CHAPTER 3

Asymptotic pointwise nonexpansive mappings in CAT(κ) spaces

In this chapter, we study convergence theorems of the Ishikawa iteration for asymptotic pointwise nonexpansive mappings on CAT(κ) spaces with $\kappa > 0$.

3.1 Basic concepts

Let C be a nonempty subset of a metric space (X, d). We shall denote by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from C into C. Let $T \in \mathcal{T}(C)$. Then there exists a sequence of mapping $\alpha_n : C \to [0, \infty)$ such that for all $x, y \in C$ and $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} \alpha_n(x) \leq 1 \text{ and } d(T^n(x), T^n(y)) \leq \alpha_n(x)d(x, y).$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. Without loss of generality, we can assume that

$$d(T^n(x), T^n(y)) \leq a_n(x)d(x, y), \text{ and}$$

$$\lim_{n\to\infty} a_n(x) = 1, \; a_n(x) \geq 1, \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}. \tag{3.1}$$

Define $b_n(x) = a_n(x) - 1$. Then, for each $x \in C$ we have $\lim_{n\to\infty} b_n(x) = 0$.

Definition 3.1.1. Define $\mathcal{T}_r(C)$ as a class of all mappings in the class $\mathcal{T}(C)$ such that

$$\sum_{n=1}^{\infty} b_n(x) < \infty \text{ for every } x \in C, \text{ and} \tag{3.2}$$

$$a_n \text{ is a bounded function for every } n \in \mathbb{N}. \tag{3.3}$$

It is clear that every nonexpansive mapping is asymptotic pointwise nonexpansive, but there exists an asymptotic pointwise nonexpansive mapping which is not nonexpansive.

Example 3.1.1. [2, p. 244] Let B_H be the closed unit ball in the Hilbert space $H = \ell_2$ and $T : B_H \to B_H$ a mapping defined by

$$T(x_1, x_2, x_3, ...) = (0, x_1^2, \alpha_2 x_2, \alpha_3 x_3, ...),$$

where $\{\alpha_i\}$ is a sequence of real numbers such that $0 < \alpha_i \leq 1$ and $\prod_{i=2}^{\infty} \alpha_i = 1/2$. Then T is asymptotic pointwise nonexpansive but T is not nonexpansive.

Let X be a complete CAT(κ) space and C be a closed convex subset of X. Let $T \in \mathcal{T}_r(C)$ and let $\{n_k\}$ be an increasing sequence of natural numbers. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$. Define a sequence $\{x_k\}$ in C as:

$$\begin{cases} x_1 \in C, \\ x_{k+1} = (1 - t_k)x_k \oplus t_k T^{n_k}(y_k), \\ y_k = (1 - s_k)x_k \oplus s_k T^{n_k}(x_k), \text{ for } k \in \mathbb{N}. \end{cases} \tag{3.4}$$

We say that the sequence $\{x_k\}$ in (3.4) is *well-defined* if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$.

Remark 3.1.2. Observe that by the definition of asymptotic pointwise nonexpansiveness, $\limsup_{k\to\infty} a_k(x) = 1$. for every $x \in C$. Hence we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well-defined.

Before proving the main convergence theorems we give the following definition and some useful lemmas.

Definition 3.1.3. [9] A strictly increasing sequence $\{n_i\} \subset \mathbb{N}$ is called *quasi-periodic* if the sequence $\{n_{i+1} - n_i\}$ is bounded, or equivalently if there exists a number $q \in \mathbb{N}$ such that any block of q consecutive natural numbers must contain a term of the sequence $\{n_i\}$. The smallest of such numbers q will be called a *quasi-period* of $\{n_i\}$.

Example 3.1.2.

(i) The sequence $\{1, 3, 5, 7, ..., 2n + 1, ...\}$ is quasi-periodic with quasi-period 2.

(ii) The sequence $\{3, 6, 9, 12, ..., 3n, ...\}$ is quasi-periodic with quasi-period 3.

