CHAPTER 3

Main Results

The purpose of this chapter is to study the methods for location problems. This chapter will be divided into three sections: 1) problem size reduction, 2) p-center problem, and 3) facility location problem. In general, the number of clients and candidate places are likely to be large in real world applications. The relaxation methods applied to minimize the size of problem and the bound of error from reduction have been considered in the first section. In section 2, the p-center problem was studied. The p-center objective function is to minimize the maximum distance between each client and the assigned facility. That is to say, the facility location has to be in the middle of all networks which are suitable for route allocation. In the last section of this chapter, 3 methods have been created to find solutions for the facility location problem. In each method, different location selections for various types of facilities will also be taken into consideration.

3.1 Problem Size Reduction

Since the facility location problem is an NP-hard problem, the problem relaxation has been applied to help solve the problem. If we consider the service nodes between the clients and each facility location in the network, it is assumed that they are connected networks. In order to get to the service node on the street (the degree of the service node equals 2), or the service node at the end of the street (the degree of the service node equals 1), it has to pass through the service node on the corner of the street (the degree of the service node is more than 2). For that particular reason, the problem-size reduction has been applied by considering the service node on the corner of the street only, and the service node on the street and the one at the end of the street with the closest corner have been examined to adjust the need value and the product transportation costs.

Next, we show the proposed method to reduce the size of the problem. First, we define some notations.

Recall $I = \{1, 2, 3, \ldots, n\}$ to be a set of clients or customers,

$J = \{1, 2, 3, \ldots, m\}$ to be a set of potential facility sites,

f_j to be facility setup cost for facility $j \in J$,

h_i to be supply of client $i \in I$,

s_j to be capacity of facility $j \in J$,

$c_{i,j}$ to be the transportation cost from client $i \in I$ to facility $j \in J$. We assume that $I \cup J$ is a node set of a complete graph, and $c_{i,j}$ is the minimum transportation cost between facility j and client i.

Additional notations are defined as follows:

$\deg(i)$ is a degree of node i,

$\tilde{I}$ is set of node who $\deg(i) > 2,\ \forall i \in I$,

$\tilde{I}_i$ is a node set combined with node i,

$\tilde{h}_i$ is a supply of client $i \in \tilde{I}$ after reducing the size of the problem,

$\tilde{n}(i)$ is the amount of the node combined with node $i, \forall i \in \tilde{I}$,

$\tilde{c}_{i,j}$ is the transportation cost between client $i \in \tilde{I}$ and facility $j \in J$ after reducing the size of the problem.

The problem size reduction methods are separated into 2 parts: creating a set with the node with the degree of more than 2 and the one with the degree less than or equal to 2, and adjusting the need value and transportation costs. The conclusion for the stated method can be summarized as follows:

Step 1: Construct a set of nodes having a degree of more than 2 and a set of nodes having a degree of less than or equal to 2.

let $\tilde{I} = \emptyset$ for $i = 1, \ldots, n$ if $\deg(i) > 2,\ \tilde{I} = \tilde{I} \cup i$.

Step 2: Where a node has a degree of less than or equal to 2, combined to the nearest node that has a degree of more than 2, (thus reducing the overall number of nodes). After that, the transportation costs of the removed nodes are estimate by multiplying the transportation costs of the remaining nodes with the number of associated removed nodes ($\tilde{c}_{k,j} = c_{k,j} + \tilde{n}(k)c_{k,j},\ \forall k \in \tilde{I}, j \in J$). The supplies of remained nodes are update by combine the supply associated removed nodes ($\tilde{h}_k = h_k + \sum_{i \in \tilde{I}_k} h_i,\ \forall k \in \tilde{I}$).

In this step, we can consider that the node that has a degree less of than or equal to 2 is then assigned to the same facility as is connected with the nearest node which has a degree of more than 2.

