CHAPTER 2

Basic Concepts and Preliminaries

In this chapter, we recall and give some useful definitions, and results which will be used in the later chapter.

2.1 System of Linear Equations

In this part, we present the necessary definitions, examples and theorems concerning system of linear equations. A linear equation in variables $x_1, x_2, ..., x_n$ of equation that can be written in the from

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, a_2, ..., a_n$ are real numbers.

A system of linear equations is a collection of one or more linear equations involving the same variables say $x_1, x_2, ..., x_n$ such as

$$\begin{aligned} 6x_1 + 5x_2 &= -6 \\ -4x_1 + 2x_2 &= 3. \end{aligned}$$

A solution of the system n variables is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$ respectively. We call $(s_1, s_2, ..., s_n)$ is an ordered n-tuple. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then n-tuple is called an ordered pair, and if $n = 3$, then it is called an ordered triple.

Linear system in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x_1 + a_2x_2 = c_1 \\ b_1x_1 + b_2x_2 = c_2 \end{aligned}$$

in which the graphs of the equations are line in the xy-plane. Each solution of this system corresponds to a point of intersection of the lines, so there are three possibilities:

1. No solution, or
2. Exactly one solution, or

3. Infinitely many solutions.

In general, a system of linear equations is said to be consistent if it has either one solution or infinitely many solutions; a system is inconsistent if it has no solution.

Matrix Notation

In order to solve a system of linear equations, we usually write it into a matrix form. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 1 \\ 3x_1 - x_2 - 5x_3 &= 2 \end{aligned}$$

which the coefficients of each variable aligned in columns the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 4 \\ 3 & -1 & -5 \end{bmatrix}$$

is called the coefficient matrix of the system and

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 2 & -1 & 4 & 1 \\ 3 & -1 & -5 & 2 \end{array}\right]$$

is called the augmented matrix of the system. An augmented matrix of a system consists of the coefficient matrix with and added column containing the constants from the right sides of the equations. The size of a matrix tells how many rows and columns it has. The augmented matrix above has 3 rows and 4 columns and is called a 3×4 (read 3 by 4) matrix. If m and n are positive integers, an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Normally, for solving a system of linear equations, we use algebraic operations as follows:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

A system of linear equations can be written in the matrix form, $AX = B$ we call $[A|B]$ an augmented matrix. In order to solve the equation $AX = B$, we usually use the following three operations on augmented matrix:

1. Multiply a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

They are called **elementary row operations** on a matrix.

Definition 2.1.1. [3] A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Matrix multiplication

Definition 2.1.2. [4] If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the product $C = AB$ is an $m \times r$ matrix. The (i, j)-entry of the product is computed as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example 2.1.1. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$. Find AB.

Solution. $AB = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+2(1) & 1(-1)+2(3) & 1(4)+2(2) \\ 1(2)+4(1) & 1(-1)+4(3) & 1(4)+4(2) \end{bmatrix}$

$$= \begin{bmatrix} 4 & 5 & 8 \\ 6 & 11 & 12 \end{bmatrix}.$$

Determinant

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix. We define the determinant of A by

$$detA = a_{11}a_{22} - a_{12}a_{21}.$$

Definition 2.1.3. [1] Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i^{th} row and j^{th} column of A. The determinant $det(M_{ij})$ is called the minor of a_{ij}. The cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} detM_{ij}.$$

Theorem 2.1.2. *[4] The Laplace Expansion Theorem*
The determinant of $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the cofactor expansion along the i^{th} row) and also as

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(which is the cofactor expansion along the j^{th} column).

Definition 2.1.4. A square matrix is called upper (lower) triangular if all its elements below (above) the main diagonal are zero.

Theorem 2.1.3. *[1] The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is $n \times n$ triangular matrix, then*

$$detA = a_{11}a_{22} \cdots a_{nn}.$$

Properties of determinants [4] : Let A be a square matrix.
1. If a multiple of one row of A is added to another row to produce a matrix B, then $detB = detA$.
2. If two row of A are interchanged to produce B, then $detB = -detA$.
3. If one row of A is multiplied by k to produce B, then $detB = kdetA$.

Inverse of Matrix

Definition 2.1.5. [1] An $n \times n$ matrix A is called nonsingular (or invertible) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called an inverse of A. If there exists no such matrix B, then A is called nonsingular (or noninvertible).

We shall now write the inverse of A, if it exist, as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n.$$

Example 2.1.4. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad CA = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $C = A^{-1}$.