(iii) The sequence $\{1, 4, 9, 16, ..., n^2, ...\}$ is not quasi-periodic.

Lemma 3.1.4. [9] *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly-index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \leq 0, \ \textit{and}\ r_{k+n} \leq r_k + d_{k,n}$$

for each $k, n \geq 1$. Then $\{r_k\}$ converges to an $r \in \mathbb{R}$.

The following lemma is a consequence of Lemma 2.2 of Khamsi and Khan [27].

Lemma 3.1.5. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X such that*

(i) $\limsup_{n\to\infty} d(u_n, w) \le r$,

(ii) $\limsup_{n\to\infty} d(v_n, w) \le r$, *and*

(iii) $\lim_{n\to\infty} d((1 - t_n)u_n \oplus t_n v_n, w) = r$,

for some $r \ge 0$. Then $\lim_{n\to\infty} d(u_n, v_n) = 0$.

The existence of fixed points for asymptotic pointwise nonexpansive mappings in CAT(κ) spaces was proved by Espínola et al. [16] as the following result.

Theorem 3.1.6. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X. Then any asymptotic pointwise nonexpansive mapping from C into C has a fixed point.*

Lemma 3.1.7. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. Let $z \in F(T)$. Then there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} d(x_k, z) = r$.*

Proof. For each $k \in \mathbb{N}$, we let $y_k = (1 - s_k)x_k \oplus s_k T^{n_k}(x_k)$. Then

$$\begin{aligned}
d(x_{k+1}, z) &= d((1 - t_k)x_k \oplus t_k T^{n_k}(y_k), z) \\
&\le (1 - t_k)d(x_k, z) + t_k d(T^{n_k}(y_k), z) \\
&\le (1 - t_k)d(x_k, z) + t_k(1 + b_{n_k}(z))d(y_k, z) \\
&\le (1 - t_k)d(x_k, z) + t_k(1 + b_{n_k}(z))\,[d((1 - s_k)x_k \oplus s_k T^{n_k}(x_k), z)] \\
&\le (1 - t_k)d(x_k, z) + t_k(1 + b_{n_k}(z))\,[(1 - s_k)d(x_k, z) + s_k d(T^{n_k}(x_k), z)] \\
&\le (1 - t_k)d(x_k, z) + t_k(1 + b_{n_k}(z))(1 + s_k b_{n_k}(z))d(x_k, z) \\
&\le (1 - t_k)d(x_k, z) + t_k(1 + b_{n_k}(z))^2 d(x_k, z) \\
&\le d(x_k, z) + t_k(2b_{n_k}(z) + b_{n_k}^2(z))d(x_k, z) \\
&\le d(x_k, z) + (2b_{n_k}(z) + b_{n_k}^2(z))d(x_k, z) \\
&= d(x_k, z) + b_{n_k}(z)(2 + b_{n_k}(z))d(x_k, z) \\
&= d(x_k, z) + b_{n_k}(z)(2 + a_{n_k}(z) - 1)d(x_k, z)
\end{aligned}$$

$$
\begin{aligned}
&= d(x_k, z) + b_{n_k}(z)(1 + a_{n_k}(z))d(x_k, z) \\
&= d(x_k, z) + (1 + a_{n_k}(z))b_{n_k}(z)d(x_k, z) \\
&\le d(x_k, z) + (1 + a_{n_k}(z))diam(C)b_{n_k}(z)
\end{aligned}
$$

Fix any $M > 1$. Since $a_{n_k}(z) \ge 1$ and $\lim_{k\to\infty} a_{n_k}(z) = 1$, it follow that there exists a $k_0 \ge 1$ such that for $k > k_0$, $a_{n_k}(z) \le M$. Then

$$d(x_{k+1}, z) \le d(x_k, z) + (1 + M)diam(C)b_{n_k}(z)$$

It follows that for each $n \in \mathbb{N}$,

$$d(x_{k+n}, z) \le d(x_k, z) + (1 + M)diam(C) \sum_{i=k}^{k+n+1} b_{n_i}(z).$$

Denote $r_k = d(x_k, z)$ for every $k \in \mathbb{N}$ and $d_{k,n} = diam(C)\sum_{i=k}^{k+n-1} b_{n_i}(z)$. Observe that since $T \in \mathcal{T}_r(C)$, it follows that $\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} = 0$. By Lemma 3.1.4 then, there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} d(x_k, z) = r$. □

