

Definition 2.3.10. A Banach space is said to satisfy *Opial's condition* ([43]) if given whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for each $y \in X$ with $y \neq x$.

2.4 Geodesic spaces

Definition 2.4.1. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map $c : [0, l] \to X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

The image $c([0, l])$ of c is called a *geodesic segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$. In this case, we write $z = \alpha x \oplus (1 - \alpha)y$.

Definition 2.4.2. The space (X, d) is said to be a *geodesic space* (D*-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (D*-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ with $d(x, y) < D$).

Definition 2.4.3. Let (X, d) be a metric space, $x \in X$, $C \subseteq X$. The *diameter* of C and the *distance* from x to C are defined, respectively, by

$$\operatorname{diam}(C) := \sup\{d(x, y) : x, y \in C\},$$

$$d(x, C) := \inf\{d(x, y) : y \in C\}.$$

Definition 2.4.4. Let (X, d) be a metric space. A subset C of X is said to be *bounded* if $\operatorname{diam}(C) < \infty$.

Definition 2.4.5. Let (X, d) be a geodesic space. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

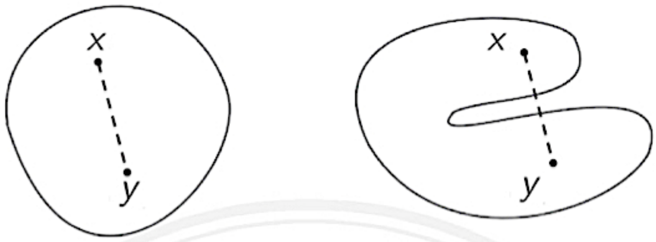

Figure 2.1: Convex set.

Now we introduce the model spaces M_κ^n, for more details on these spaces the reader is referred to [5]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance.

Let $\mathbb{S}^n$ denote the *n-dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \langle x, x \rangle = 1\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Let $T_x\mathbb{S}^n$ be a tangent space of $\mathbb{S}^n$ at x. The normalized geodesic $c : \mathbb{R} \to \mathbb{S}^n$ starting from $x \in \mathbb{S}^n$ is given by

$$c(t) = (\cos t)x + (\sin t)u, \quad \forall t \in \mathbb{R}$$

where $u \in T_x\mathbb{S}^n$ is the unit vector; while the distance d on $\mathbb{S}^n$ is

$$d(x, y) = \arccos(\langle x, y \rangle), \quad \forall x, y \in \mathbb{S}^n.$$

Then iteration process (1.1) has the form

$$x_{n+1} = (\cos (1 - \alpha_n)\, r(x_n, x_n))\, x_n + (\sin (1 - \alpha_n)\, r(x_n, x_n))\, V(x_n, x_n), \quad \forall n \geq 0,$$

and iteration process (1.2) has the form

$$y_n = (\cos(1 - \beta_n) r(x_n, x_n))x_n + (\sin(1 - \beta_n) r(x_n, x_n))V(x_n, x_n),$$
$$x_{n+1} = (\cos(1 - \alpha_n) r(x_n, y_n))x_n + (\sin(1 - \alpha_n) r(x_n, y_n))V(x_n, y_n), \quad \forall n \geq 0,$$

where

$$r(x, y) = \arccos(\langle x, Ty \rangle) \quad \text{and} \quad V(x, y) = \frac{Ty - \langle x, Ty \rangle x}{\sqrt{1 - \langle x, Ty \rangle^2}}, \quad \forall x, y \in \mathbb{R}^{n+1}.$$

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, ..., u_{n+1})$ and $v = (v_1, ..., v_{n+1})$, the real number $\langle u|v\rangle$ is defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic n-space* defined by

$$\mathbb{H}^n = \{u = (u_1, ..., u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\}.$$

Let $T_x\mathbb{H}^n$ be a tangent space of $\mathbb{H}^n$ at x. The normalized geodesic $c : \mathbb{R} \to \mathbb{H}^n$ starting from $x \in \mathbb{H}^n$ is given by

$$c(t) = (\cosh t)x + (\sinh t)v, \quad \forall t \in \mathbb{R}$$

where $v \in T_x\mathbb{H}^n$ is the unit vector; while the distance d on $\mathbb{H}^n$ is

$$d(x, y) = \operatorname{arccosh}(-\langle x, y\rangle), \quad \forall x, y \in \mathbb{H}^n.$$

