CHAPTER 4

Generalized hybrid mappings in CAT(κ) spaces

In this chapter, we study fixed point theorems and convergence theorems of the Ishikawa iteration for generalized hybrid mappings on CAT(κ) spaces with $\kappa > 0$.

4.1 Basic concepts

Let C be a nonempty subset of a CAT(κ) space (X, d). Recall that a mapping $T : C \to X$ is called *generalized hybrid* if there exist functions $a_1, a_2, a_3, k_1, k_2 : C \to [0, 1)$ such that

(P1) $d^2(T(x), T(y)) \leq a_1(x)d^2(x, y) + a_2(x)d^2(T(x), y) + a_3(x)d^2(T(y), x) + k_1(x)d^2(T(x), x) + k_2(x)d^2(T(y), y)$ for all $x, y \in C$;

(P2) $a_1(x) + a_2(x) + a_3(x) \leq 1$ for all $x, y \in C$;

(P3) $2k_1(x) < 1 - a_2(x)$ and $k_2(x) < 1 - a_3(x)$ for all $x \in C$.

It is clear that every nonexpansive mapping is generalized hybrid, but the converse is not true.

Example 4.1.1. [12] Define a mapping $T : [0, 3] \to [0, 3]$ by

$$T(x) = \begin{cases} 0, & x \neq 3 \\ 2, & x = 3. \end{cases}$$

Then T is generalized hybird but T is not nonexpansive.

Let C be a nonempty closed convex subset of a complete CAT(κ) space (X, d) and let

$$P_C(x) = \{y \in C \ : \ d(x, y) = \inf_{z \in C} d(x, z)\}$$

be the *metric projection* from X onto C.

Lemma 4.1.1. ([15, Proposition 3.5]) *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $< \frac{\pi}{2\sqrt{\kappa}}$. Let $x \in X$ and C be a nonempty closed convex subset of X. Then*

(i) for each $x \in X$, $P_C(x)$ is a singleton;

(ii) for each $y \in C$, $d(P_C(x), P_C(y)) \leq d(x, y)$.

4.2 Demiclosed principle

Now, we prove the demiclosed principle for generalized hybrid mappings on CAT(κ) spaces.

Theorem 4.2.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in C$ where $R = (\pi - 2\varepsilon)tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in C with $\Delta-\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Then $z \in C$ and $z = T(z)$.*

Proof. Since $\Delta-\lim_{n\to\infty} x_n = z$, by Lemma 2.4.19, $z \in C$. Since T is a generalized hybrid mapping,

$$\begin{aligned} d^2(T(x_n), T(z)) &\leq a_1(z)d^2(z, x_n) + a_2(z)d^2(T(z), x_n) + a_3(z)d^2(T(x_n), z) \\ &\quad + k_1(z)d^2(T(z), z) + k_2(z)d^2(T(x_n), x_n) \\ &\leq a_1(z)d^2(z, x_n) + a_2(z)\left[d(T(z), T(x_n)) + d(T(x_n), x_n)\right]^2 \\ &\quad + a_3(z)\left[d(T(x_n), x_n) + d(x_n, z)\right]^2 \\ &\quad + k_1(z)d^2(T(z), z) + k_2(z)d^2(T(x_n), x_n), \end{aligned}$$

yielding

$$\limsup_{n\to\infty} d^2(T(x_n), T(z)) \leq \limsup_{n\to\infty} d^2(z, x_n) + \frac{k_1(z)}{1 - a_2(z)} d^2(z, T(z)).$$

This implies that

$$\begin{aligned} \limsup_{n\to\infty} d^2(x_n, T(z)) &\leq \limsup_{n\to\infty} \left[d(x_n, T(x_n)) + d(T(x_n), T(z))\right]^2 \\ &\leq \limsup_{n\to\infty} \left[d^2(x_n, T(x_n)) + 2d(x_n, T(x_n))d(T(x_n), T(z)) + d^2(T(x_n), T(z))\right] \\ &\leq \limsup_{n\to\infty} d^2(T(x_n), T(z)) \\ &\leq \limsup_{n\to\infty} d^2(z, x_n) + \frac{k_1(z)}{1 - a_2(z)} d^2(z, T(z)). \qquad (4.1) \end{aligned}$$