Definition 2.1.6. [1] Let $A = [a_{ij}]$ be an $n \times n$ matrix and let B be a matrix of cofactors of A. Then the adjoint of A, written $adjA$, is the transpose of $n \times n$ matrix B, that is

$$adjA = B^t = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

Theorem 2.1.5. *[4] If A is an invertible $n \times n$ matrix, then the system of linear equations given by $AX = B$ has unique solution $X = A^{-1}B$ for any matrix B with size $n \times 1$.*

Theorem 2.1.6. *[Gauss-Jordan Elimination] If augmented $[A|B]$ is row reduce to $[I|P]$, then P is the solution of the equation $AX = B$.*

Elementary Matrices

Definition 2.1.7. [4] An elementary matrix is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

Example 2.1.7. Elementary matrices and Row operations: Listed below are four elementary matrices and the operations that produce them.

$\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}$ Multiply the second row of I_2 by -7.

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Interchange the first and second rows of I_4.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ Add 2 times the second row of I_3 to the third row.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Multiply the third row of I_3 by -1.

Theorem 2.1.8. *[4] Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.*

Theorem 2.1.9. *[4] Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.*

Definition 2.1.8. [2] The number of nonzero rows in the row-echelon form of a matrix is known as its rank.

Theorem 2.1.10. *[2] Consider the nonhomogeneous equations $AX = B$ where A is $m \times n$. One of the following possibilities must hold:*
(a). If the rank of the augmented matrix $[A|B]$ is greater than the rank of A, then the system of equations is inconsistent.
(b). If the rank of $[A|B]$ is equal to the rank of A, this begin equal to the number of unknowns, then the equations have a unique solution.
(c). If the rank of $[A|B]$ is equal to the rank of A, this begin less than the number of unknows, then the equations have an infinity of solutions.

Computation of A^{-1}[6]

To calculate the inverse of a nonsingular size $n \times n$ matrix A, we can proceed as follows:
Step 1. From the size $n \times 2n$ matrix $[A|I]$.
Step 2. Use elementary row operations to transform $[A|I]$ to the form $[I|B]$.
In this final form, $B = A^{-1}$.

Example 2.1.11. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

Solution.
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 := r_2 - 2r_1 \\ r_3 := r_3 - r_1 \end{array}$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ r_3 := r_3 + 2r_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad r_3 := (-1)r_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \begin{array}{l} r_1 := r_1 - 3r_3 \\ r_2 := r_2 + 3r_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad r_1 := r_1 - 2r_2$$

Thus $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

2.2 Solving System of Linear Equations

2.2.1 Gaussian Elimination

Row-reduce the coefficient matrix to row echelon form, solve for the last unknown, and then use back substitution to solve for the other unknown.

Example 2.2.1. Find the solution of linear system

$$\begin{aligned} x + y + z &= 2 \\ 2x - 3y + 4z &= -1 \\ x + 5y + 3z &= 6. \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -3 & 4 & -1 \\ 1 & 5 & 3 & 6 \end{array}\right].$$

Row-reduce the coefficient matrix to row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -3 & 4 & -1 \\ 1 & 5 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -5 & 2 & -5 \\ 0 & 4 & 2 & 4 \end{array}\right] \quad \begin{array}{l} r_2 := r_2 - 2r_1 \\ r_3 := r_3 - r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{18}{5} & 0 \end{array}\right] \; r_3 := r_3 + \frac{4}{5}r_2 .$$

Then we have

$$\frac{18}{5}z = 0 \tag{2.1}$$

$$-5y + 2z = -5 \tag{2.2}$$

$$x + y + z = 2 \tag{2.3}$$

From (2.1) we have $z = 0$.

Substitute $z = 0$ in (2.2), we have $y = 1$.

Substitute $z = 0$ and $y = 1$ in (2.3), we have $x = 1$.

Hence the solution are $x = 1, y = 1$ and $z = 0$.

2.2.2 Gauss-Jordan Elimination

Row-reduce the coefficient matrix to reduced row echelon form.

Example 2.2.2. Find the solutions of linear system

$$2x + y = 7$$
$$x + 2y = 8.$$

Solution. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 2 & 8 \end{array}\right].$$

Row-reduce the coefficient matrix to reduced row echelon form.

$$\begin{aligned}
\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 2 & 8 \end{array}\right] &\sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{7}{2} \\ 1 & 2 & 8 \end{array}\right] \; r_1 := \tfrac{1}{2}r_1 \\
&\sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{7}{2} \\ 0 & \frac{3}{2} & \frac{9}{2} \end{array}\right] \; r_2 := r_2 - r_1 \\
&\sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & 3 \end{array}\right] \; r_2 := \tfrac{2}{3}r_2 \\
&\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \; r_1 := r_1 - \tfrac{1}{2}r_2 .
\end{aligned}$$

Hence $x = 2$ and $y = 3$.