Then iteration process (1.1) has the form

$$x_{n+1} = (\cosh(1-\alpha_n)r(x_n, x_n))x_n + (\sinh(1-\alpha_n)r(x_n, x_n))V(x_n, x_n), \quad \forall n \geq 0,$$

and iteration process (1.2) has the form

$$\begin{aligned} y_n &= (\cosh(1-\beta_n)r(x_n, x_n))x_n + (\sinh(1-\beta_n)r(x_n, x_n))V(x_n, x_n), \\ x_{n+1} &= (\cosh(1-\alpha_n)r(x_n, y_n))x_n + (\sinh(1-\alpha_n)r(x_n, y_n))V(x_n, y_n), \quad \forall n \geq 0, \end{aligned}$$

where

$$r(x, y) = \operatorname{arccosh}(-\langle x, Ty\rangle) \text{ and } V(x, y) = \frac{Ty + \langle x, Ty\rangle x}{\sqrt{\langle x, Ty\rangle^2 - 1}}, \quad \forall x, y \in \mathbb{R}^{n+1}.$$

Definition 2.4.6. Given $\kappa \in \mathbb{R}$, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$ then M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(iii) if $\kappa < 0$ then M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,d) consists of three points x,y,z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x,y,z)$ in (X,d) is a triangle $\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ in M_κ^2 such that

$$d(x,y)=d_{M_\kappa^2}(\overline{x},\overline{y}),\ d(y,z)=d_{M_\kappa^2}(\overline{y},\overline{z}) \text{ and } d(z,x)=d_{M_\kappa^2}(\overline{z},\overline{x}).$$

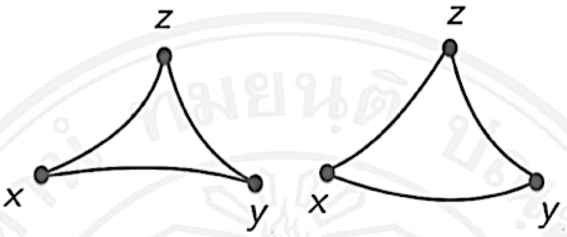

Figure 2.2: Geodesic triangle.

If $\kappa \leq 0$ then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$ then such a triangle exists whenever $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\overline{x},\overline{p})$.

2.4.1 CAT(0) Spaces

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the *CAT(0) inequality* if for any $p,q \in \triangle(x,y,z)$ and for their comparison points $\overline{p},\overline{q} \in \overline{\triangle}(\overline{x},\overline{y},\overline{z})$, one has

$$d(p,q) \leq d_{\mathbb{R}^2}(\overline{p},\overline{q}).$$

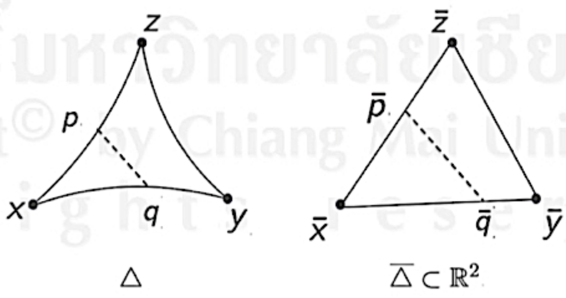

Figure 2.3: The CAT(0) inequality.

Definition 2.4.7. A geodesic space (X,d) is said to be a *CAT(0) space* if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(0) inequality.

Complete CAT(0) spaces are often called Hadamard spaces. Examples of CAT(0) spaces include, among others, Hilbert spaces, classical hyperbolic spaces, Euclidean buildings, $\mathbb{R}$-trees, etc.

The following lemma is the *(CN) inequality* of Bruhat and Tits [10].