On the other hand, by Lemma 2.4.14 we have

$$d^2\left(x_n, \frac{1}{2}z \oplus \frac{1}{2}T(z)\right) \leq \frac{1}{2}d^2(x_n, z) + \frac{1}{2}d^2(x_n, T(z)) - \frac{R}{8}d^2(z, T(z)). \qquad (4.2)$$

By (4.1) and (4.2), we get

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, \tfrac{1}{2}z \oplus \tfrac{1}{2}T(z)) &\leq \frac{1}{2}\limsup_{n\to\infty} d^2(x_n, z) + \frac{1}{2}\limsup_{n\to\infty} d^2(x_n, T(z)) \\ &\quad - \frac{R}{8}d^2(z, T(z)) \\ &\leq \limsup_{n\to\infty} d^2(x_n, z) + \frac{k_1(z)}{2(1-a_2(z))}\limsup_{n\to\infty} d^2(z, T(z)) \\ &\quad - \frac{R}{8}d^2(z, T(z)).\end{aligned}$$

Thus

$$\left(\frac{R}{8} - \frac{k_1(z)}{2(1-a_2(z))}\right) d^2(z, T(z)) \leq \limsup_{n\to\infty} d^2(x_n, z) - \limsup_{n\to\infty} d^2\left(x_n, \frac{1}{2}z \oplus \frac{1}{2}T(z)\right) \leq 0.$$

Since $\frac{2k_1(z)}{1-a_2(x)} < \frac{R}{2}$, we get $\frac{k_1(z)}{2(1-a_2(z))} < \frac{R}{8}$ and so $d^2(z, T(z)) = 0$. Hence $z = T(z)$. □

Remark 4.2.2. From Theorem 4.2.1, we can find an optimal value of $R = (\pi - 2\varepsilon)\tan(\varepsilon)$, which depends on the value κ. For example, if the value $\kappa = 1$, then the optimal value of R is 1.07148718 as illustrated in Figure 2.6.

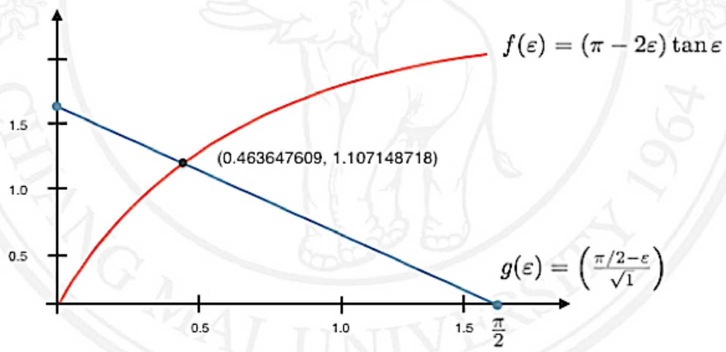

Figure 2.6: The optimal value of $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.

Corollary 4.2.3. *Let (X, d) be a complete CAT(0) space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a generalized hybrid mapping. Let $\{x_n\}$ be a sequence in C with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Then $z \in C$ and $z = T(z)$.*

Proof. It is well known that very convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space. Then (C, d) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that C is R-convex for $R = 2$. Since C is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $\kappa > 0$ so that $\mathrm{diam}(C) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 4.2.1. □

4.3 Fixed point theorems

Theorem 4.3.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in C$. Then T has a fixed point.*