Lemma 3.1.8. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. Then*

$$\lim_{k\to\infty} d(T^{n_k}(y_k), x_k) = 0, \tag{3.5}$$

and

$$\lim_{k\to\infty} d(x_{k+1}, x_k) = 0. \tag{3.6}$$

Proof. By Theorem 3.1.6, $F(T) \neq \emptyset$. Let $z \in F(T)$. By Lemma 3.1.7, we get $\lim_{k\to\infty} d(x_k, z)$ exists. Let

$$\lim_{k\to\infty} d(x_k, z) = c. \tag{3.7}$$

Since $z \in F(T)$, $T \in \mathcal{T}_r(C)$ and $\lim_{k\to\infty} d(x_k, z) = c$, by Lemma 3.1.7 we have the following

$$
\begin{aligned}
\limsup_{k\to\infty} d(T^{n_k}(y_k), z) &= \limsup_{k\to\infty} d(T^{n_k}(y_k), T^{n_k}(z)) \\
&\le \limsup_{k\to\infty} a_{n_k}(z)d(y_k, z) \\
&\le \limsup_{k\to\infty} a_{n_k}(z)d(s_kT^{n_k}(x_k) \oplus (1 - s_k)x_k, z) \\
&\le \limsup_{k\to\infty} (s_ka_{n_k}(z)d(T^{n_k}(x_k), z) + (1 - s_k)a_{n_k}(z)d(x_k, z)) \\
&\le \limsup_{k\to\infty} \big(s_ka^2_{n_k}(z)d(x_k, z) + (1 - s_k)a_{n_k}(z)d(x_k, z)\big) \le c. \qquad (3.8)
\end{aligned}
$$

Note that

$$\lim_{k\to\infty} d(t_kT^{n_k}(y_k) \oplus (1-t_k)x_k, z) = \lim_{k\to\infty} d(x_{k+1}, z) = c. \tag{3.9}$$

It follows from (3.7), (3.8), (3.9) and Lemma 3.1.5 that

$$\lim_{k\to\infty} d(T^{n_k}(y_k), x_k) = 0.$$

Observe that (3.5) and the construction of the sequence $\{x_k\}$ yield

$$\lim_{k\to\infty} d(x_{k+1}, x_k) = 0.$$

This completes the proof. □

Lemma 3.1.9. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. Then*

$$\lim_{k\to\infty} d(T^{n_k}(x_k), x_k) = 0.$$

Proof. Let $y_k = s_kT^{n_k}(x_k) \oplus (1-s_k)x_k$. Since

$$\begin{aligned} d(T^{n_k}(x_k), x_k) &\leq d(T^{n_k}(x_k), T^{n_k}(y_k)) + d(T^{n_k}(y_k), x_k) \\ &\leq a_{n_k}(x_k)d(x_k, y_k) + d(T^{n_k}(y_k), x_k) \\ &= s_k a_{n_k}(x_k)d(T^{n_k}(x_k), x_k) + d(T^{n_k}(y_k), x_k), \end{aligned}$$

it follows that

$$d(T^{n_k}(x_k), x_k) \leq \frac{d(T^{n_k}(y_k), x_k)}{(1 - s_k a_{n_k}(x_k))}.$$