After using the problem size reduction method, the capaciated facility location problem (CFP) model has been evaluated using the $C\tilde{F}P$ model presented below.

$$
\begin{aligned}
(\tilde{CFP}):\ \min & \sum_{j\in J} f_j\tilde{y}_j + \sum_{i\in\tilde{I}}\sum_{j\in J}\tilde{c}_{i,j}\tilde{x}_{i,j} && \\
s.t. \qquad & \sum_{j\in J}\tilde{x}_{i,j} = 1, && \forall i\in\tilde{I}, \\
& \sum_{i\in\tilde{I}}\tilde{h}_i\tilde{x}_{i,j} \le s_j\tilde{y}_j, && \forall j\in J, \\
& \tilde{x}_{i,j}\in\{0,1\}, && \forall i\in\tilde{I}, j\in J, \\
& \tilde{y}_j\in\{0,1\}, && \forall j\in J,
\end{aligned}
$$

where $\tilde{x}_{i,j}$ is a binary decision variable; it is equal to 1 if the client i assigned to facility j, it is equal to 0 otherwise, $\tilde{y}_j$ is equal to 1 if the facility j is opened, it is equal to 0 otherwise. Next we will verify the bound of error derived from reducing the problem size using the stated method with the notations as follows:

Let $x^*_{i,j},\ y^*_j,\ \forall i\in I, j\in J$ be the optimal solution of CFP,

$\tilde{x}^*_{i,j},\ \tilde{y}^*_j,\ \forall i\in\tilde{I}, j\in J$ be the optimal solution of $\tilde{CFP}$,

z^* be the optimal value of CFP,

$\tilde{z}^*$ be the optimal value of the problem after reducing the size by the proposed method,

$|\cdot|$ be the number of element in the set,

abs$(\cdot)$ be the absolute value of a number.

Proposition 3.1.1. *The bound of error from reducing the size of a problem less than or equal to*

$$\mathrm{abs}(r|\tilde{I}| + R|I-\tilde{I}| + K|J|)$$

where

$$r = \max_{k\in\tilde{I}, j\in J}\{c_{k,j}\} - \min_{k\in\tilde{I}, j\in J}\{c_{k,j}\},$$

$$R = \max_{k\in\tilde{I}, j\in J}\{c_{k,j}\} - \min_{i\in I-\tilde{I}, j\in J}\{c_{i,j}\},$$

$$K = \max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}.$$

Proof. Let z^*, $\tilde{z}^*$, $x^*_{i,j},\ y^*_j,\ \forall i \in I, j \in J$ and $\tilde{x}^*_{i,j},\ \tilde{y}^*_j,\ \forall i \in \tilde{I}, j \in J$ be defined as above.

The bound of error is

$$\begin{aligned}\mathrm{abs}(\tilde{z}^* - z^*) &= \mathrm{abs}((\sum_{j\in J} f_j\tilde{y}^*_j + \sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{c}_{k,j}\tilde{x}^*_{k,j}) - (\sum_{j\in J} f_j y^*_j + \sum_{i\in I}\sum_{j\in J} c_{i,j}x^*_{i,j}))\\ &= \mathrm{abs}((\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j) + (\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{c}_{k,j}\tilde{x}^*_{k,j} - \sum_{i\in I}\sum_{j\in J} c_{i,j}x^*_{i,j})).\end{aligned}$$

Since $\tilde{c}_{k,j} = 1 + \tilde{n}(k)$,

$$\begin{aligned}\mathrm{abs}(\tilde{z}^* - z^*) &= \mathrm{abs}((\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j) + \sum_{k\in\tilde{I}}\sum_{j\in J}(1+\tilde{n}(k))c_{k,j}\tilde{x}^*_{k,j} - \sum_{i\in I}\sum_{j\in J} c_{i,j}x^*_{i,j}))\\ &= \mathrm{abs}((\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j) + \sum_{k\in\tilde{I}}\sum_{j\in J}(1+\tilde{n}(k))c_{k,j}\tilde{x}^*_{k,j} - \sum_{i\in I-\tilde{I}}\sum_{j\in J} c_{i,j}x^*_{i,j}\\ &\quad - \sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}x^*_{k,j})\\ &= \mathrm{abs}((\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j) + (\sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}\tilde{x}^*_{k,j} - \sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}x^*_{k,j})\\ &\quad + (\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{n}(k)c_{k,j}\tilde{x}^*_{k,j} - \sum_{i\in I-\tilde{I}}\sum_{j\in J} c_{i,j}x^*_{i,j}).\end{aligned}$$

Consider the first term $(\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j)$.

Since $\sum_{j\in J} f_j y^*_j \geq |J|\min_{j\in J}\{f_j\}$ and $\sum_{j\in J} f_j\tilde{y}^*_j \leq |J|\max_{j\in J}\{f_j\}$ then

$$(\sum_{j\in J} f_j\tilde{y}^*_j - \sum_{j\in J} f_j y^*_j) \leq |J|(\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}) = K|J|,$$

where $K = \max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}$.