Proof. Fix $x \in C$ and define $x_n := T^n(x)$ for $n \in \mathbb{N}$. Suppose that $A(\{x_n\}) = \{z\}$. Then by Lemma 2.4.19, $z \in C$. Since T is generalized hybrid and $k_1(z) = k_2(z) = 0$,

$$\begin{aligned} d^2(x_n, T(z)) &= d^2(T^n(x), T(z)) \\ &= d^2(T(T^{n-1}(x)), T(z)) \\ &= d^2(T(x_{n-1}), T(z)) \\ &\leq a_1(z)d^2(z, x_{n-1}) + a_2(z)d^2(T(z), x_{n-1}) + a_3(z)d^2(x_n, z). \end{aligned}$$

Taking the limit superior on both sides, we get

$$\begin{aligned} \limsup_{n\to\infty} d^2(x_n, T(z)) \leq\ & a_1(z) \limsup_{n\to\infty} d^2(z, x_{n-1}) + a_2(z) \limsup_{n\to\infty} d^2(T(z), x_{n-1}) \\ & + a_3(z) \limsup_{n\to\infty} d^2(x_n, z) \\ \leq\ & (a_1(z) + a_3(z)) \limsup_{n\to\infty} d^2(x_n, z) + a_2(z) \limsup_{n\to\infty} d^2(x_n, T(z)). \end{aligned}$$

This implies by (P2) that $\limsup_{n\to\infty} d^2(x_n, T(z)) \leq \limsup_{n\to\infty} d^2(x_n, z)$. But, Since $A(\{x_n\}) = \{z\}$, it must be the case that $z = T(z)$ and the proof is complete. □

As a consequence of Theorem 4.3.1, we obtain:

Corollary 4.3.2. *Let (X, d) be a complete CAT(0) space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in C$. Then T has a fixed point.*

4.4 Δ-convergence theorems

We begin this section by proving a crucial lemma.

Lemma 4.4.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in C$ where $R = (\pi - 2\varepsilon)tan(\varepsilon)$. Suppose $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and*

$\{d(x_n, v)\}$ *converges for all* $v \in F(T)$*, then* $\omega_w(x_n) \subseteq F(T)$*. Here* $\omega_w(x_n) := \bigcup A(\{u_n\})$ *where the union is taken over all subsequences* $\{u_n\}$ *of* $\{x_n\}$*. Moreover,* $\omega_w(x_n)$ *consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4.19, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v \in C$. By Theorem 4.2.1, $v \in F(T)$. By Lemma 2.4.20, $u = v$. This shows that $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, we have $\{d(x_n, u)\}$ converges. Again, by Lemma 2.4.20, $x = u$. This completes the proof. □

Theorem 4.4.2. *Let* $\kappa > 0$ *and* (X, d) *be a complete CAT(*κ*) space with diam(*X*)* $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \pi/2)$*. Let* C *be a nonempty closed convex subset of* X*, and* $T : C \to X$ *be a generalized hybrid mapping with* $F(T) \neq \emptyset$*. Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$ *and define a sequence* $\{x_n\}$ *in* C *by*

$$\begin{cases} x_1 \in C & \textit{chosen arbitrary} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n \oplus \alpha_n T(x_n)\right), & n \in \mathbb{N}. \end{cases}$$

Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$ *and suppose that*

(i) $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ *for all* $x \in C$*,*

(ii) $\liminf_{n\to\infty} \alpha_n \left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right] > 0$ *for all* $z \in F(T)$*.*

Then $\{x_n\}$ Δ*-converges to an element of* $F(T)$*.*

Proof. Let $z \in F(T)$. Since T is generalized hybrid,

$$\begin{aligned} d^2(T(x), z) &= d^2(T(x), T(z)) \\ &\leq a_1(z)d^2(z, x) + a_2(z)d^2(T(z), x) + a_3(z)d^2(T(x), z) \\ &\quad + k_1(z)d^2(T(z), z) + k_2(z)d^2(T(x), x). \end{aligned}$$

Thus

$$\begin{aligned} (1 - a_3(z))\, d^2(T(x), z) &\leq (a_1(z) + a_2(z))\, d^2(z, x) + k_2(z)d^2(T(x), x) \\ &\leq (1 - a_3(z))\, d^2(z, x) + k_2(z)d^2(T(x), x). \end{aligned}$$

