CHAPTER 2

Preliminaries

In this chapter, we will briefly review some concepts and some results of Semigroup Theory.

2.1 Elementary Concepts

In this thesis, the cardinality of a set X is denoted by $|X|$ and $X = A \dot{\cup} B$ means X is a disjoint union of A and B.

Definition 2.1.1. A semigroup is a pair $(S, \cdot)$ in which S is a nonempty set and $\cdot$ is a binary associative operation on S, i.e., the equation $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all $x, y, z \in S$.

Definition 2.1.2. Let S be a semigroup.

(i) If there exists an element 1 of S such that

$$x1 = x = 1x \quad \text{for all } x \in S,$$

then 1 is called an *identity* element of S and S is called a *semigroup with identity* or a *monoid*.

(ii) If there exists an element 0 of S such that

$$x0 = 0 = 0x \quad \text{for all } x \in S,$$

then 0 is called a *zero* element of S and S is called a *semigroup with zero*.

A nonempty subset T of a semigroup S is called a *subsemigroup* of S if $xy \in T$ for all $x, y \in T$.

Definition 2.1.3. An element e of a semigroup S is called an *idempotent* if $e = e^2$. The set of all idempotents in S is denoted by $E(S)$.

Definition 2.1.4. Let S be a semigroup with identity 1. An element $a \in S$ is called a *unit* of S if there exists $b \in S$ such that $ab = 1 = ba$.

Lemma 2.1.1. *Let S be a semigroup with identity 1 and*

$$G = \{a \in S \mid a \text{ is a unit of } S\}.$$

Then G is a maximal subgroup of S having 1 as the identity.

Proof. Let $a, b \in G$. Then there exist $a', b' \in S$ such that $aa' = 1 = a'a$ and $bb' = 1 = b'b$. So $(ab)(b'a') = 1 = (b'a')(ab)$, that is $ab \in G$. It is clear that $1 \in G$, thus G is a monoid. Let $c \in G$. Then $c'c = 1 = cc'$ for some $c' \in S$, it follows that $c' \in G$ and c' is the inverse of c. Thus G is a subgroup of S. Let G' be a subgroup of S containing 1 and $d \in G'$. So there exists $d^{-1} \in G'$ such that $dd^{-1} = 1 = d^{-1}d$, which implies that d is a unit of S and thus $G' \subseteq G$. Therefore, G is a maximal subgroup of S having 1 as the identity. □

We call the subgroup G of S (in Lemma 2.1.1) the *group of units* of S.

Definition 2.1.5. Let $A \neq \emptyset$. Then a relation R on A is an *equivalence relation* on A provided R is:

reflexive: $(a, a) \in R$ for all $a \in A$;

symmetric: if $(a, b) \in R$, then $(b, a) \in R$ for all $a, b \in A$;

transitive: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$.

Definition 2.1.6. Let S and T be semigroups. A mapping φ from S into T is called a *homomorphism* if

$$(xy)\varphi = (x\varphi)(y\varphi) \quad \text{for all} \quad x, y \in S.$$

An injective homomorphism is called a *monomorphism*. A surjective homomorphism is called an *epimorphism*, and if a homomorphism is bijective, then we call it an *isomorphism*. If there exists an isomorphism from S onto T, then we say that S and T are *isomorphic* and write $S \cong T$. If φ is a homomorphism from S into S, then we call it an *endomorphism* of S. An isomorphism from S onto S is called an *automorphism* of S.

2.2 Regularity of Semigroups

Definition 2.2.1. An element a of a semigroup S is called *regular* if there exists x in S such that $a = axa$. We denote the set of all regular elements of S by Reg S.

An element a of a semigroup S is called *left* [*right*] *regular* if there exists x in S such that $a = xa^2$ [$a = a^2x$].

An element a of a semigroup S is called *completely regular* if there exists x in S such that $a = axa$ and $ax = xa$.

Theorem 2.2.1. *Let S be a semigroup and $a \in S$. Then a is completely regular if and only if a is left regular and right regular.*

Proof. Assume that a is completely regular. Then $a = axa$ and $ax = xa$ for some $x \in S$. So $a = axa = xa^2$ and $a = axa = a^2x$. Thus a is left regular and right regular.

Conversely, assume that a is left regular and right regular. Then there are $x, y \in S$ such that $a = xa^2$ and $a = a^2y$. So

$$aya = (xa^2)ya = x(a^2y)a = xaa = a,$$
$$axa = ax(a^2y) = a(xa^2)y = aay = a,$$

and $ay = xa^2y = xa$. Then

$$a(xay)a = (axa)ya = aya = a, \text{ and}$$
$$a(xay) = (axa)y = ay = xa = x(aya) = (xay)a.$$

Thus a is completely regular. □

2.3 Ideals and Green's Relations

Definition 2.3.1. A nonempty subset A of a semigroup S is called a *left ideal* of S if $SA \subseteq A$, a *right ideal* of S if $AS \subseteq A$, and an (*two-sided*) *ideal* of S if it is both a left and a right ideal.