Consider the second term $(\sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}\tilde{x}^*_{k,j} - \sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}x^*_{k,j})$.

Since $x^*_{k,j},\ \tilde{x}^*_{k,j},\ \forall k \in \tilde{I}, j \in J$ is a solution of P and $\tilde{P}$ respectively then

$$\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{x}^*_{k,j} = \sum_{k\in\tilde{I}}\sum_{j\in J} x^*_{k,j} = 1,$$

.

$$\therefore \sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}\tilde{x}^*_{k,j} \leq \sum_{k\in\tilde{I}}\max_{k\in\tilde{I},j\in J}\{c_{k,j}\} = |\tilde{I}|\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}$$

$$\text{and } \sum_{k\in\tilde{I}}\sum_{j\in J} c_{k,j}x^*_{k,j} \geq \sum_{k\in\tilde{I}}\min_{k\in\tilde{I},j\in J}\{c_{k,j}\} = |\tilde{I}|\min_{k\in\tilde{I},j\in J}\{c_{k,j}\}.$$

$$(\sum_{k\in\tilde{I}}\sum_{j\in J}c_{k,j}\tilde{x}^*_{k,j}-\sum_{k\in\tilde{I}}\sum_{j\in J}c_{k,j}x^*_{k,j}) \leq |\tilde{I}|\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-|\tilde{I}|\min_{k\in\tilde{I},j\in J}\{c_{k,j}\}$$

$$= |\tilde{I}|(\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-\min_{k\in\tilde{I},j\in J}\{c_{k,j}\})$$

$$= r|\tilde{I}|$$

where $r = \max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-\min_{k\in\tilde{I},j\in J}\{c_{k,j}\}$.

Consider the last term $(\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{n}(k)c_{k,j}\tilde{x}^*_{k,j}-\sum_{i\in I-\tilde{I}}\sum_{j\in J}c_{i,j}x^*_{i,j})$

Consider $\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{n}(k)c_{k,j}\tilde{x}^*_{k,j}$.

Since $\tilde{x}^*_{k,j},\ \forall k\in\tilde{I}, j\in J$ is a solution of $C\tilde{P}F$, $\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{x}^*_{k,j}=1$.

$$\therefore \sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{n}(k)c_{k,j}\tilde{x}^*_{k,j} \leq \sum_{k\in\tilde{I}}\tilde{n}(k)\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}.$$

From $\tilde{n}(k)$ is the number of nodes that are integrated to node $k, \forall k\in\tilde{I}$ therefore

$$\sum_{k\in\tilde{I}}\tilde{n}(k)=|I-\tilde{I}|,$$

$$\therefore \sum_{k\in\tilde{I}}\tilde{n}(k)\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}=|I-\tilde{I}|\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}.$$

Consider $\sum_{i\in I-\tilde{I}}\sum_{j\in J}c_{i,j}x^*_{i,j}$. Since $x^*_{i,j},\ \forall i\in I, j\in J$ is a solution of P then

$$\sum_{i\in I}\sum_{j\in J}x^*_{i,j}=1.$$

$$\therefore \sum_{i\in I-\tilde{I}}\sum_{j\in J}c_{i,j}x^*_{i,j} \geq \sum_{i\in I-\tilde{I}}\min_{i\in I-\tilde{I},j\in J}c_{i,j}=|I-\tilde{I}|\min_{i\in I-\tilde{I},j\in J}\{c_{i,j}\}.$$

$$\therefore (\sum_{k\in\tilde{I}}\sum_{j\in J}\tilde{n}(k)c_{k,j}\tilde{x}^*_{k,j}-\sum_{i\in I-\tilde{I}}\sum_{j\in J}c_{i,j}x^*_{i,j}) \leq |I-\tilde{I}|\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-|I-\tilde{I}|\min_{i\in I-\tilde{I},j\in J}\{c_{i,j}\}$$

$$= |I-\tilde{I}|(\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-\min_{i\in I-\tilde{I},j\in J}\{c_{i,j}\})$$

$$= R|I-\tilde{I}|,$$

where $R=\max_{k\in\tilde{I},j\in J}\{c_{k,j}\}-\min_{i\in I-\tilde{I},j\in J}\{c_{i,j}\}$.

$$\therefore \mathrm{abs}(\tilde{z}^*-z^*) \leq \mathrm{abs}(r|\tilde{I}|+R|I-\tilde{I}|+K|J|).$$

$\square$

In the next section, we will present the method for p-center problems.