So

$$d^2(T(x), z) \leq d^2(z, x) + \frac{k_2(z)}{1 - a_3(z)} d^2(T(x), x) \quad \text{for all } x \in C.$$

By Lemmas 2.4.14 and 4.1.1, we have

$$\begin{aligned}
d^2(x_{n+1}, z) &= d^2\left(P_C((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), z\right) \\
&\leq d^2\left((1-\alpha_n)x_n \oplus \alpha_n T(x_n), z\right) \\
&\leq (1-\alpha_n)d^2(x_n, z) + \alpha_n d^2(T(x_n), z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n, T(x_n)) \\
&\leq (1-\alpha_n)d^2(x_n, z) + \alpha_n[d^2(z, x_n) + \frac{k_2(z)}{1-a_3(z)}d^2(T(x_n), x_n)] \\
&- \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n, T(x_n)) \\
&\leq d^2(x_n, z) + \alpha_n\left[\frac{k_2(z)}{1-a_3(z)} - \frac{R(1-\alpha_n)}{2}\right]d^2(x_n, T(x_n)). \qquad (4.3)
\end{aligned}$$

By (ii), there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right] \geq \delta > 0 \quad \text{for all } n \geq N.$$

Without loss of generality, we may assume that

$$\alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right] > 0 \quad \text{for all } n \geq N. \qquad (4.4)$$

It follows from (4.3) and (4.4) that $\{d(x_n, z)\}$ is a nonincreasing sequence and hence $\lim_{n\to\infty} d(x_n, z)$ exists. Again, by (4.3), we have

$$\lim_{n\to\infty} \alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right]d^2(x_n, T(x_n)) = 0.$$

This implies by (ii) that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. By Lemma 4.4.1, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

Theorem 4.4.3. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ and define a sequence $\{x_n\}$ in C by*

$$\begin{cases}
x_1 \in C \quad \text{chosen arbitrary,} \\
x_{n+1} := P_C\left((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n)\right), \\
y_n := P_C\left((1-\beta_n)x_n \oplus \beta_n T(x_n)\right).
\end{cases}$$

Assume that

(i) $k_2(z) = 0$ *for all* $z \in F(T)$,

(ii) $\liminf_{n\to\infty} \alpha_n > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ Δ-converges to an element of $F(T)$.

Proof. Fix $z \in F(T)$. By (i), we have $d(T(x), z) \leq d(x, z)$ for all $x \in C$. Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. By Lemmas 2.4.14 and 4.1.1, we have

$$\begin{aligned}
d^2(y_n, z) &= d^2(P_C\left((1-\beta_n)x_n \oplus \beta_n T(x_n)\right), z) \\
&\leq d^2\left((1-\beta_n)x_n \oplus \beta_n T(x_n), z\right) \\
&\leq (1-\beta_n)d^2(x_n, z) + \beta_n d^2(T(x_n), z) - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n, T(x_n)) \\
&\leq (1-\beta_n)d^2(x_n, z) + \beta_n d^2(x_n, z) - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n, T(x_n)) \\
&\leq d^2(x_n, z) - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n, T(x_n)) \\
&\leq d^2(x_n, z). \qquad (4.5)
\end{aligned}$$

This implies that

$$\begin{aligned}
d^2\left(x_{n+1}, z\right) &= d^2(P_C\left((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n)\right), z) \\
&\leq d^2\left((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n), z\right) \\
&\leq (1-\alpha_n)d^2(T(x_n), z) + \alpha_n d^2(T(y_n), z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(x_n), T(y_n)) \\
&\leq (1-\alpha_n)d^2(x_n, z) + \alpha_n d^2(y_n, z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(x_n), T(y_n)) \\
&\leq d^2(x_n, z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(x_n), T(y_n)) \\
&\leq d^2(x_n, z).
\end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, z)$ exists and

$$0 \leq \frac{R}{2}\alpha_n(1-\alpha_n)d^2(T(x_n), T(y_n)) \leq d^2(x_n, z) - d^2(x_{n+1}, z) + \alpha_n\left[d^2(y_n, z) - d^2(x_n, z)\right].$$