2.2.3 Using $X = A^{-1}B$

By Theorem 2.1.5 if A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix B, the system of equation $AX = B$ has exactly one solution, namely, $X = A^{-1}B$.

Example 2.2.3. Find the solutions of linear system

$$\begin{aligned} x_1 + 3x_2 &= 8 \\ x_1 - 2x_2 &= 3. \end{aligned}$$

Solution. From the linear system we have

$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}.$$

We know that $A = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 8 \\ 3 \end{bmatrix}.$

First, we find A^{-1} by Row-reduce A to I_n.

Consider

$$\begin{aligned} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] &\sim \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{array}\right] \quad r_2 := r_2 - r_1 \\ &\sim \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \quad r_2 := r_2 - \frac{1}{5}r_2 \\ &\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \quad r_1 := r_1 - 3r_2. \end{aligned}$$

Then we have $A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}.$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Hence $x_1 = 5$ and $x_2 = 1$.

2.2.4 Cramer's rule

Let [1]

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

be a linear system of n equations in n unknowns and let $A = [a_{ij}]$ be a coefficient matrix so that we can write the given system as $AX = B$, where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $detA \neq 0$, then the system has unique solution. This solution is

$$x_1 = \frac{detA_1}{detA}, x_2 = \frac{detA_2}{detA}, \cdots, x_n = \frac{detA_n}{detA},$$

where A_i is the matrix obtained by replacing the i^{th} column of A by B.

Example 2.2.4. Find the solutions of linear system

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0.$$

Solution. From the linear system we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

We know that

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}.$$

First, we find $detA$, $detA_1$, $detA_2$, $detA_3$.

$$detA = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{vmatrix} = -1, \quad detA_1 = \begin{vmatrix} 9 & 1 & 2 \\ 1 & 4 & -3 \\ 0 & 6 & -5 \end{vmatrix} = -1,$$

$$detA_2 = \begin{vmatrix} 1 & 9 & 2 \\ 2 & 1 & -3 \\ 3 & 0 & -5 \end{vmatrix} = -2, \quad detA_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 4 & 1 \\ 3 & 6 & 0 \end{vmatrix} = -3.$$

Then the solutions are

$$x = \frac{detA_1}{detA} = \frac{-1}{-1} = 1, y = \frac{detA_2}{detA} = \frac{-2}{-1} = 2, z = \frac{detA_3}{detA} = \frac{-3}{-1} = 3.$$

There for $x = 1, y = 2, z = 3$.

2.2.5 LU-Decomposition

Definition 2.2.1. A factorization of a square matrix A as $A = LU$, where L is lower triangular and U is upper triangular is called an LU-decomposition of A.

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row below it. In this case, there exist unit lower triangular elementary matrices $E_1, ..., E_p$ such that

$$E_p \cdots E_2 E_1 A = U.$$

Since elementary matrices are invertible, we can solve for A as

$$A = E_1^{-1} E_2^{-1} \cdots E_p^{-1} U.$$

or more briefly as

$$A = LU,$$

where

$$L = E_1^{-1} E_2^{-1} \cdots E_p^{-1}.$$

The Method of LU-Decomposition

Step 1. Rewrite the system $AX = B$ as

$$LUX = B. \tag{2.4}$$

Find U and L from $E_p \cdots E_2 E_1 A = U$ and $L = E_1^{-1} E_2^{-1} \cdots E_p^{-1}$.

Step 2. Define a new $n \times 1$ matrix Y by

$$UX = Y. \tag{2.5}$$

Use (2.5) to rewrite (2.4) as $LY = B$ and solve this system for Y.

Step 3. Substitute Y in (2.5) and solve for X.