3.2 Method for P-center Problem

Since the aim of the p-center problem is to minimize the maximum distance between each client and the facility they are assigned to, the optimal solution of this problem is a measure of some distance between the client and the facility.

If we consider the elements in the distance matrix, the optimal solution has to be one of the matrix elements and the amount of elements in that set is mn when m represents any number of candidate locations for facilities and n is the amount of client nodes. We have invented the method according to the above information. The p-center problem has been demonstrated in 2 parts. In the first part, we will predict the radius value in order to verify the possibility for facility construction in p places in which the location must be able to serve all the clients under the predicted radius. We will execute that by bisecting the members which are arranged in ascending order. The bisection method is used to determine the radius. This method is guaranteed to converge to the optimal solution of p center problem in $\log_2(mn)$ iterations. In the second part, we will examine the stated possibilities. On one hand, if it covers all the customers needs, the predict radius value will be on the upper bound of the result chart. On the other hand, if it cannot cover all the needs, the radius value will be on the lower bound of the chart. We will keep examining that until the margin values between the upper bound are almost the same as these in the lower bound. The main idea of this method construct is based on the following. First, we sort all the distances between the client and the facility. Next, we use a series of the covering problems as well as using the bisection method to select the coverage radius. To explain the method, we define certain notations as follows:

Recall that $d_{i,j}$ is the distance from client i to facility j. We assume $I \cup J$ to be a node set of a complete graph, and $d_{i,j}$ to be the minimum distance (or shortest path) between the facility j and the client i. In order to solve the problem, we denote $D_1 < D_2 < \ldots < D_{max}$ to be the sorted distinct entries of the distance $d_{i,j}$. Obviously, the value of the optimal solution is one of the elements in $D = \{D_1, D_2, \ldots, D_{max}\}$.

D_u and D_l are upper bound and lower bound where u and l are the indices of the upper bound and lower bound in the set D.

We carried out our idea for solving the p-center problem as follows:

Step 0: Set the upper bound which is the member with the most value in the distance matrix and the lower bound is the member with the least value in distance matrix.

Step 1: Evaluating radius which is the middle value between the upper and lower bounds of members in the distance matrix arranged in ascending order.

Step 2: Represents the possibility for the mentioned radius by solving the maximal client coverage problem (MCP1) with the given radius.

Step 3: If the solution of the maximal client coverage problem allows all clients to be covered with p facilities, set the upper bound to be the radius. Else, set the lower bound to be the radius.

Step 4: If the lower and upper bounds are close enough to each other, the upper bound is the solution of the p-center problem. Else, go to Step 1.

The capacitated maximal client coverage problem in Step 2 can be formulated as:

$$\text{(MCP1):} \quad \max \sum_{i \in I} \sum_{j \in J} x_{i,j} \tag{3.1}$$

$$s.t. \quad \sum_{j \in J} x_{i,j} \leq 1, \qquad \forall i \in I, \tag{3.2}$$

$$\sum_{j \in J} y_j = p, \tag{3.3}$$

$$d_{i,j} x_{i,j} \leq \delta y_j, \qquad \forall i \in I, j \in J, \tag{3.4}$$

$$\sum_{i \in I} h_i x_{i,j} \leq s_j y_j, \qquad \forall j \in J, \tag{3.5}$$

$$x_{i,j} \in \{0,1\}, \qquad \forall i \in I, j \in J, \tag{3.6}$$

$$y_j \in \{0,1\}, \qquad \forall j \in J. \tag{3.7}$$

The objective of function (3.1) is to maximize the covered clients. Recall that n is the number of clients: if the solution of the maximum coverage problem is less than n, then all clients cannot be covered by p opened facilities. The constrains (3.2) guarantee that the each client can be assigned to only 1 facility. Constrain (3.3) requires that the exact p facilities must be opened. The constraints (3.4) ensure that a client must be assigned to a facility such that the distance from client i to facility j is less than or equal to the coverage radius δ. The last constraints (3.5) are the capacities restriction.

For the uncapacitated case, we assume that $h_i = 1$, $\forall i \in I$ and $s_j = n$, $\forall j \in J$. Now, we show a step by step method to solve the p-center problem.

Step 1: Set an initial $D_l = D_1$ and $D_u = D_{max}$ $(l = 1, u = |D|)$ and set $\varepsilon = \lceil (u+l)/2 \rceil$, $\delta = D_\varepsilon$.