So,

$$\alpha_n\left[d^2(x_n, z) - d^2(y_n, z)\right] \leq d^2(x_n, z) - d^2(x_{n+1}, z).$$

Since $\liminf_{n\to\infty} \alpha_n > 0$, $\limsup_{n\to\infty}\left[d^2(y_n, z) - d^2(x_n, z)\right] = 0$. By (4.5), we have

$$\frac{R}{2}\beta_n(1-\beta_n)d^2(x_n, T(x_n)) \leq d^2(x_n, z) - d^2(y_n, z).$$

This implies by (ii) that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. By Lemma 4.4.1, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

The following lemma is also needed (cf. [39, Lemma 4.2]).

Theorem 4.4.4. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X with $\lim_{n\to\infty} d(x_n, y_n) = 0$. If $\Delta - \lim_{n\to\infty} x_n = x$ and $\Delta - \lim_{n\to\infty} y_n = y$, then $x = y$.*

Proof. Since $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x$, we know that

$$r(\{x_n\}) = r(x, \{x_{n_j}\}) = \limsup_{j\to\infty} d(x_{n_j}, x)$$

for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Now, take any subsequence $\{y_{n_j}\}$ of $\{y_n\}$. Then, there exists $y \in X$ such that $A(\{y_{n_j}\}) = \{y\}$. Hence,

$$\begin{aligned}
\limsup_{j\to\infty} d(y_{n_j}, y) &\leq \limsup_{j\to\infty} d(y_{n_j}, x) \\
&\leq \limsup_{j\to\infty} d(y_{n_j}, x_{n_j}) + \limsup_{j\to\infty} d(x_{n_j}, x) \\
&= \limsup_{j\to\infty} d(x_{n_j}, x) \\
&= r(\{x_n\}) \\
&\leq \limsup_{j\to\infty} d(x_{n_j}, y) \\
&\leq \limsup_{j\to\infty} d(x_{n_j}, y_{n_j}) + \limsup_{j\to\infty} d(y_{n_j}, y) \\
&\leq \limsup_{j\to\infty} d(y_{n_j}, y)
\end{aligned}$$

Hence, $\limsup_{j\to\infty} d(y_{n_j}, y) = \limsup_{j\to\infty} d(y_{n_j}, x)$. And this implies that $x \in A(\{y_{n_j}\})$. Since $A(\{y_{n_j}\}) = \{y\}$, $x = y$. This completes the proof. □

Theorem 4.4.5. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam(X) $\leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T, S : C \to X$ be two generalized hybrid mappings with $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence in $[0, 1]$ and define a sequence $\{x_n\}$ in C by*

$$\begin{cases}
x_1 \in C \quad \text{chosen arbitrary}, \\
x_{n+1} := P_C\left((1-\alpha_n)x_n \oplus \alpha_n T(y_n)\right), \\
y_n := P_C\left((1-\beta_n)x_n \oplus \beta_n S(x_n)\right).
\end{cases}$$

Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$ and suppose that

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,

(ii) $k_2^T(z) = 0$ *and* $\liminf_{n\to\infty} \beta_n \left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right] > 0$ *for all* $z \in F(T) \cap F(S)$.

Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

Proof. Let $z \in F(T) \cap F(S)$. Since $k_2^T(z) = 0$, $d(T(x), z) \leq d(x, z)$ for all $x \in C$. By Lemmas 2.4.14 and 4.1.1, we have

$$\begin{aligned}
d^2(y_n, z) &= d^2(P_C\left((1-\beta_n)x_n \oplus \beta_n S(x_n)\right), z) \\
&\leq d^2\left((1-\beta_n)x_n \oplus \beta_n S(x_n), z\right) \\
&\leq (1-\beta_n)d^2(x_n, z) + \beta_n d^2(S(x_n), z) - \frac{R}{2}\beta_n(1-\beta_n)d^2(x_n, S(x_n))
\end{aligned}$$

$$\leq (1-\beta_n)d^2(x_n,z) + \beta_n\left[d^2(x_n,z) + \frac{k_2^S(z)}{1-a_3^S(z)}d^2(S(x_n),x_n)\right]$$
$$-\frac{R}{2}\beta_n(1-\beta_n)d^2(x_n,S(x_n))$$
$$\leq d^2(x_n,z) - \beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right]d^2(S(x_n),x_n). \tag{4.6}$$

By (ii), there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right] \geq \delta > 0 \text{ for all } n \geq N.$$

Without loss of generality, we may assume that

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right] > 0 \text{ for all } n \geq N.$$

By (4.6), $d(y_n,z) \leq d(x_n,z)$. Thus

$$\begin{aligned}
d^2(x_{n+1},z) &= d^2(P_C((1-\alpha_n)x_n \oplus \alpha_n T(y_n)),z)\\
&\leq d^2((1-\alpha_n)x_n \oplus \alpha_n T(y_n),z)\\
&\leq (1-\alpha_n)d^2(x_n,z) + \alpha_n d^2(T(y_n),z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n,T(y_n))\\
&\leq (1-\alpha_n)d^2(x_n,z) + \alpha_n d^2(y_n,z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n,T(y_n))\\
&\leq d^2(x_n,z) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n,T(y_n))\\
&\leq d^2(x_n,z).
\end{aligned} \tag{4.7}$$

Hence $\lim_{n\to\infty} d(x_n,z)$ exists and

$$\lim_{n\to\infty} \alpha_n(1-\alpha_n)d^2(x_n,T(y_n)) = 0.$$

By (i), $\lim_{n\to\infty} d^2(x_n,T(y_n)) = 0$. It follows from (4.7) that

$$0 \leq \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n,T(y_n)) \leq d^2(x_n,z) - d^2(x_{n+1},z) + \alpha_n\left[d^2(y_n,z) - d^2(x_n,z)\right].$$

Thus

$$\alpha_n(1-\alpha_n)\left[d^2(x_n,z) - d^2(y_n,z)\right] \leq d^2(x_n,z) - d^2(x_{n+1},z).$$

Again, by (i), $\limsup_{n\to\infty}\left[d^2(x_n,z) - d^2(y_n,z)\right] = 0$. By (4.6), we have

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right]d^2(x_n,S(x_n)) \leq d^2(x_n,z) - d^2(y_n,z).$$

This implies by (ii) that $\lim_{n\to\infty} d(x_n, S(x_n)) = 0$. Hence,

$$\begin{aligned}
\limsup_{n\to\infty} d(y_n, x_n) &= \limsup_{n\to\infty} d(P_C\left((1-\beta_n)x_n \oplus \beta_n S(x_n)\right), P_C(x_n)) \\
&\leq \limsup_{n\to\infty} d\left((1-\beta_n)x_n \oplus \beta_n S(x_n), x_n\right) \\
&= \limsup_{n\to\infty} \beta_n d(x_n, S(x_n)) \\
&\leq \limsup_{n\to\infty} d(x_n, S(x_n)) \\
&= 0.
\end{aligned}$$

So, $\lim_{n\to\infty} d(y_n, T(y_n)) = 0$. By Lemma 4.4.1, There exist $u, v \in C$ such that $\omega_w(x_n) = \{u\} \subseteq F(S)$ and $\omega_w(y_n) = \{v\} \subseteq F(T)$. This means that $\Delta - \lim_{n\to\infty} x_n = u$ and $\Delta - \lim_{n\to\infty} y_n = v$. Hence, by Lemma 4.4.4, $u = v$ and the proof is complete. □