Example 2.2.5. Find the solution of linear system

$$3x_1 + 2x_2 + x_3 = 7$$
$$x_1 - 2x_2 + 2x_3 = -3$$
$$4x_1 - x_2 + 3x_3 = 2.$$

Solution. From the linear system, we can write

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 2 \\ 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 2 \\ 4 & -1 & 3 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}.$$

First, we find U by row reducing of A,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 2 \\ 4 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & -\frac{11}{3} & \frac{5}{3} \end{bmatrix} \begin{matrix} \\ r_2 := r_2 - \frac{1}{3}r_1 \\ r_3 := r_3 - \frac{4}{3}r_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{5}{8} \end{bmatrix} \begin{matrix} \\ \\ r_3 := r_3 - \frac{11}{8}r_2 \end{matrix} = U.$$

By row operation we have E_1, E_2, E_3 that is :

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{11}{8} & 1 \end{bmatrix}.$$

Then $E_1^{-1}, E_2^{-1}, E_3^{-1}$ are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{3} & 0 & 1 \end{bmatrix}, \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{8} & 1 \end{bmatrix}.$$

We can find L by $E_1^{-1}, E_2^{-1}, E_3^{-1}$ so that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{8} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{4}{3} & \frac{11}{8} & 1 \end{bmatrix}.$$

So $A = LU$, we get

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 2 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{4}{3} & \frac{11}{8} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{5}{8} \end{bmatrix}.$$

Second, from $AX = B$, we have that $LUX = B$.

define a new $n \times 1$ matrix Y by $UX = Y$, then $LY = B$ that is

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{4}{3} & \frac{11}{8} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$$

So we have

$$\begin{aligned} y_1 &= 7 \\ \frac{1}{3}y_1 + y_2 &= -3 \\ \frac{4}{3}y_1 + \frac{11}{8}y_2 + y_3 &= 2. \end{aligned}$$

Solve this system for Y we have $y_1 = 7, y_2 = -\frac{16}{3}, y_3 = 0$.

Finally, substitute Y in $UX = Y$ and solve for X, then

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{5}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -\frac{16}{3} \\ 0 \end{bmatrix}.$$

So we have

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 7 \\ -\frac{8}{3}x_2 + \frac{5}{3}x_3 &= -\frac{16}{3} \\ -\frac{5}{8}x_3 &= 0. \end{aligned}$$

Solve this system for X we have $x_1 = 1, x_2 = 2, x_3 = 0$.

Hence the solutions of the system are $x_1 = 1, x_2 = 2, x_3 = 0$.

Algorithm for finding A^{-1}

In 2014, Jafree et al.[5] introduced an algorithm for finding A^{-1} by constructing the dictionary of matrix as follows : For matrix A of size $n \times n$, we define the dictionary matrix $D(A)$ by

$$D(A) = \begin{array}{c} \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \end{array} \\ \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array} \right] \end{array}.$$

Step 1. Set $H := \{1, 2, ..., n\}, B := \{y_1, y_2, ..., y_n\}, N := \{x_1, x_2, ..., x_n\}$. Construct dictionary of the matrix A, i.e. $D(A)$.

Step 2. Set $P := \{p : y_p \in B\}$.

Step 3. If $P = \emptyset$, goto Step 6, otherwise, choose $p \in P$ and let $L := \{k : a_{pk} \neq 0, x_k \in N\}$.

Step 4. If $L = \emptyset$, then inverse does not exist and exit. Otherwise, for any $p \in P, k \in L$ taking

$a_{pj} := \dfrac{a_{pj}}{a_{pk}}, \forall j \in H - \{k\}$,

$m_i := -\dfrac{a_{ik}}{a_{pk}}, \forall i \in H - \{p\}$,

$a_{ik} := m_i, \forall i \in H - \{p\}$,

$a_{ij} := a_{ij} + a_{pj} \times m_i, \forall i \in H - \{p\}, \forall j \in H - \{k\}$,

$a_{pk} := \dfrac{1}{a_{pk}}$.

Step 5. $B := (B \cup \{x_k\}) - \{y_p\}, N := (N - \{x_k\}) \cup \{y_p\}$. Update $D(A)$ and goto Step 2.

Step 6. $A^{-1} = [a_{ij}], x_i \in B, y_j \in N$ for all $i, j \in H$ and exit.

Consider Algorithm for finding A^{-1} and finding A^{-1} by row operaton $[A|I]$ to $[I|A^{-1}]$

Algorithm for finding A^{-1}

We have

$$D(A) = \begin{array}{c} \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \end{array} \\ \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array} \right] \end{array}.$$

If we choose $p = 1, k = 1$ such that $a_{11} \neq 0$. By step 4, we get

$$D(A) = \begin{array}{c} \\ x_1 \\ y_2 \\ \vdots \\ y_n \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & x_2 & x_3 & \cdots & x_n \end{array} \\ \left[\begin{array}{ccccc} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{31}}{a_{11}} a_{13} & \cdots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{11}} & a_{n2} - \frac{a_{n1}}{a_{11}} a_{12} & a_{n3} - \frac{a_{n1}}{a_{11}} a_{13} & \cdots & a_{nn} - \frac{a_{n1}}{a_{11}} a_{1n} \end{array} \right] \end{array}.$$

Finding A^{-1} by row operaton $[A|I]$ to $[I|A^{-1}]$.