Note that if S has the identity, then A is an ideal of S if SAS is contained in A.

For any semigroup S, the notation S^1 means S itself if S contains the identity element, otherwise, we let $S^1 = S \cup \{1\}$ and define the binary operation on S^1 by

$$1 \cdot s = s = s \cdot 1 \text{ for all } s \in S,\ 1 \cdot 1 = 1 \text{ and}$$
$$a \cdot b = ab \text{ for all } a, b \in S.$$

Then S^1 becomes a semigroup with the identity element 1.

For any element a in S,

the smallest left ideal of S containing a is $Sa \cup \{a\} = S^1a$,

the smallest right ideal of S containing a is $aS \cup \{a\} = aS^1$, and

the smallest ideal of S containing a is $SaS \cup aS \cup Sa \cup \{a\} = S^1aS^1$,

which we call the *principal left ideal*, *principal right ideal* and *principal ideal generated by* a, respectively.

In 1951, J. A. Green defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on S by the rules that, for $a, b \in S$,

$$a\mathcal{L}b \quad \text{if and only} \quad S^1a = S^1b,$$
$$a\mathcal{R}b \quad \text{if and only} \quad aS^1 = bS^1, \text{ and}$$
$$a\mathcal{J}b \quad \text{if and only} \quad S^1aS^1 = S^1bS^1.$$

Then he defined the equivalence relations

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R},$$

and obtained that the composition of $\mathcal{L}$ and $\mathcal{R}$ is commutative. This follows that $\mathcal{D}$ is the *join* $\mathcal{L} \vee \mathcal{R}$, that is, $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L} \cup \mathcal{R}$. Moreover, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. But, in commutative semigroups, we have $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are called *Green's relations* on S. For each $a \in S$, we denote $\mathcal{L}-$class, $\mathcal{R}-$class, $\mathcal{H}-$class, $\mathcal{D}-$class and $\mathcal{J}-$class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

Theorem 2.3.1. [5] *Let S be a semigroup and $a, b \in S$. Then*

(1) *$a\mathcal{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$.*

(2) *$a\mathcal{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$.*

(3) *$a\mathcal{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.*

Corollary 2.3.2. [5] *If e is an idempotent in a semigroup S, then H_e is a subgroup of S.*

Lemma 2.3.3. *Let S be a semigroup with identity* 1. *Then H_1 is the group of units of S.*

Proof. To prove that for all elements in H_1 are units of S, let $x \in H_1$. Then $x \in L_1 \cap R_1$ and hence $S^1x = S^11$ and $xS^1 = 1S^1$. Since $1 \in S$, we have $Sx = S$ and $xS = S$. So $yx = 1 = xz$ for some $y, z \in S$. Since $y = y1 = y(xz) = (yx)z = 1z = z$, we get $yx = 1 = xy$, which implies that x is a unit of S. Conversely, let a be a unit of S. Then $ab = 1 = ba$ for some $b \in S$. From $ba = 1$, $a1 = a$ and $ab = 1$, $1a = a$, we have $a\mathcal{L}1$ and $a\mathcal{R}1$. Thus $a\mathcal{H}1$ and therefore $a \in H_1$. □

2.4 Transformation Semigroups

In this section, we list some known results, definitions and notations about transformation semigroups that will be used throughout this thesis.

2.4.1 The Semigroups $T(X)$

Let X be a nonempty set and $T(X)$ denote the set of all transformations from X into itself. Then $T(X)$ is a semigroup under the composition of maps, that is, if $\alpha, \beta \in T(X)$, then $\alpha\beta \in T(X)$ is defined by

$$x(\alpha\beta) = (x\alpha)\beta \text{ for all } x \in X,$$

and it is called the *full transformation semigroup* on X. It is known that $T(X)$ is a regular semigroup, that is, for every $\alpha \in T(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X)$.

For a nonempty subset A of X, we let id_A denote the identity map on A. Then it is clear that id_X is the identity element of $T(X)$.

In 1955, C. G. Doss and D. D. Miller [3] described Green's relations and group $\mathcal{H}-$classes of $T(X)$.