Step 2: Solve MCP1 using radius $= \delta$.

Step 3: If the solution of MCP1 $< n$, set $D_l = \delta,\ l = \varepsilon$.

Else, D_u is set to be δ and $u = \varepsilon$.

Step 4: If $u - l > 1$, set $\varepsilon = \lceil (u+l)/2 \rceil,\ \delta = D_\varepsilon$, go to Step 2.

Else, the solution of PC $= D_u$.

The idea for solving a maximal client coverage problem was carried out as follows:

Step 1: Find clients who connect to only one facility in the given coverage radius, then open that facility.

Step 2: Assign clients who connect exclusively to the opened facility in step 1 by using the knapsack problem (KP), delete all clients who have been assigned a facility.

Step 3: If the number of opened facilities is fewer than p, go to Step 4. Else, stop.

Step 4: If there are clients who connect to only one facility in the given coverage radius go to Step 1. Else, choose one unopened facility having a maximum number of connecting clients to be opened and go to Step 2.

The knapsack problem used in Step 2 can be formulated as follows:

$$\text{(KP):} \quad \max \sum_{i \in I} h_i x_{i,j} \tag{3.8}$$

$$s.t. \quad \sum_{i \in I} h_i x_{i,j} \leq s_j, \tag{3.9}$$

$$x_{i,j} \in \{0,1\}, \qquad \forall i \in I, j \in J. \tag{3.10}$$

Our method for solving the knapsack problem was carried out as follows:

Step 1: Construct a candidate set whose distances to the facility are less than the given radius and whose supplies are less than the facility's capacities.

Step 2: Assign clients in the candidate set who have a maximum supplies to the facility.

Step 3: Update the capacity of the facility and update the candidate set.

Step 4: If the candidate set is empty, stop. Else, go to Step 2.

3.3 Method for Facility Location Problem

Capacitated facility location problems are considered hard to solve due to a lack of the following information:

i) the number of opened facilities,

ii) the location where the facilities should be opened,

iii) how to assign clients to a facility.

The idea of solving the facility location problem is built around the fact that the problems are relatively easy to solve so long as the number and locations of facilities to be opened are known. Hence, the number of opened facilities are predetermined by the minimum number of β that allows for the feasibility of the problem. For that, supplies of facilities are sorted and added to a set in descending order until the total supplies can be covered by the total capacities of the facilities in the set. Next, the bisection method is applied to determine the given number of opened facilities. After that, the facility location problems are solved using the given number of the opened facilities. Since the facility location problem consists of setup and transportation costs, the difference between setup and transportation initiates the utilization of the 3 methods to solve the facility location problem. When considering setup cost as a function depending on the amount of the prospected facilities, the function characteristics should increase function. However, the stated function might not be any of the functions shown in example 1.

Example 1, in building one facility, the initial setup cost include facility A 5 units, facility B 15 units, and facility C 8 units. If the selection has been made to open 1 facility which is facility B, the overall initial setup cost is 15 units. If choosing to open 2 facilities which are facilities A and C, the overall initial setup cost is 13 units. As demonstrated, the exampled function is not an increasing function according to the increasing amount of facilities.

From the example, it is demonstrated that if the setup cost for each facility is not highly different from each of the others, the value for the setup cost function will be raised. Therefore, we have proposed a proposition and have proved it to guarantee that the mentioned setup cost function will be the increasing function due to the number of facilities represented in this part. Conversely, the minimum transportation cost is a non-increasing function in relation to a number of facilities. The function characteristic should be the non-increasing function and it should be bounded below.

If the setup cost is an increasing function while the transportation cost is a non-increasing function, the total cost function has local optima as shown in Figure 3.1 a). On the other hand, if the setup cost is increasing but the transportation cost is non-increasing and are also convex, the total cost function has a global optimal as seen in Figure 3.1 b).