If we use the row operations for finding A^{-1} in Step 4, we obtain the following :

$$[A|I] = \left[\begin{array}{ccccc|ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 1 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & 0 & 0 & 0 & \cdots & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccccc|ccccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & 0 & 0 & 0 & \cdots & 1 \end{array} \right] \begin{array}{l} r_1 := \frac{1}{a_{11}} r_1 \\ \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|ccccc} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} - a_{21}\frac{a_{12}}{a_{11}} & \cdots & a_{2n} - a_{21}\frac{a_{1n}}{a_{11}} & -\frac{a_{21}}{a_{11}} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - a_{n1}\frac{a_{12}}{a_{11}} & \cdots & a_{nn} - a_{n1}\frac{a_{1n}}{a_{11}} & -\frac{a_{n1}}{a_{11}} & 0 & 0 & \cdots & 1 \end{array} \right] \begin{array}{l} \\ r_2 := r_2 - a_{21} r_1 \\ \\ r_n := r_n - a_{n1} r_1. \end{array}$$

We note that the first column of $D(A)$ is the same as the $(n+1)^{th}$ column of $[A|I]$. After the n^{th} iteration, we obtain $[A|I] \sim [I|A^{-1}]$ and $D(A) = A^{-1}$.

Example 2.2.6. Find the inverse of $A = \begin{bmatrix} 1 & 3 & -3 & 2 \\ 1 & 3 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 4 & 1 & 3 & 5 \end{bmatrix}$.

Solution. Step 1. Set $D(A)$ by

$$D(A) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 3 & -3 & 2 \\ 1 & 3 & 1 & 4 \\ 0 & 1 & 3 & -1 \\ 4 & 1 & 3 & 5 \end{bmatrix} \end{array}$$

and $H := \{1,2,3,4\}, N := \{x_1, x_2, x_3, x_4\}, B := \{y_1, y_2, y_3, y_4\}$.

Iteration 1 :

Step 2. Set $P := \{1,2,3,4\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 1$, we get $L := \{1,2,3,4\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 1$, then

$a_{12} := \frac{a_{12}}{a_{11}} = 3$, $\quad a_{13} := \frac{a_{13}}{a_{11}} = -3$, $\quad a_{14} := \frac{a_{14}}{a_{11}} = 2$,

$m_2 = -\frac{a_{21}}{a_{11}} = -1$, $\quad m_3 = -\frac{a_{31}}{a_{11}} = 0$, $\quad m_4 = -\frac{a_{41}}{a_{11}} = -4$,

$a_{21} := m_2 = -1$, $\quad a_{31} := m_3 = 0$, $\quad a_{41} := m_4 = -4$,

$a_{22} := a_{22} + a_{12}m_2 = 0$, $\quad a_{23} := a_{23} + a_{13}m_2 = 4$,

$a_{24} := a_{24} + a_{14}m_2 = 2$, $\quad a_{32} := a_{32} + a_{12}m_3 = 1$,

$a_{33} := a_{33} + a_{13}m_3 = 3$, $\quad a_{34} := a_{34} + a_{14}m_3 = -1$,

$a_{42} := a_{42} + a_{12}m_4 = -11$, $\quad a_{43} := a_{43} + a_{13}m_4 = 15$,

$a_{44} := a_{44} + a_{14}m_4 = -3$, $\quad a_{11} := \frac{1}{a_{11}} = 1$.

Step 5. $B := (B \cup \{x_1\}) - \{y_1\} = \{x_1, y_2, y_3, y_4\}$,

$N := (N - \{x_1\}) \cup \{y_1\} = \{y_1, x_2, x_3, x_4\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 3 & -3 & 2 \\ -1 & 0 & 4 & 2 \\ 0 & 1 & 3 & -1 \\ -4 & -11 & 15 & -3 \end{bmatrix} \end{array}.$$

Iteration 2 :

Step 2. Set $P = \{2,3,4\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 2$, we get $L = \{3,4\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 3$, then

$a_{21} := \frac{a_{21}}{a_{23}} = -\frac{1}{4}$, $a_{22} := \frac{a_{22}}{a_{23}} = 0$, $a_{24} := \frac{a_{24}}{a_{23}} = \frac{1}{2}$,

$m_1 = -\frac{a_{13}}{a_{23}} = \frac{3}{4}$, $m_3 = -\frac{a_{33}}{a_{23}} = -\frac{3}{4}$, $m_4 = -\frac{a_{43}}{a_{23}} = -\frac{15}{4}$,