Lemma 2.4.8. (cf. [5, p. 163]) *A geodesic space (X, d) is a CAT(0) space if and only if for $x, y_1, y_2 \in X$ and if y_0 is the midpoint of the segment $[y_1, y_2]$ then*

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \qquad \text{(CN)}$$

Lemma 2.4.9. (cf. [14, p, 2574]) *Let (X, d) be a CAT(0) space. Then for $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

Lemma 2.4.10. (cf. [14, p. 2574]) *Let (X, d) be a CAT(0) space. Then for $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y).$$

2.4.2 CAT(κ) spaces

A geodesic triangle $\triangle(x, y, z)$ in X is said to satisfy the *CAT(κ) inequality* if for any $p, q \in \triangle(x, y, z)$ and for their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$, one has

$$d(p, q) \leq d_{M_\kappa^2}(\overline{p}, \overline{q}).$$

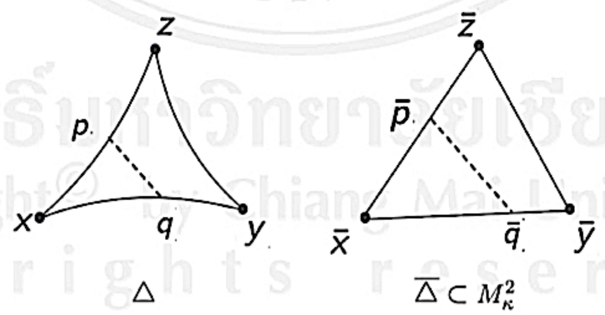

Figure 2.4: The CAT(κ) inequality.

Definition 2.4.11. If $\kappa \leq 0$, then X is called a *CAT(κ) space* if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then X is called a *CAT(κ) space* if X is D_κ-geodesic, where $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ and any geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Theorem 2.4.12. [5, p. 165] *The following statements hold:*

(i) If X is a CAT(κ) space, then it is a CAT(κ') space for every $\kappa' \geq \kappa$.

(ii) If X is a CAT(κ') for every $\kappa' \geq \kappa$, then it is a CAT(κ) space.

The following example shows that there exists a CAT(κ) space with $\kappa > 0$ which is not a CAT(0) space.

Example 2.4.1. [15] Let $(\mathbb{S}^2, d)$ be the spherical space and $e_i \in \mathbb{S}^2$, for $i = 1, 2, 3$ be each of the elements of the canonical basis of $\mathbb{R}^3$. Let C be the closed convex hull over the sphere of $\{e_i : i = 1, 2, 3\}$, i.e, the positive octant of the sphere. Then C is CAT(1) space but C is not CAT(0) space.

Lemma 2.4.13. ([5, p.176]) *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) < \frac{\pi}{2\sqrt{\kappa}}$. Then*

$$d\left(x, \alpha y \oplus (1-\alpha)z\right) \leq \alpha d(x, y) + (1-\alpha)d(x, z),$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Lemma 2.4.14. [44] *Let $\kappa > 0$ and (X, d) be a CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then for any $x, y, z \in X$ and $\alpha \in [0, 1]$, we have*

$$d^2\left(x, (1-\alpha)y \oplus \alpha z\right) \leq (1-\alpha)d^2(x, y) + \alpha d^2(x, z) - \frac{R}{2}\alpha(1-\alpha)d^2(y, z), \tag{2.1}$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.

Definition 2.4.15. [37] A geodesic space (X, d) is called *uniformly convex* if for any $r > 0$, and $\eta \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$,

$$\left.\begin{aligned} d(x, a) &\leq r \\ d(y, a) &\leq r \\ d(x, y) &\leq \eta r \end{aligned}\right\} \Longrightarrow d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r.$$

A mapping $\theta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \theta(r, \eta)$ for any $r > 0$ and $\eta \in (0, 2]$ is called a *modulus of uniform convexity.*

Lemma 2.4.16. *Let $\kappa > 0$ and (X, d) be a CAT(κ) space with diam$(X) < \frac{\pi}{2\sqrt{\kappa}}$. Then X is uniformly convex.*

Proof. Let $r > 0, \eta \in (0, 2]$ and $a, x, y \in X$ be such that $d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq \eta r$. Since $\operatorname{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, then there exists $\varepsilon \in (0, \pi/2)$ such that $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. By Lemma 2.4.14, we have

$$\begin{aligned}
d(\frac{1}{2}x \oplus \frac{1}{2}y, a) &\leq \sqrt{\frac{1}{2}d^2(x,a) + \frac{1}{2}d^2(y,a) - \frac{R}{8}d^2(x,y)} \\
&\leq \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r^2 - \frac{R}{8}\eta^2 r^2} \\
&= \sqrt{r^2 - \frac{R}{8}\eta^2 r^2} \\
&= \sqrt{1 - \frac{R}{8}\eta^2} \cdot r \\
&\leq \left(1 - \frac{R\eta^2}{16}\right) \cdot r.
\end{aligned}$$