Since $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, by Lemma 3.1.8 we get that

$$\lim_{k\to\infty} d(T^{n_k}(x_k), x_k) = 0. \tag{3.10}$$

This completes of the proof. □

Theorem 3.1.10. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. If the set $\mathcal{J} = \{j \in \mathbb{N} : n_{j+1} = 1 + n_j\}$ is quasi-periodic, then*

$$\lim_{k\to\infty} d(T(x_k), x_k) = 0. \tag{3.11}$$

Proof. It is enough to prove that $d(T(x_k), x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. Indeed, let q be a quasi-period of $\mathcal{J}$ and $\varepsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that

$$d(T(x_k), x_k) < \frac{\varepsilon}{3}, \text{ for all } k \in \mathcal{J} \text{ such that } k \geq N_1. \tag{3.12}$$

By the quasi-periodicity of $\mathcal{J}$, for each $l \in \mathbb{N}$ there exists $i_l \in \mathcal{J}$ such that $|l - i_l| \leq q$. Without loss of generality, we can assume that $l \leq i_l \leq l + q$ (the proof for the other case is identical). Let $M = \sup\{a_1(x) : x \in C\}$. Then $M \geq 1$. Since $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$ by (3.6), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{l+1}, x_l) < \frac{\varepsilon}{3qM}, \text{ for all } l \geq N_2. \tag{3.13}$$

This implies that for all $l \geq N_2$,

$$d(x_{i_l}, x_l) \leq d(x_{i_l}, x_{i_l - 1}) + \ldots + d(x_{l+1}, x_l) \leq q\left(\frac{\varepsilon}{3qM}\right) = \frac{\varepsilon}{3M}. \tag{3.14}$$

By the definition of T, we have

$$d(T(x_{i_l}), T(x_l)) \leq M d(x_{i_l}, x_l) \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{3.15}$$

Let $N = \max\{N_1, N_2\}$. Then for $l \geq N$, we have from (3.12), (3.14) and (3.15) that

$$d(x_l, T(x_l)) \leq d(x_l, x_{i_l}) + d(x_{i_l}, T(x_{i_l})) + d(T(x_{i_l}), T(x_l)) < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

To prove that $d(T(x_k), x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. Since $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = n_k + 1\}$ is quasi-periodic then for each $k \in \mathcal{J}$, we have

$$\begin{aligned} d(x_k, Tx_k) &\leq d\left(x_k, x_{k+1}\right) + d\left(x_{k+1}, T^{n_{k+1}}(x_{k+1})\right) + d(T^{n_{k+1}}(x_{k+1}), T^{n_{k+1}}(x_k)) \\ &\quad + d(T^{n_k+1}(x_k), T(x_k)) \\ &\leq d(x_k, x_{k+1}) + d(x_{k+1}, T^{n_{k+1}}(x_{k+1})) + a_{n_{k+1}}(x_{k+1}) d(x_{k+1}, x_k) \\ &\quad + a_1(x_k) d(T^{n_k}(x_k), x_k). \end{aligned}$$

This, together with (3.6) and (3.10), we can obtain that $d(T(x_k), x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. □

Lemma 3.1.11. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. If $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, then $\lim_{n\to\infty} d(x_n, T^l(x_n)) = 0$ for every $l \in \mathbb{N}$.*

Proof. It follows from (3.3) that there exists $M > 0$ such that

$$\sum_{i=1}^{l-1} \sup\{a_i(x) \; ; \; x \in C\} \leq M. \tag{3.16}$$

It follows from

$$
\begin{aligned}
d(T^l(x_n), x_n) &\leq \sum_{i=1}^{l-1} d(T^{i+1}(x_n), T^i(x_n)) + d(T(x_n), x_n) && (3.17)\\
&\leq d(T(x_n), x_n) \left(\sum_{i=1}^{l-1} a_i(x_n) + 1 \right) && (3.18)\\
&\leq (M+1) d(T(x_n), x_n) && (3.19)
\end{aligned}
$$

that

$$\lim_{n\to\infty} d(T^l(x_n), x_n) = 0. \tag{3.20}$$

This completes the proof. □

The following result is the demiclosed principal for asymptotic pointwise nonexpansive mapping in CAT(κ) spaces.