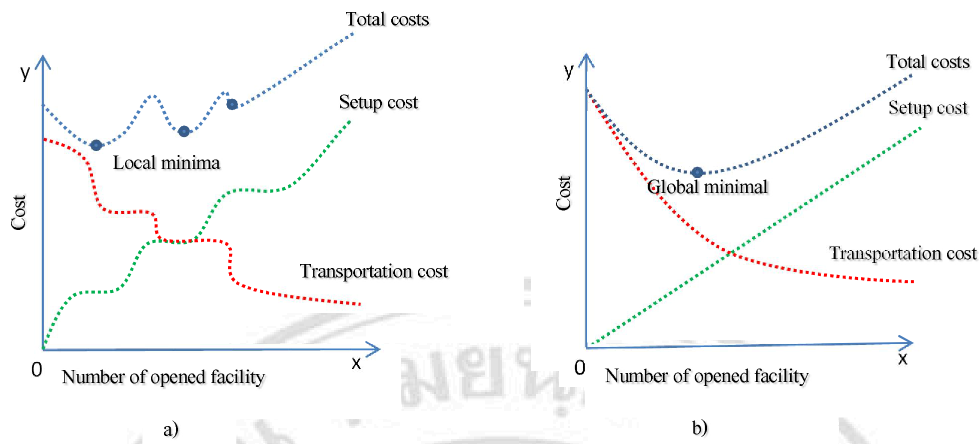

Figure 3.1: Characteristic of the total cost function.

Accordingly, in this part, each method will be separately discussed. The facility location method divides into 3 parts: 1) predict of the opening facility numbers, 2) find candidate sites for facilities, and 3) assign clients to facilities. Next, we will explain the three methods for facility location problems.

Step 0: Let β be the lower bound of the facilities amounts and γ be the upper bound of the facilities amounts.

Step 1: Determine the prediction of the opening facility numbers by using bisection $\lambda = (\beta + \gamma)/2$.

Step 2: Determine the selection to open the facility λ and $\lambda + 1$ facility through the use of the different methods. In this step, we propose three criteria to select sites for the facilities. The method of selection will be discussed later.

Step 3: Use the location from Step 2 for client distribution and the total cost calculation.

Step 4: If the total cost of the facility with the amount λ less than the total cost of the facility with the amount $\lambda + 1$, we will adjust the upper bound value of the facilities numbers. Conversely, if the total of facility with the amount λ greater than or equal to the total of the facility with the amount $\lambda + 1$, we will adjust the lower bound value.

In the next part, we will explain the second step about the principles for facility selection for each method.

3.3.1 Method A (Setup Cost Method)

Since the objective of the facility location problem consists of the setup cost and the transportation cost, if the setup cost is larger than the transportation cost, the optimization routine will give priority to the setup cost. The setup cost method chooses to open facilities that minimize the total setup cost. This method is, therefore, suitable for problems in which the setup cost is larger than the transportation cost.

In this method, the problem is considered as a multi-objective problem. The first objective is to minimize the setup cost and the second objective is to minimize the transportation cost. The problem is solved by using a greedy algorithm to choose p opened facilities that minimize the setup cost. Then, clients are assigned to the opened facilities such that the total transportation cost is minimized. If the solution that minimizes the setup cost is not unique, the local search is used to find the solutions that minimize the total transportation costs.

The principle for facility selection in Method A can be summarized as follows:

Step 0: Initialize the set of locations to the empty set and sort candidate place in non-descending order.

Step 1: Find the node that relates to the minimum setup cost and add it to the solution.

Step 2: If fewer than λ facilities have been added to the solution, the method continues with Step 1; if not, the method stops.

3.3.2 Method B (Modified P-median Method)

When solving a capacitated facility location problem, it can be useful at some point to know the minimum of the total transportation cost with p opened facilities. The objective of the p-median problem is to find that minimum.

For a facility location problem where the transportation cost is higher than the setup cost, the optimization routine will give priority to the transportation cost. The facility with the lowest transportation cost found from the p-median problem will be chosen by this modified method. The local search and greedy algorithm are used to solve the p-median problem.

The principle for facility selection in method B can be summarized as follows:

Step 0: Initialize the set of locations to the empty set and sort candidate place in non-descending order.

Step 1: Find the node that relates to the minimum transportation cost and add it to the solution.

Step 2: If fewer than λ facilities have been added to the solution, the method continues with Step 1; if not, the method stops.

3.3.3 Method C (Modified P-center Method)

The objective of the p-center problem is to locate p facilities so as to minimize the maximum distance between these facilities and their assigned clients. In this method, the farthest distances between the clients and the facilities that they are assigned to are minimized. The p-center problem is similar to the p-cluster problem where clients in each cluster are presented in terms of the minimum farthest distance to the opened facilities in their cluster. As a result, the facilities from the p-center problem are well distributed; therefore, they are opened by this modified method of facility location problem. For the principle selection in Method C, we have selected the location for facility using the solution from p-center problem with the solution-finding method presented in Section 3.2.