Hence, X is uniformly convex. □

The following figure shows the relationship between Banach spaces, Hilbert spaces, CAT(0) spaces and CAT(κ) spaces with $\kappa > 0$.

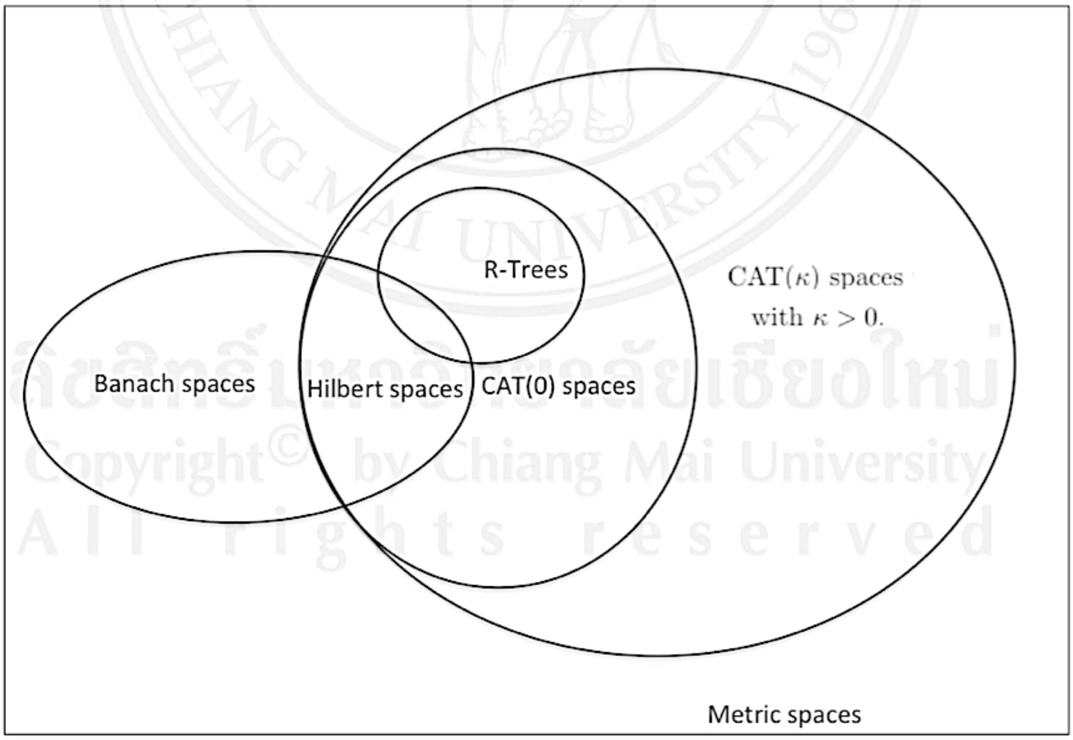

Figure 2.5: The relations of spaces.

Definition 2.4.17. Let (X, d) be a complete CAT(κ) space, let $\{x_n\}$ be a bounded sequence in X and for $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 4.1 of [15] that in a CAT(κ) space X with diameter smaller than $\frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.4.18. ([32], [38]) A sequence $\{x_n\}$ in X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 2.4.19. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) < \frac{\pi}{2\sqrt{\kappa}}$. Then the following statements hold:*

(i) [15, Corollary 4.4] *Every sequence in X has a Δ-convergence subsequence.*

(ii) [15, Proposition 4.5] *Let C be a closed convex subset of X. If $\{x_n\} \subset C$ and $\Delta - \lim_{n\to\infty} x_n = x$, then $x \in C$.*

By the uniqueness of asymptotic centers, we can obtain the following lemma.