$a_{13} := m_1 = \frac{3}{4}$, $a_{33} := m_3 = -\frac{3}{4}$, $a_{43} := m_4 = -\frac{15}{4}$,

$a_{11} := a_{11} + a_{21}m_1 = \frac{1}{4}$, $a_{12} := a_{12} + a_{22}m_1 = 3$,

$a_{14} := a_{14} + a_{24}m_1 = \frac{7}{2}$, $a_{31} := a_{31} + a_{21}m_3 = \frac{3}{4}$,

$a_{32} := a_{32} + a_{22}m_3 = 1$, $a_{34} := a_{34} + a_{24}m_3 = -\frac{5}{2}$,

$a_{41} := a_{41} + a_{21}m_4 = \frac{1}{4}$, $a_{42} := a_{42} + a_{22}m_4 = -11$,

$a_{44} := a_{44} + a_{24}m_4 = -\frac{21}{2}$, $a_{23} := \frac{1}{a_{23}} = \frac{1}{4}$.

Step 5. $B := (B \cup \{x_3\}) - \{y_2\} = \{x_1, x_3, y_3, y_4\}$,

$N := (N - \{x_3\}) \cup \{y_2\} = \{y_1, x_2, y_2, x_4\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ x_3 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & x_2 & y_2 & x_4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & 3 & \frac{3}{4} & \frac{7}{2} \\ -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 1 & -\frac{3}{4} & -\frac{5}{2} \\ -\frac{1}{4} & -11 & -\frac{15}{4} & -\frac{21}{2} \end{bmatrix} \end{array}.$$

Iteration 3 :

Step 2. Set $P = \{3, 4\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 3$, we get $L = \{2, 4\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 2$, then

$a_{31} := \frac{a_{31}}{a_{32}} = \frac{3}{4}$, $a_{33} := \frac{a_{33}}{a_{32}} = -\frac{3}{4}$, $a_{34} := \frac{a_{34}}{a_{32}} = -\frac{5}{2}$,

$m_1 = -\frac{a_{12}}{a_{32}} = -3$, $m_2 = -\frac{a_{22}}{a_{32}} = 0$, $m_4 = -\frac{a_{42}}{a_{32}} = 11$,

$a_{12} := m_1 = -3$, $a_{22} := m_2 = 0$, $a_{42} := m_4 = 11$,

$a_{11} := a_{11} + a_{31}m_1 = -2$, $a_{13} := a_{13} + a_{33}m_1 = 3$,

$a_{14} := a_{14} + a_{34}m_1 = 11$, $a_{21} := a_{21} + a_{31}m_2 = -\frac{1}{4}$,

$a_{23} := a_{23} + a_{33}m_2 = \frac{1}{4}$, $a_{24} := a_{24} + a_{34}m_2 = \frac{1}{2}$,

$a_{41} := a_{41} + a_{31}m_4 = 8$, $a_{43} := a_{43} + a_{33}m_4 = -12$,

$a_{44} := a_{44} + a_{34}m_4 = -38$, $a_{32} := \frac{1}{a_{32}} = 1$.

Step 5. $B := (B \cup \{x_2\}) - \{y_3\} = \{x_1, x_3, x_2, y_4\}$,

$N := (N - \{x_2\}) \cup \{y_3\} = \{y_1, y_3, y_2, x_4\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ x_3 \\ x_2 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_3 & y_2 & x_4 \end{array} \\ \left[\begin{array}{cccc} -2 & -3 & 3 & 11 \\ -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 1 & -\frac{3}{4} & -\frac{5}{2} \\ 8 & 11 & -12 & -38 \end{array} \right] \end{array}.$$

Iteration 4 :

Step 2. Set $P = \{4\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 4$, we get $L = \{4\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 4$, then

$a_{41} := \frac{a_{41}}{a_{44}} = -\frac{4}{19}$, $\quad a_{42} := \frac{a_{42}}{a_{44}} = -\frac{11}{38}$, $\quad a_{43} := \frac{a_{43}}{a_{44}} = \frac{6}{19}$,

$m_1 = -\frac{a_{14}}{a_{44}} = \frac{11}{38}$, $\quad m_2 = -\frac{a_{24}}{a_{44}} = \frac{1}{76}$, $\quad m_3 = -\frac{a_{34}}{a_{44}} = -\frac{5}{76}$,

