CHAPTER 5

Fundamentally nonexpansive mappings in CAT(κ) spaces

In this chapter, we study fixed point theorems and Δ-convergence theorems of the Ishikawa iteration for fundamentally nonexpansive mappings on CAT(κ) spaces with $k > 0$.

5.1 Basic concepts

Let C be a nonempty subset of a metric space (X, d). Recall that a mapping $T : C \to C$ is said to be *fundamentally nonexpansive* if

$$d(T^2(x), T(y)) \leq d(T(x), y), \quad \text{for all } x, y \in C.$$

It is clear that every nonexpansive mapping is fundamentally nonexpansive, but the converse is not true.

Example 5.1.1. [47] Define a mapping $T : [0, 2] \to [0, 2]$ by

$$T(x) = \begin{cases} 0, & x \neq 2, \\ 1, & x = 2. \end{cases}$$

Then T is fundamentally nonexpansive, but T is not nonexpansive.

Proposition 5.1.1. [47] *Every mapping which satisfies condition* (C) *is fundamentally nonexpansive, but the inverse is not true.*

Example 5.1.2. [20] Suppose $X = \{(0,0), (0,1), (1,1), (1,2)\}$. Define

$$d\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{|x_1 - x_2|, |y_1 - y_2|\right\}$$

Define T on X by

$$T(0,0) = (1,2), \quad T(0,1) = (0,0), \quad T(1,1) = (1,1), \quad T(1,2) = (0,1).$$

Then T is fundamentally nonexpansive but does not satisfy condition (C).

Example 5.1.3. Define a mapping $T : [0,2] \to [0,2]$ by

$$T(x) = \begin{cases} 0, & x \in [0,1), \\ \frac{3}{4}, & x \in [1,2]. \end{cases}$$

Then T is generalized hybrid but T is not fundamentally nonexpansive.

Proof. By taking $x = 2, y = 1$, we have

$$d(T^2(x), T(y)) = d\left(0, \frac{3}{4}\right) = \frac{3}{4}$$

but

$$d(T(x), y) = d\left(\frac{3}{4}, 1\right) = \frac{1}{4} \ngeq d(T^2(x), T(y)).$$

Therefore T is not fundamentally nonexpansive. Next, we show that T is generalized hybrid. If $x \in [0,1), y \in [1,2]$ then $d^2(T(x), T(y)) = d^2\left(0, \frac{3}{4}\right) = \frac{9}{16}$. We choose $a_1(x) = \frac{1}{16}, a_2(x) = \frac{9}{16}, a_3(x) = \frac{6}{16}, k_1(x) = \frac{6}{32}, k_2(x) = \frac{9}{16}$, we get that

$$\begin{aligned} d^2(T(x), T(y)) = \frac{9}{16} &\leq a_2(x) d^2(T(x), y) \\ &\leq a_1(x) d^2(x, y) + a_2(x) d^2(T(x), y) + a_3(x)(T(y), x) + k_1(x) d^2(T(x), x) \\ &+ k_2(x) d^2(T(y), y). \end{aligned}$$

Therefore T is generalized hybrid. □

Lemma 5.1.2. *Let C be a nonempty bounded closed convex subset of a complete CAT(κ) space (X, d), and $T : C \to C$ be a fundamentally nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ converging to some point $z \in C$. Since

$$d(x_n, T(z)) = d(T^2(x_n), T(z)) \leq d(T(x_n), z) = d(x_n, z),$$

$$\limsup_{n\to\infty} d(x_n, T(z)) \leq \limsup_{n\to\infty} d(x_n, z) = 0.$$

That is $\{x_n\}$ converges to $T(z)$. This implies that $T(z) = z$. Therefore $F(T)$ is closed. □

Lemma 5.1.3. *Let C be a nonempty subset of a CAT(κ) space (X, d), and $T : C \to C$ be a fundamentally nonexpansive mapping. Then*

$$d(x, T(y)) \leq 3d(T(x), x) + d(x, y),$$

for all $x, y \in C$.