We note that for any $\alpha \in T(X)$, the symbol π_α denotes the decomposition of X induced by the map α, namely

$$\pi_\alpha = \{x\alpha^{-1} \mid x \in X\alpha\}.$$

Theorem 2.4.1. [2] *Let $\alpha, \beta \in T(X)$. Then*

(1) $\alpha\mathcal{L}\beta$ *if and only if* $X\alpha = X\beta$.

(2) $\alpha\mathcal{R}\beta$ *if and only if* $\pi_\alpha = \pi_\beta$.

(3) $\alpha\mathcal{H}\beta$ *if and only if* $X\alpha = X\beta$ *and* $\pi_\alpha = \pi_\beta$.

(4) $\alpha\mathcal{D}\beta$ *if and only if* $|X\alpha| = |X\beta|$.

(5) $\mathcal{D} = \mathcal{J}$.

A subset A of X is said to be a *cross−section* of $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$ if each $x\alpha^{-1}$ contains exactly one element of A.

Theorem 2.4.2. [2] *Let ϵ be an idempotent in $T(X)$. Then the group $\mathcal{H}-$class H_ϵ is isomorphic to a permutation group $G(A)$ for some $A \subseteq X$. In this case, A is a cross−section of π_ϵ.*

2.4.2 Transformation Semigroups with Invariant Sets

For a fixed nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of total transformations on X which leave the subset Y of X invariant. Note that id_X, the identity map on X, belong to $S(X,Y)$. K. D. Magill [6] introduced and studied the semigroup $S(X,Y)$. The author started in 1966 with an arbitrary topological space X and let $S(X)$ denote the semigroup of all continuous functions from X into itself where the binary operation is composition, and let $S(X,Y)$ denote the subsemigroup of $S(X)$ consisting of all continuous functions on X which leave a given subset Y of X invariant, that is, map Y into itself. The author investigated the question of when $S(X,Y)$ is isomorphic to $S(Z)$ for some space Z.

In 1975, J. S. V. Symons [8] considered a special case of Magill's idea: namely, when X is endowed with the discrete topology, that is, when $S(X,Y)$ is simply the set of all maps from X into X which leave $Y \subseteq X$ invariant, and he described the automorphism group of this semigroup.

Later in 2005, S. Nenthein, P. Youngkhong and Y. Kemprasit [7] characterized the regular elements of $S(X,Y)$ and gave a necessary and sufficient condition for $S(X,Y)$ to be a regular semigroup, and applied the result to determine the number of regular elements in $S(X,Y)$ for a finite set X. Then they gave the following results.

Theorem 2.4.3. [7] *The following statements hold for the semigroup $S(X,Y)$.*

(1) *For $\alpha \in S(X,Y)$, α is a regular element of $S(X,Y)$ if and only if $X\alpha \cap Y = Y\alpha$.*

(2) *$S(X,Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.*

Let Reg $S(X,Y) = \{\alpha \in S(X,Y) \mid X\alpha \cap Y = Y\alpha\}$. Then Reg $S(X,Y)$ is the set of all regular elements of $S(X,Y)$.

For positive numbers n and r with $r \leq n$, the number of partitions of $\{1,2,...,n\}$ into r blocks is called the *stirling number of the second kind*, denoted by $S(n,r)$. It is known that

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n.$$

Then the number of maps from $\{1,2,...,n\}$ onto $\{1,2,...,r\}$ is $r!S(n,r)$.

Theorem 2.4.4. [7] *If $|X| = n$ and $|Y| = m$, then the number of regular elements in $S(X,Y)$ is*

$$\sum_{r=1}^{m} \binom{m}{r} r! S(m,r)(n-m+r)^{n-m}.$$

In 2011, P. Honyam and J. Sanwong [4] gave a necessary and sufficient condition for Reg $S(X,Y)$ to be a regular subsemigroup of $S(X,Y)$.

Lemma 2.4.5. [4] *The following statements are equivalent:*

(1) Reg $S(X,Y)$ *is a regular subsemigroup of* $S(X,Y)$.

(2) $S(X,Y)$ *is a regular semigroup.*

(3) $X = Y$ *or* $|Y| = 1$.

Lemma 2.4.6. *$S(X,Y)$ has the zero element if and only if $|Y| = 1$.*

Proof. Assume that $|Y| = 1$. Let $Y = \{a\}$ and define

$$\alpha = \begin{pmatrix} X \\ a \end{pmatrix} \in S(X,Y).$$

Then $\alpha\beta = \alpha = \beta\alpha$ for all $\beta \in S(X,Y)$ and therefore α is the zero element of $S(X,Y)$.