$a_{14} := m_1 = \frac{11}{38}$, $\quad a_{24} := m_2 = \frac{1}{76}$, $\quad a_{34} := m_3 = -\frac{5}{76}$,

$a_{11} := a_{11} + a_{41}m_1 = \frac{6}{19}$, $\quad a_{12} := a_{12} + a_{42}m_1 = \frac{7}{38}$,

$a_{13} := a_{13} + a_{43}m_1 = -\frac{9}{19}$, $\quad a_{21} := a_{21} + a_{41}m_2 = -\frac{11}{76}$,

$a_{22} := a_{22} + a_{42}m_2 = \frac{11}{76}$, $\quad a_{23} := a_{23} + a_{43}m_2 = \frac{7}{76}$,

$a_{31} := a_{31} + a_{41}m_3 = \frac{17}{76}$ $\quad a_{32} := a_{32} + a_{42}m_3 = \frac{21}{76}$,

$a_{33} := a_{33} + a_{43}m_3 = \frac{3}{76}$, $\quad a_{44} := \frac{1}{a_{44}} = -\frac{1}{38}$.

Step 5. $B := (B \cup \{x_4\}) - \{y_4\} = \{x_1, x_3, x_2, x_4\}$,

$N := (N - \{x_4\}) \cup \{y_4\} = \{y_1, y_3, y_2, y_4\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ x_3 \\ x_2 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_3 & y_2 & y_4 \end{array} \\ \left[\begin{array}{cccc} \frac{6}{19} & \frac{7}{38} & -\frac{9}{19} & \frac{11}{38} \\ -\frac{11}{76} & \frac{11}{76} & \frac{7}{76} & \frac{1}{76} \\ \frac{17}{76} & \frac{21}{76} & \frac{3}{76} & -\frac{5}{76} \\ -\frac{4}{19} & -\frac{11}{38} & \frac{6}{19} & -\frac{1}{38} \end{array} \right] \end{array}.$$

So $P = \emptyset$. Now place the elements with respect to indices of variables in B and N. For example, here $H = \{1, 2, 3, 4\}$, so $x_1 \in B, y_1 \in N$, implies $a_{11} = \frac{6}{19}$. Also $x_1 \in B, y_3 \in N$ implies $a_{13} = \frac{7}{38}$. Similarly placing the remaining elements we get

$$D(A) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \left[\begin{array}{cccc} \frac{6}{19} & -\frac{9}{19} & \frac{7}{38} & \frac{11}{38} \\ \frac{17}{76} & \frac{3}{76} & \frac{21}{76} & -\frac{5}{76} \\ -\frac{11}{76} & \frac{7}{76} & \frac{11}{76} & \frac{1}{76} \\ -\frac{4}{19} & \frac{6}{19} & -\frac{11}{38} & -\frac{1}{38} \end{array} \right] \end{array}.$$

Hence $A^{-1} = D(A) = \begin{bmatrix} \frac{6}{19} & -\frac{9}{19} & \frac{7}{38} & \frac{11}{38} \\ \frac{17}{76} & \frac{3}{76} & \frac{21}{76} & -\frac{5}{76} \\ -\frac{11}{76} & \frac{7}{76} & \frac{11}{76} & \frac{1}{76} \\ -\frac{4}{19} & \frac{6}{19} & -\frac{11}{38} & -\frac{1}{38} \end{bmatrix}$.

Example 2.2.7. Find the solution of linear systems

$$x + 5y + z = 10$$
$$4x + 3y + 4z = 6$$
$$2x + y + 7z = -3.$$

Solution. From the linear system, we have

$$\begin{bmatrix} 1 & 5 & 1 \\ 4 & 3 & 4 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ -3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 4 & 3 & 4 \\ 2 & 1 & 7 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad B = \begin{bmatrix} 10 \\ 6 \\ -3 \end{bmatrix}.$$

First, we find A^{-1} by the algorithm of Jafree et al.

Step 1. Set $D(A)$ by

$$D(A) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 5 & 1 \\ 4 & 3 & 4 \\ 2 & 1 & 7 \end{bmatrix} \end{array}$$

and $H := \{1, 2, 3\}, N := \{x_1, x_2, x_3\}, B := \{y_1, y_2, y_3\}$.