Theorem 3.1.12. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Suppose $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\Delta - \lim_{n\to\infty} x_n = z$. Then $T(z) = z$.*

Proof. Define $\phi(x) = \limsup_{n\to\infty} d(x_n, x)$ for $x \in C$. Let $m \in \mathbb{N}$ be such that $m > 2$. Thus

$$
\begin{aligned}
d(T^m(x_n), x) &\leq \sum_{i=1}^{m} d(T^i(x_n), T^{i-1}(x_n)) + d(x_n, x)\\
&\leq d(T(x_n), x_n) \sum_{i=1}^{m} a_i(x_n) + d(x_n, x).
\end{aligned}
$$

Since all functions a_i are bounded and $d(T(x_n), x_n) \to 0$, we have

$$\limsup_{n\to\infty} d(T^m(x_n), x) \leq \limsup_{n\to\infty} d(x_n, x) = \phi(x).$$

On the other hand, by Lemma 3.1.11, we have

$$\phi(x) \leq \limsup_{n\to\infty} d(x_n, T^m(x_n)) + \limsup_{n\to\infty} d(T^m(x_n), x) = \limsup_{n\to\infty} d(T^m(x_n), x).$$

Hence,

$$\phi(x) = \limsup_{n\to\infty} d(T^m(x_n), x).$$

Since T is asymptotic pointwise nonexpansive,

$$
\begin{aligned}
\phi(T^m(x)) &= \limsup_{n\to\infty} d(T^m(x_n), T^m(x)) \\
&\le \limsup_{n\to\infty} a_m(x) d(x_n, x) \\
&\le a_m(x) \left[\limsup_{n\to\infty} d(x_n, x)\right] \\
&\le a_m(x)\phi(x).
\end{aligned}
$$

That is

$$\phi(T^m(x)) \le a_m(x)\phi(x), \quad \text{for every} \quad x \in C.$$

Applying this to z and passing with $m \to \infty$, we have

$$\lim_{m\to\infty} \phi(T^m(z)) \le \phi(z). \tag{3.21}$$

Since $\Delta - \lim_{n\to\infty} x_n = z$, for $x \neq z$ we have

$$\phi(z) = \limsup_{m\to\infty} d(x_n, z) < \limsup_{m\to\infty} d(x_n, x) = \phi(x), \tag{3.22}$$

which implies that $\phi(z) = \inf\{\phi(x)\ ;\ x \in C\}$. This together with (3.21) gives us

$$\lim_{m\to\infty} \phi(T^m(z)) = \phi(z). \tag{3.23}$$

Since $\operatorname{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, then there exists $\varepsilon \in (0, \pi/2)$ such that $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. By Lemma 2.4.14, we have

$$d^2\left(x_n, \frac{z \oplus T^m(z)}{2}\right) \le \frac{1}{2}d^2(x_n, z) + \frac{1}{2}d^2(x_n, T^m(z)) - \frac{R}{8}d^2(z, T^m(z))$$

for any $n, m \ge 1$. If we take $\limsup_{n\to\infty}$ to both side, we get that

$$
\begin{aligned}
\limsup_{n\to\infty} d^2\left(x_n, \frac{z \oplus T^m(z)}{2}\right) \le \frac{1}{2}\limsup_{n\to\infty} d^2(x_n, z)^2 + \frac{1}{2}\limsup_{n\to\infty} d(x_n, T^m(z)) \\
- \frac{R}{8}\limsup_{n\to\infty} d^2(z, T^m(z)).
\end{aligned}
$$

This implies that

$$\phi\left(\frac{z \oplus T^m(z)}{2}\right)^2 \le \frac{1}{2}\phi(z)^2 + \frac{1}{2}\phi(T^m(z))^2 - \frac{R}{8}d^2(z, T^m(z))$$

for any $m \ge 1$. The definition of z implies

$$\phi(z)^2 \le \frac{1}{2}\phi(z)^2 + \frac{1}{2}\phi(T^m(z))^2 - \frac{R}{8}d^2(z, T^m(z))$$

for any $m \ge 1$. Thus

$$d^2(z, T^m z) \le \frac{4}{R}\phi(T^m(z))^2 - \frac{4}{R}\phi(z)^2.$$

Take $m \to \infty$, we have $\lim_{m\to\infty} d(z, T^m(z)) = 0$. Hence $T(z) = z$ since T is continuous.