Proof. Since T is fundamentally nonexpansive, we have

$$\begin{aligned} d(x, T(y)) &\le d(x, T(x)) + d(T(x), T^2(x)) + d(T^2(x), T(y)) \\ &\le 2d(x, T(x)) + d(T(x), y) \\ &\le 3d(x, T(x)) + d(x, y). \end{aligned}$$

This completes the proof. □

Lemma 5.1.4. (cf. [45]) *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space such that diam$(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and let $\{z_n\}$ and $\{w_n\}$ be two sequences in X. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $z_{n+1} = \beta_n z_n + (1-\beta_n)w_n$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty}(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \le 0$. Then $\lim_{n\to\infty} d(w_n, z_n) = 0$.*

Lemma 5.1.5. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping, then there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} d(T(x_n), x_n) = 0$.*

Proof. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = \alpha T(x_n) \oplus (1-\alpha)x_n$ for $n \in \mathbb{N}$, where α is a real number belonging to $(0, 1)$. Then we have

$$d(T(x_{n+1}), T(x_n)) = \alpha d(T^2(x_n), T(x_n)) \le \alpha d(T(x_n), x_n) = d(x_{n+1}, x_n)$$

for all $n \in \mathbb{N}$ and hence

$$d(T(x_{n+1}), T(x_n)) \le d(x_{n+1}, x_n).$$

This implies that

$$\limsup_{n\to\infty} (d(T(x_{n+1}), T(x_n)) - d(x_{n+1}, x_n)) \le 0.$$

So by Lemma 5.1.4, we have

$$\lim_{n\to\infty} d(T(x_n), x_n) = 0.$$

This completes the proof. □

5.2 Fixed point theorems

Theorem 5.2.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Proof. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = \alpha T(x_n) \oplus (1-\alpha)x_n$ for $n \in \mathbb{N}$, where α is a real number belonging to $(0,1)$ and let $A(\{x_n\}) = \{z\}$. It follows from Lemma 2.4.19 that $z \in C$. By Lemma 5.1.5, we have $\limsup_{n\to\infty} d(T(x_n), x_n) = 0$. By Lemma 5.1.3, we have

$$d(x_n, T(z)) \leq 3d(T(x_n), x_n) + d(x_n, z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty} d(x_n, T(z)) \leq \limsup_{n\to\infty} d(x_n, z).$$

Since $A(\{x_n\}) = \{z\}$, it must be the case that $z = T(z)$. □

As a consequence of Theorem 5.2.1, we obtain

Corollary 5.2.2. *Let (X,d) be a complete CAT(0) space. Let C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Corollary 5.2.3. *Let $\kappa > 0$ and (X,d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let C be a nonempty compact convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Lemma 5.2.4. *Let $\kappa > 0$ and (X,d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. Suppose $x_1 \in C$, and $\{x_n\}$ defined by*

$$\begin{cases} y_n = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n T(y_n) \oplus (1-\beta_n)x_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, z)$ exists for all $z \in F(T)$.

Proof. Let $z \in F(T)$. By Lemma 5.1.3, we have

$$\begin{aligned} d(x_{n+1}, z) &= d(\beta_n T(y_n) \oplus (1-\beta_n)x_n, z) \\ &\leq \beta_n d(T(y_n), z) + (1-\beta_n)d(x_n, z) \\ &\leq \beta_n \left[3d(T(z), z) + d(y_n, z)\right] + (1-\beta_n)d(x_n, z) \\ &\leq \beta_n d(y_n, z) + (1-\beta_n)d(x_n, z) \\ &= \beta_n d(\alpha_n T(x_n) \oplus (1-\alpha_n)x_n, z) + (1-\beta_n)d(x_n, z) \end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n\alpha_n d(T(x_n), z) + \beta_n(1-\alpha_n)d(x_n, z) + (1-\beta_n)d(x_n, z) \\
&\leq \beta_n\alpha_n \left[3d(T(z), z) + d(x_n, z)\right] + \beta_n(1-\alpha_n)d(x_n, z) + (1-\beta_n)d(x_n, z) \\
&\leq \beta_n\alpha_n d(x_n, z) + \beta_n(1-\alpha_n)d(x_n, z) + (1-\beta_n)d(x_n, z) \\
&\leq d(x_n, z). \qquad (5.1)
\end{aligned}
$$