Iteration 1 :

Step 2. Set $P := \{1, 2, 3\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 1$, we get $L := \{1, 2, 3\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 1$, then

$a_{12} := \frac{a_{12}}{a_{11}} = 5,$ $\qquad a_{13} := \frac{a_{13}}{a_{11}} = 1,$

$m_2 = -\frac{a_{21}}{a_{11}} = -4,$ $\qquad m_3 = -\frac{a_{31}}{a_{11}} = -2,$

$a_{21} := m_2 = -4,$ $\qquad a_{31} := m_3 = -2,$

$a_{22} := a_{22} + a_{12}m_2 = -17,$ $\qquad a_{23} := a_{23} + a_{13}m_2 = 0,$

$a_{32} := a_{32} + a_{12}m_3 = -9, \qquad a_{33} := a_{33} + a_{13}m_3 = 5, \quad a_{11} := \frac{1}{a_{11}} = 1.$

Step 5. $B := (B \cup \{x_1\}) - \{y_1\} = \{x_1, y_2, y_3\}$,

$N := (N - \{x_1\}) \cup \{y_1\} = \{y_1, x_2, x_3\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 5 & 1 \\ -4 & -17 & 0 \\ -2 & -9 & 5 \end{bmatrix} \end{array}.$$

Iteration 2 :

Step 2. Set $P := \{2, 3\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 2$, we get $L := \{2, 3\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 2$, then

$a_{21} := \frac{a_{21}}{a_{22}} = \frac{4}{17}, \qquad a_{23} := \frac{a_{23}}{a_{22}} = 0,$

$m_1 = -\frac{a_{12}}{a_{22}} = \frac{5}{17}, \qquad m_3 = -\frac{a_{32}}{a_{22}} = -\frac{9}{17},$

$a_{12} := m_1 = \frac{5}{17}, \qquad a_{32} := m_3 = -\frac{9}{17},$

$a_{11} := a_{11} + a_{21}m_1 = -\frac{3}{17}, \qquad a_{13} := a_{13} + a_{23}m_1 = 1,$

$a_{31} := a_{31} + a_{21}m_3 = \frac{2}{17}, \qquad a_{33} := a_{33} + a_{23}m_3 = 5, \quad a_{22} := \frac{1}{a_{22}} = -\frac{1}{17}.$

Step 5. $B := (B \cup \{x_2\}) - \{y_2\} = \{x_1, x_2, y_3\}$,

$N := (N - \{x_2\}) \cup \{y_2\} = \{y_1, y_2, x_3\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ x_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & x_3 \end{array} \\ \begin{bmatrix} -\frac{3}{17} & \frac{5}{17} & 1 \\ \frac{4}{17} & -\frac{1}{17} & 0 \\ \frac{2}{17} & -\frac{9}{17} & 5 \end{bmatrix} \end{array}.$$

Iteration 3 :

Step 2. Set $P = \{3\}$.

Step 3. Since $P \neq \emptyset$, taking $p = 3$, we get $L = \{3\}$.

Step 4. Since $L \neq \emptyset$, taking $k = 3$, then

$a_{31} := \frac{a_{31}}{a_{33}} = \frac{2}{85}, \qquad a_{32} := \frac{a_{32}}{a_{33}} = -\frac{9}{85},$

$m_1 = -\frac{a_{13}}{a_{33}} = -\frac{1}{5}, \qquad m_2 = -\frac{a_{23}}{a_{33}} = 0,$

$a_{13} := m_1 = -\frac{1}{5}, \qquad a_{23} := m_2 = 0,$

$a_{11} := a_{11} + a_{31}m_1 = -\frac{1}{5}, \qquad a_{12} := a_{12} + a_{32}m_1 = \frac{2}{5},$

$a_{21} := a_{21} + a_{31}m_2 = \frac{4}{17}, \qquad a_{22} := a_{22} + a_{32}m_2 = -\frac{1}{17}, \quad a_{33} := \frac{1}{a_{33}} = \frac{1}{5}.$

Step 5. $B := (B \cup \{x_3\}) - \{y_3\} = \{x_1, x_2, x_3\}$,

$N := (N - \{x_3\}) \cup \{y_3\} = \{y_1, y_2, y_3\}$ and

$$D(A) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{4}{17} & -\frac{1}{17} & 0 \\ \frac{2}{85} & -\frac{9}{85} & \frac{1}{5} \end{bmatrix} \end{array}.$$

So $P = \emptyset$.

Step 6. Inverse exits.

Hence

$$A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{4}{17} & -\frac{1}{17} & 0 \\ \frac{2}{85} & -\frac{9}{85} & \frac{1}{5} \end{bmatrix}.$$

Finally, we can find X by use A^{-1} that is $X = A^{-1}B$.

So, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{4}{17} & -\frac{1}{17} & 0 \\ \frac{2}{85} & -\frac{9}{85} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 10 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Therefore the solution are $x = 1, y = 2, z = -1$.