Method A, B, and C are constructed based on different aspects. The optimization routines of Method A, B and C give priorities to the setup cost, to the transportation cost and to the distribution of facility location, respectively. For information segregation, the assumptions are created, in this thesis, from the relationship between setup cost and transportation cost. As the average slope of setup costs is the average between the setup cost of 1 opened facility and that of all opened facilities, the average slope of transportation cost is calculated in the same way using the transportation costs. If the minimum and the average slope of the setup costs are more than double of the minimum and the average slope of the transportation costs, respectively, it is considered, in this thesis, the "*higher setup*" cost case (or Method A). If, in fact, the minimum and the average slope of transportation costs are more than double of the minimum and the average slope of setup costs, respectively, this case is considered, in this work, the "*higher transportation*" cost (or Method B). If the minimum of the setup costs is between 0.75 and 1.25 multiply that of the transportation costs and the average slope of the setup costs is between 0.75 and 1.25 multiply that of the transportation costs, the case is defined to be "*balanced*" cost and Method C is used. If these assumptions are not met, the problem is not categorized. The following hypotheses are then constructed to match all methods with problems in each categories.

Hypothesis 1) Method A works better than the Method B and C for the problems with higher setup cost.

Hypothesis 2) Method B works better than the Method A and C for problems with higher transportation cost.

Hypothesis 3) Method C works better than the Method A and B for problems with balanced cost.

The hypothesis testing will be given in the next chapter.

In general, the characteristic of setup cost function depending on the amount of the prospected facilities cannot guarantee that is an increasing function. As shown in Example 1. Therefore, we propose conditions to guarantee that the characteristic of setup cost function depending on the amount of the prospected facilities is an increasing function. To prove conditions is shown in the next section.

First, we define some notations of significance.

Let $C_r^n = \dfrac{n!}{(n-r)!r!}$ be a combinatorial coefficient,

$\lfloor x \rfloor$ be largest integer that does not exceed x,

$p(k)$ be the setup cost with k opened facilities where $k \in \{1, 2, \ldots, m\}$. Given that m represents the maximum number of opened facilities. It is prominent that there is not only the $p(k)$ function amount but there will also be C_k^m functions.

Proposition 3.3.1. *The setup cost function $p(k)$ is an increasing function if there exist an integer r that satisfies one of these conditions*

1) $r = m$, if $f_i = f_j$, $\forall i, j \in J$

2) $2 \leq r \leq \lfloor \dfrac{\min_{j\in J}\{f_j\}}{\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}} \rfloor$

Proof. Let $p(k)$ be a setup cost with k opened facilities where $k \in \{1, 2, \ldots, m\}$.

Case I: $f_i = f_j$, $\forall i, j \in J$

It is easy to see that, in this case $p(k) = kf$ is a linear function. Therefore, the setup cost function $p(k)$, $\forall k \in \{1, 2, \ldots, m\}$ is an increasing function.

Case II: $2 \leq r \leq \lfloor \dfrac{\min_{j\in J}\{f_j\}}{\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}} \rfloor$

Consider $k \in \{1, 2, \ldots, r-1\}$.

Since $r \leq \lfloor \dfrac{\min_{j\in J}\{f_j\}}{\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}} \rfloor$ and $k < r$,

$$k < \lfloor \frac{\min_{j\in J}\{f_j\}}{\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}} \rfloor \leq \frac{\min_{j\in J}\{f_j\}}{\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}}.$$

Since $\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\} > 0$,

$$\begin{aligned} k(\max_{j\in J}\{f_j\} - \min_{j\in J}\{f_j\}) &< \min_{j\in J}\{f_j\} \\ k(\max_{j\in J}\{f_j\}) &< \min_{j\in J}\{f_j\} + k(\min_{j\in J}\{f_j\}) \\ &< (k+1)(\min_{j\in J}\{f_j\}) \end{aligned}$$

Since $p(k)$ is a setup cost with k opened facilities, $k(\min_{j \in J}\{f_j\}) \leq p(k) \leq k(\max_{j \in J}\{f_j\})$.

We have that $p(k) < k(\max_{j \in J}\{f_j\}) \leq p(k+1)$.