□

Corollary 3.1.13. *Let (X, d) be a complete CAT(0) space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n\to\infty} d(T(x_n), x_n) = 0$ and $\Delta-\lim_{n\to\infty} x_n = z$. Then $z \in C$ and $z = T(z)$.*

Proof. It well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space. Then (C, d) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that C is R-convex for $R = 2$. Since C is bounded, we can choose $\kappa > 0$ so that $\mathrm{diam}(C) < \frac{\pi}{2\sqrt{\kappa}}$. The conclusion follows from Theorem 3.1.12. □

Lemma 3.1.14. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Suppose $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\{d(x_n, v)\}$ converges for each $v \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$. Here $\omega_w(x_n) = \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4.19, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v \in C$. By Theorem 3.1.12, $v \in F(T)$. By Lemma 2.4.20, $u = v$. This shows that $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, we have $\{d(x_n, u)\}$ converges. Again, by Lemma 2.4.20, $x = u$. This completes the proof. □

3.2 Δ and strong convergence theorems

Theorem 3.2.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. If the set $\mathcal{J} = \{j\,;\, n_{j+1} = 1 + n_j\}$ is quasi-periodic, then $\{x_k\}$ Δ-converge to a fixed point of T.*

Proof. By Theorem 3.1.6, $F(T) \neq \emptyset$. Let $z \in F(T)$. By Lemma 3.1.7, $\lim_{k\to\infty} d(x_k, z)$ exists and hence $\{x_k\}$ is bounded. We have from Lemma 3.1.10 that $\lim_{k\to\infty} d(x_k, T(x_k)) = 0$. It follows from Lemma 3.1.14 that $\omega_w(x_k) \subseteq F(T)$. Since $\omega_w(x_k)$ consists of exactly one point, $\{x_k\}$ Δ-converges to an element of $F(T)$ as desired. □

Theorem 3.2.2. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C)$. Assume that T^m is compact for some $m \geq 1$. Let $\{t_k\}, \{s_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.4) is well-defined. If the set $\mathcal{J} = \{j\,;\, n_{j+1} = 1 + n_j\}$ is quasi-periodic, then $\{x_k\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1.10,

$$\lim_{k\to\infty} d(T(x_k), x_k) = 0. \tag{3.24}$$

By Lemma 3.1.11,

$$\lim_{k\to\infty} d(T^m(x_k), x_k) = 0. \tag{3.25}$$

By the compactness of T^m we can select a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\lim_{j\to\infty} d(T^m(x_{k_j}), z) = 0, \text{ for all } z \in C. \tag{3.26}$$

Since

$$d(x_{k_j}, z) \leq d(x_{k_j}, T^m(x_{k_j})) + d(T^m(x_{k_j}), z),$$

it follows from (3.25) and (3.26) that

$$\lim_{j\to\infty} d(x_{k_j}, z) = 0. \tag{3.27}$$

Since

$$\begin{aligned} d(T(z), z) &\leq d(T(z), T(x_{k_j})) + d(T(x_{k_j}), x_{k_j}) + d(x_{k_j}, z) \\ &\leq a_1(z) d(z, x_{k_j}) + d(T(x_{k_j}), x_{k_j}) + d(x_{k_j}, z), \end{aligned}$$

it follows from (3.24) and (3.27) that $z \in F(T)$. Applying Lemma 3.1.7 we conclude that $\lim_{k\to\infty} d(x_k, z)$ exists. In view of (3.27), $\lim_{k\to\infty} d(x_k, z) = 0$. □