Conversely, assume that $S(X,Y)$ has the zero element, say α. Suppose that $|Y| > 1$. Let $b, c \in Y$ be such that $b \neq c$ and define

$$\beta = \begin{pmatrix} X \\ b \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} X \\ c \end{pmatrix}.$$

Then $\beta, \gamma \in S(X,Y)$ and $\beta \neq \gamma$. Since α is the zero element, we have $\alpha\beta = \alpha = \alpha\gamma$. But $\beta = \alpha\beta$ and $\gamma = \alpha\gamma$. Thus $\beta = \alpha\beta = \alpha = \alpha\gamma = \gamma$ which is a contradiction. Therefore, $|Y| = 1$. □

We note that for any $\alpha \in S(X,Y)$ if $Z \subseteq X$, we will denote $\pi_\alpha(Z)$ by

$$\pi_\alpha(Z) = \{x\alpha^{-1} \mid x \in X\alpha \cap Z\}.$$

Thus $\pi_\alpha(Y) = \{y\alpha^{-1} \mid y \in X\alpha \cap Y\}$.

For each partitions $\mathcal{A}$ and $\mathcal{B}$ of a set X, we say that $\mathcal{A}$ *refines* $\mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

Green's relations on $S(X,Y)$ were given by P. Honyam and J. Sanwong [4]. For convenience, we present $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ relations here.

Theorem 2.4.7. [4] *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(X,Y)$ if and only if $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$. Consequently, $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*

Theorem 2.4.8. [4] *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(X,Y)$ if and only if π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Consequently, $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.*

Corollary 2.4.9. [4] *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta$, $Y\alpha = Y\beta$ and $\pi_\alpha = \pi_\beta$, $\pi_\alpha(Y) = \pi_\beta(Y)$.*

We note that for any $\alpha \in S(X,Y)$, the notation $\alpha|_Z : Z \to X$ where $Z \subseteq X$ and $G(X)$ is a permutation group on X.

Recall that each group $\mathcal{H}-$class of $T(X)$ is isomorphic to a permutation group $G(A)$ for some $A \subseteq X$ (Theorem 2.4.2). Here, for the semigroup $S(X,Y)$, the result depends on the group which is denoted by $G(A,B)$ and

$$G(A,B) = \{\rho \in G(A) \mid \rho|_B \in G(B)\}$$

where $B \subseteq A$ for some $A \subseteq X$ and $B \subseteq Y$. Then $G(A,B)$ is a subgroup of the permutation group $G(A)$.

Theorem 2.4.10. [4] *Let ϵ be an idempotent in $S(X,Y)$. Then the group $\mathcal{H}-$class H_ϵ is isomorphic to $G(A,B)$ for some $A \subseteq X$ and $B \subseteq Y \cap A$. In this case, A is a cross$-$section of π_ϵ.*

Since $S(X,Y)$ is a semigroup with identity id_X, its group of units is as follows.

Lemma 2.4.11. [4] *Let $G(X,Y) = \{\alpha \in G(X) \mid \alpha|_Y \in G(Y)\}$. Then $G(X,Y)$ is the group of units of $S(X,Y)$.*

The following theorem is given by W. Choomanee, P. Honyam and J. Sanwong [1].

Theorem 2.4.12. [1] *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) *α is left regular;*

(2) *$X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$;*

(3) *$\alpha^2 \in L_\alpha$.*

Theorem 2.4.13. [1] *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent:*

(1) α *is right regular;*

(2) $\pi_\alpha = \pi_{\alpha^2}$ *and* $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$*;*

(3) $\alpha^2 \in R_\alpha$.

We note that for $\alpha \in S(X,Y)$, the notation Y' and X' are for $Y\alpha$ and $X\alpha \setminus Y\alpha$, respectively.

Lemma 2.4.14. [1] *Let* $\alpha \in S(X,Y)$*. If* $X\alpha = X\alpha^2$ *is finite and* $Y\alpha = Y\alpha^2$*, then*

(1) $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$*;*

(2) $X' \subseteq X \setminus Y'\alpha^{-1}$*;*

(3) $|y'\alpha^{-1} \cap Y'| = 1$ *for all* $y' \in Y'$*;*

(4) $|x'\alpha^{-1} \cap X'| = 1$ *for all* $x' \in X'$.

Lemma 2.4.15. [1] *Let* $\alpha \in S(X,Y)$*. If* $\pi_\alpha = \pi_{\alpha^2}$ *is finite and* $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$*, then*

(1) $(X \setminus Y)\alpha \subseteq (X \setminus Y) \cup Y'$*;*

(2) $X' \subseteq X \setminus Y'\alpha^{-1}$*;*

(3) $|y'\alpha^{-1} \cap Y'| = 1$ *for all* $y' \in Y'$*;*

(4) $|x'\alpha^{-1} \cap X'| = 1$ *for all* $x' \in X'$.