This implies that $\{d(x_n, z)\}$ is bounded and nonincreasing for all $z \in F(T)$. Hence $\lim_{n\to\infty} d(x_n, z)$ exists. □

Lemma 5.2.5. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$
\begin{cases}
y_n = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n, \\
x_{n+1} = \beta_n T(y_n) \oplus (1-\beta_n)x_n,
\end{cases}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\beta_n \in [a, b]$ and $\alpha_n \in [0, b]$ or $\beta_n \in [a, 1]$ and $\alpha_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(T(x_n), x_n) = 0$.

Proof. Suppose that $F(T) \neq \emptyset$ and let $z \in F(T)$. Then by Lemma 5.2.4, $\lim_{n\to\infty} d(x_n, z)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n, z) = c. \qquad (5.2)$$

By Lemma 5.1.3, we have

$$
\begin{aligned}
d(T(y_n), z) &\leq 3d(T(z), z) + d(y_n, z) \\
&= d(y_n, z) \\
&= d(\alpha_n T(x_n) \oplus (1-\alpha_n)x_n, z) \\
&\leq \alpha_n d(T(x_n), z) + (1-\alpha_n)d(x_n, z) \\
&\leq \alpha_n \left[3d(T(z), z) + d(x_n, z)\right] + (1-\alpha_n)d(x_n, z) \\
&= \alpha_n d(x_n, z) + (1-\alpha_n)d(x_n, z) = d(x_n, z).
\end{aligned}
$$

Thus

$$\limsup_{n\to\infty} d\left(T(y_n), z\right) \leq \limsup_{n\to\infty} d\left(y_n, z\right) \leq \limsup_{n\to\infty} d\left(x_n, z\right) = c. \qquad (5.3)$$

Further, we have

$$c = \lim_{n\to\infty} d\left(x_{n+1}, z\right) = \lim_{n\to\infty} d\big(\beta_n T(y_n) \oplus (1-\beta_n)x_n, z\big). \qquad (5.4)$$

We first consider case: If $0 < a \leq \beta_n \leq b < 1$ and $0 \leq \alpha_n \leq b < 1$.
By (5.2), (5.3), (5.4) and using Lemma 3.1.5, we obtain $\lim_{n\to\infty} d(T(y_n), x_n) = 0$.
Since

$$\begin{aligned}
d(T(x_n), x_n) &\leq d(T(x_n), T(y_n)) + d(T(y_n), x_n) \\
&\leq d\left(T(x_n), T^2(y_n)\right) + d\left(T^2(y_n), T(y_n)\right) + d(T(y_n), x_n) \\
&\leq d(x_n, T(y_n)) + d(T(y_n), y_n) + d(T(y_n), x_n) \\
&\leq 2d(x_n, T(y_n)) + d(T(y_n), y_n) \\
&\leq 2d(x_n, T(y_n)) + d(T(y_n), x_n) + d(x_n, y_n) \\
&\leq 3d(x_n, T(y_n)) + d(x_n, y_n) \\
&= 3d(T(y_n), x_n) + d\big(x_n, \alpha_n T(x_n) \oplus (1-\alpha_n)x_n\big) \\
&\leq 3d(T(y_n), x_n) + \alpha_n d(x_n, T(x_n)) + (1-\alpha_n) d(x_n, x_n) \\
&= \alpha_n d(T(x_n), x_n) + 3d(T(y_n), x_n).
\end{aligned}$$