Lemma 2.4.20. (cf. [14]) *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) < \frac{\pi}{2\sqrt{\kappa}}$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Proof. Suppose that $x \neq u$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, u) \\
&= \limsup_{n\to\infty} d(u_n, u),
\end{aligned}$$

a contradiction, and hence $x = u$. □

Definition 2.4.21. Let C be a nonempty subset of a metric space (X, d).

(1) A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$d(T(x), p) \leq d(x, p),$$

for all $x \in C$ and $p \in F(T)$.

(2) A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exist a sequence $k_n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and

$$d(T^n(x), T^n(y)) \leq k_n d(x, y),$$

for all $x, y \in C, n \in \mathbb{N}$.

(3) A mapping $T : C \to C$ is said to be *asymptotic pointwise nonexpansive* if there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ such that

$$\limsup_{n\to\infty} \alpha_n(x) \leq 1 \text{ and } d(T^n(x), T^n(y)) \leq \alpha_n(x) d(x, y),$$

for all $x, y \in C, n \in \mathbb{N}$.

(4) A mapping $T : C \to C$ is said to be *nonspreading* if

$$2d^2(T(x), T(y)) \leq d^2(T(x), y) + d^2(T(y), x)$$

for all $x, y \in C$.

(5) A mapping $T : C \to C$ is said to be *hybrid* if

$$3d^2(T(x), T(y)) \leq d^2(x, y) + d^2(T(x), y) + d^2(T(y), x)$$

for all $x, y \in C$.

(6) A mapping $T : C \to C$ is said to be *generalized hybrid* if there exist functions $a_1, a_2, a_3, k_1, k_2 : C \to [0, 1)$ such that

(P1) $d^2(T(x), T(y)) \leq a_1(x)d^2(x, y) + a_2(x)d^2(T(x), y) + a_3(x)d^2(T(y), x) + k_1(x)d^2(T(x), x) + k_2(x)d^2(T(y), y)$ for all $x, y \in C$;

(P2) $a_1(x) + a_2(x) + a_3(x) \leq 1$ for all $x, y \in C$;

(P3) $2k_1(x) < 1 - a_2(x)$ and $k_2(x) < 1 - a_3(x)$ for all $x \in C$.

(7) A mapping $T : C \to C$ is said to satisfy *condition (C)* if

$$\frac{1}{2}d(x, T(x)) \leq d(x, y) \text{ implies } d(T(x), T(y)) \leq d(x, y)$$

for all $x, y \in C$.

(8) A mapping $T : C \to C$ is said to be *fundamentally nonexpansive* if

$$d(T^2(x), T(y)) \leq d(T(x), y)$$

for all $x, y \in C$.

(9) A mapping $T : C \to C$ is said to be *compact* if for every bounded sequence $\{x_n\}$ in C, $\{T(x_n)\}$ has convergent subsequence in C.

(10) A mapping $T : C \to C$ is said to satisfy *condition (I)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, T(x)) \geq f\big(d(x, F(T))\big)$ for all $x \in C$.

Remark 2.4.22.

(1) Every mapping which satisfies condition (C) is fundamentally nonexpansive, but the inverse is not true (see [47]).

(2) Every nonexpansive mapping satisfies condition (C) (see [52]).

(3) Every nonexpansive mapping is asymptotically nonexpansive (see Definition 2.4.21).

(4) Every asymptotically nonexpansive is asymptotic pointwise nonexpansive (see Definition 2.4.21).

(5) Every nonexpansive mapping is generalized hybrid (see [39]).

(6) Every nonspreading mapping is generalized hybrid (see [39]).

(7) Every hybrid mapping is generalized hybrid (see [39]).

The following diagram shows the relationship between nonexpansive, condition (C), fundamentally nonexpansive, asymptotically nonexpansive, asymptotic pointwise nonexpansive, nonspreading, generalized hybrid and hybrid mappings.

T is fundamentally nonexpansive		
$\Uparrow$		
T is satisfy condition (C)		T is nonspreading
$\Uparrow$		$\Downarrow$
T is nonexpansive	$\Longrightarrow$	T is generalized hybrid
$\Downarrow$		$\Uparrow$
T is asymptotically nonexpansive		T is hybrid
$\Downarrow$		
T is asymptotic pointwise nonexpansive		