Hence, $p(k) < p(k+1), \ \forall \ k \in \{1, 2, \ldots, r-1\}$. This implies that the setup cost function $p(k), \ k \in \{1, 2, \ldots, r-1\}$ is an increasing function. $\square$

It is easy to see that, the condition to guarantee that the characteristic of setup cost function depending on the amount of the prospected facilities is an increasing function if and only if the setup cost of each facility are similar. The problem in accordance with such conditions is often a problem finding location for the same type of facilities in the same area. For example, the problem to find the location for the ATM in the Thonburi area of Bangkok. The setup cost of this problem is a rental. This rental for each area are the same prize. Then this problem has an increasing setup cost function. In the next section the proposition to guarantee that the characteristic of transportation cost function depending on the amount of the prospected facilities is a non-increasing function.

Let $q(k)$ be a transportation cost with k opened facilities where $k \in \{1, 2, \ldots, m\}$. Given that m represents a maximum number of opened facilities, $q(k)$ is not unique and $q(k)$ has the C_k^m functions of $q(k)$,

$q^*(k)$ would then be the minimum transportation cost with k opened facilities,

$Y_k = \{j \mid y_j = 1, \forall j \in J\}$ would then be the set of k opened facilities of the feasible solution,

$Y_k^* = \{j \mid y_j^* = 1, \forall j \in J\}$ would then be the set of k opened facilities of the optimal solution.

Proposition 3.3.2. *The minimum transportation cost function with k opened facilities, $q^*(k)$, $k \in \{1, 2, \ldots, m\}$, is a non-increasing function.*

Proof. Let $q(k)$, $q^*(k)$, Y_k and Y_k^* be defined as above.

Since $q^*(k)$ is the minimum transportation cost with k opened facilities, $q^*(k) \leq q(k)$.

Considering Y_k^*, there exists Y_{k+1} such that $Y_k^* \subset Y_{k+1}$.

Therefore, there exists $q(k+1) \leq q^*(k)$

Since $q^*(k+1) \leq q(k+1)$, it implies that $q^*(k+1) \leq q^*(k)$. Therefore, $q^*(k)$ is a non-increasing function. $\square$

In general, the facility location problem is according to the proposition 3.3.2. The initial idea for solving facility location is considered the setup cost function depending on the amount of the prospected facilities is an increasing function and transportation cost function depending on the amount of the prospected facilities is a non-increasing function. If the facility location problem is according to the proposition 3.31 and 3.3.2, the solution is obtained from the proposed method convert to the local optimal. To guarantee that, the prove conditions is shown in the next section.

Proposition 3.3.3. *The total cost obtained by the proposed method is a local minimum.*

Proof. Recall that α and β are the minimum and maximum number of opened facilities.

Let $[\alpha_i, \beta_i]$ be the searching interval at i iteration.

Let $C^*(\alpha)$ be the minimum total cost with α opened facilities.

Thus, the proposed method will terminate if $\beta_i = \alpha_i + 1$.

From the method, α_i is updated if and only if $C^*(\alpha_i+1) < C^*(\alpha_i)$, and β_i is updated if and only if $C^*(\beta_i) \leq C^*(\beta_i + 1)$.

Hence, at the final step $C^*(\beta_i) = C^*(\alpha_i + 1) < C^*(\alpha_i) = C^*(\beta_i - 1)$ and $C^*(\beta_i) \leq C^*(\beta_i + 1)$. Therefore, the total cost obtained from the proposed method is $C^*(\beta_i)$ and that cost is a local minimum.

If α_i in all iterations are not updated ($\alpha_i = \alpha$) at all iterations, we have determined that at the final step, $C^*(\alpha_i) \leq C^*(\alpha_i + 1) = C^*(\beta_i) \leq C^*(\beta_i + 1)$. Therefore, the total cost obtained from the proposed method, $C^*(\alpha_i)$, is a local minimum. □

Proposition 3.3.4. *If the setup cost with k opened facilities, $p^*(k)$, is an increasing convex function and the minimum transportation cost with k opened facilities, $q^*(k)$, is a convex function where $k \in \{1, 2, \ldots, m\}$, the solution obtained by the proposed method is a global minimum.*

Proof. Since $p^*(k)$ and $q^*(k)$ are convex functions, the minimum total cost $C^*(k) = p^*(k)+ q^*(k)$ is also a convex function. From Theorem 2.1.5 therefore, a local minimum is a global minimum. From Proposition 3, our method always converges to a local minimum. This implies that the obtained solution is also a global minimum. □