Then $(1-\alpha_n) d(T(x_n), x_n) \leq 3d(T(y_n), x_n)$.
Since $0 \leq \alpha_n \leq b < 1$, we obtain

$$(1-b) d(T(x_n), x_n) \leq (1-\alpha_n) d(T(x_n), x_n) \leq 3d(T(y_n), x_n).$$

Thus

$$d(T(x_n), x_n) \leq \frac{3}{1-b} d(T(y_n), x_n).$$

Therefore

$$\lim_{n\to\infty} d(T(x_n), x_n) \leq \frac{3}{1-b} \lim_{n\to\infty} d(T(y_n), x_n) = 0.$$

On the other hand, if $0 < a \leq \beta_n \leq 1$ and $0 < a \leq \alpha_n \leq b < 1$, then by Lemma 5.1.3 we have $d(T(x_n), z) \leq 3d(T(z), z) + d(x_n, z) = d(x_n, z)$ for all $n \in \mathbb{N}$. This implies that

$$\limsup_{n\to\infty} d(T(x_n), z) \leq c. \tag{5.5}$$

Now,

$$\begin{aligned}
d(x_{n+1}, z) &= d\big(\beta_n T(y_n) \oplus (1-\beta_n)x_n, z\big) \\
&\leq \beta_n d(T(y_n), z) + (1-\beta_n) d(x_n, z) \\
&\leq \beta_n \left[3d(T(z), z) + d(y_n, z)\right] + (1-\beta_n) d(x_n, z) \\
&= \beta_n d(y_n, z) + (1-\beta_n) d(x_n, z) \\
&= \beta_n d(y_n, z) + d(x_n, z) - \beta_n d(x_n, z).
\end{aligned}$$

This implies that

$$\frac{d\left(x_{n+1}, z\right)-d\left(x_n, z\right)}{\beta_n} \leq d\left(y_n, z\right)-d\left(x_n, z\right).$$

Hence

$$c \leq \liminf_{n \rightarrow \infty} d\left(y_n, z\right).$$

By (5.3) we have, $\limsup_{n \rightarrow \infty} d\left(y_n, z\right) \leq c$ thus

$$\liminf_{n \rightarrow \infty} d\left(y_n, z\right)=c=\limsup_{n \rightarrow \infty} d\left(y_n, z\right),$$

yielding

$$c=\lim_{n \rightarrow \infty} d\left(y_n, z\right)=\lim_{n \rightarrow \infty} d\big(\alpha_n T(x_n) \oplus(1-\alpha_n) x_n, z\big). \tag{5.6}$$

Since $\lim_{n \rightarrow \infty} d\left(x_n, z\right)$ exists, by (5.2), (5.5) (5.6) and using Lemma 3.1.5, we have

$$\lim_{n \rightarrow \infty} d\left(x_n, T(x_n)\right)=0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} d\left(T(x_n), x_n\right)=0$.
Let $A\left(\{x_n\}\right)=\{z\}$. By Lemma 5.1.3, we have

$$d(x_n, T(z)) \leq 3 d(T(x_n), x_n)+d(x_n, z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n \rightarrow \infty} d\left(x_n, T(z)\right) \leq \limsup_{n \rightarrow \infty}\big(3 d(T(x_n), x_n)+d(x_n, z)\big).$$

Since $\limsup_{n \rightarrow \infty} d\left(x_n, T(x_n)\right)=0$, $\limsup_{n \rightarrow \infty} d\left(x_n, T(z)\right) \leq \limsup_{n \rightarrow \infty} d\left(x_n, z\right)$. By the uniqueness of asymptotic center, we obtain $T(z)=z$. Therefore, z is a fixed point of T. □

Theorem 5.2.6. *Let $\kappa>0$ and (X, d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in(0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T: C \rightarrow C$ be a fundamentally nonexpansive mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n \rightarrow \infty} d(T(x_n), x_n)=0$ and $\Delta-\lim_{n \rightarrow \infty} x_n=z$. Then $z \in C$ and $z=T(z)$.*

Proof. Since $\Delta-\lim_{n \rightarrow \infty} x_n=z$, by Lemma 2.4.19, we have $z \in C$. It follows from Lemma 5.1.3 that

$$d(x_n, T(z)) \leq 3 d(T(x_n), x_n)+d(x_n, z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty} d(x_n, T(z)) \leq \limsup_{n\to\infty} d(x_n, z).$$

By the uniqueness of asymptotic center, we obtain $z = T(z)$. □

Lemma 5.2.7. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Suppose $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(T(x_n), x_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4.19, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v \in C$. By Theorem 5.2.6, $v \in F(T)$. By Lemma 2.4.20, $u = v$. This shows that $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, we have $\{d(x_n, u)\}$ converges. Again, by Lemma 2.4.20, $x = u$. This completes the proof. □

5.3 Δ and strong convergence theorems

Theorem 5.3.1. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n T(y_n) \oplus (1-\beta_n)x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\beta_n \in [a, b]$ and $\alpha_n \in [0, b]$ or $\beta_n \in [a, 1]$ and $\alpha_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. By Theorem 5.2.1, $F(T) \neq \emptyset$. Let $z \in F(T)$, by (5.1) we have $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \geq 1$. Then $\{d(x_n, z)\}$ is bounded and nonincreasing for each $z \in F(T)$, so it is convergent, by Lemma 5.2.5 we have $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. By using Lemma 5.2.7, we obtain that $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$.

□

Theorem 5.3.2. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty compact convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n T(y_n) \oplus (1-\beta_n)x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\beta_n \in [a, b]$ and $\alpha_n \in [0, b]$ or $\beta_n \in [a, 1]$ and $\alpha_n \in [a, b]$ for some a, b with $0 < a \le b < 1$. Then $\{x_n\}$ converge strongly to a fixed point of T.

Proof. By Corollary 5.2.3, $F(T) \neq \emptyset$. Then by Lemma 5.2.5, we have $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to z for some $z \in C$. By Lemma 5.1.3 we have

$$d(x_{n_k}, T(z)) \le 3d(T(x_{n_k}), x_{n_k}) + d(x_{n_k}, z), \qquad \forall k \in \mathbb{N}.$$

Therefore $\{x_{n_k}\}$ converges to $T(z)$. This implies $T(z) = z$. That is, z is a fixed point of T. By Lemma 5.2.4, we have $\lim_{n\to\infty} d(x_n, z)$ exists. Thus $\{x_n\}$ converges strongly to a fixed point of T. $\square$

Theorem 5.3.3. *Let $\kappa > 0$ and (X, d) be a complete CAT(κ) space with diam$(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n T(y_n) \oplus (1-\beta_n)x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\beta_n \in [a, b]$ and $\alpha_n \in [0, b]$ or $\beta_n \in [a, 1]$ and $\alpha_n \in [a, b]$ for some a, b with $0 < a \le b < 1$. Suppose T satisfies condition (I). Then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. By condition (I), we have

$$f(d(x_n, F(T))) \le d(x_n, T(x_n)) \quad \text{for all } n \in \mathbb{N}.$$

It follows from Lemma 5.2.5 that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

We can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{z_k\}$ in $F(T)$ such that

$$d(x_{n_k}, z_k) \le \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}. \tag{5.7}$$

By (5.1), we have

$$d(x_{n_{k+1}}, z_k) \leq d(x_{n_k}, z_k) \leq \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(z_{k+1}, z_k) &\leq d(z_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, z_k) \\ &\leq \frac{1}{2^{(k+1)}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

This shows that $\{z_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed in X, there exists a point $z \in F(T)$ such that $\lim_{k\to\infty} z_k = z$. It follows from (5.7) that $\lim_{k\to\infty} x_{n_k} = z$. Since $\lim_{n\to\infty} d(x_n, z)$ exists, it must be the case that $\lim_{n\to\infty} d(x_n, z) = 0$. Therefore we obtain the desired result. □