# CHAPTER 3

# Main Results

In this chapter, we study transformation semigroups with two invariant subsets and describe Green's relations $\mathcal{L}$ and $\mathcal{R}$, and group $\mathcal{H}-$classes of $S(X, Y_1, Y_2)$. Moreover, we characterize regular, left regular, right regular and completely regular elements on $S(X, Y_1, Y_2)$ and consider the relationships of these elements. Moreover, we count the numbers of regular elements of $S(X, Y_1, Y_2)$ when $X$ is a finite set.

## 3.1 Transformation Semigroups with Two Invariant Subsets

For two nonempty subsets $Y_1$, $Y_2$ of $X$ with $Y_1 \cap Y_2 = \emptyset$, let

$$S(X, Y_1, Y_2) = \{\alpha \in T(X) \mid Y_1\alpha \subseteq Y_1, Y_2\alpha \subseteq Y_2\}.$$

For $\alpha, \beta \in S(X, Y_1, Y_2)$, we have $Y_1\alpha \subseteq Y_1, Y_2\alpha \subseteq Y_2, Y_1\beta \subseteq Y_1$ and $Y_2\beta \subseteq Y_2$. So $Y_1\alpha\beta \subseteq Y_1\beta \subseteq Y_1$ and $Y_2\alpha\beta \subseteq Y_2\beta \subseteq Y_2$, hence $\alpha\beta \in S(X, Y_1, Y_2)$. Then $S(X, Y_1, Y_2)$ is a semigroup of total transformations on $X$ which leave subsets $Y_1, Y_2$ of $X$ invariant. Note that $id_X$, the identity map on $X$, belong to $S(X, Y_1, Y_2)$ and $S(X, Y_1, Y_2) = S(X, Y_1) \cap S(X, Y_2)$.

As in A. H. Clifford and G. B. Preston [2] vol. 2, p. 241, we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript $i$ belongs to some (unmentioned) index set $I$, the abbreviation $\{a_i\}$ denotes $\{a_i \mid i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$ for all $i \in I$.

With the above notation, for any $\alpha \in S(X, Y_1, Y_2)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k & D_l & E_m \\ a_i & b_j & c_k & d_l & e_m \end{pmatrix},$$

where $A_i \cap Y_1 \neq \emptyset \neq B_j \cap Y_2$; $C_k, D_l, E_m \subseteq X \setminus (Y_1 \cup Y_2)$; $\{a_i\} \subseteq Y_1, \{b_j\} \subseteq Y_2$, $\{c_k\} \subseteq Y_1 \setminus \{a_i\}, \{d_l\} \subseteq Y_2 \setminus \{b_j\}$ and $\{e_m\} \subseteq X \setminus (Y_1 \cup Y_2)$. Here, $I$ and $J$ are nonempty sets, but $K, L$ or $M$ can be empty.

**Example 1.** Let $X = \{1,2,3,4\},\ Y_1 = \{1,2\},\ Y_2 = \{3\}$ and $Y = \{1,2,3\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix},\ \beta = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 3 & 1 \end{pmatrix}.$$

Hence $Y_1\alpha = \{1\} \subseteq Y_1,\ Y_2\alpha = \{3\} = Y_2$ and $Y\alpha = \{1,3\} \subseteq Y$, so $\alpha \in S(X,Y_1,Y_2) \cap S(X,Y)$. Since $Y\beta = \{1,3\} \subseteq Y,\ Y_1\beta = \{3\} \nsubseteq Y_1$ and $Y_2\beta = \{1\} \nsubseteq Y_2$, we have $\beta \in S(X,Y) \setminus S(X,Y_1,Y_2)$. Thus $S(X,Y_1,Y_2) \subsetneq S(X,Y)$.

Note that if $Y = Y_1 \cup Y_2$, then $S(X,Y_1,Y_2)$ is a proper subsemigroup of $S(X,Y)$ since $S(X,Y_1,Y_2) \subseteq S(X,Y)$ and there exists

$$\alpha = \begin{pmatrix} Y_1 & X \setminus Y_1 \\ a & b \end{pmatrix} \in S(X,Y) \setminus S(X,Y_1,Y_2),$$

where $a \in Y_2$ and $b \in Y_1$.

**Remark 3.1.1.** *We note that if $|X| = 2$, then $|Y_1| = 1 = |Y_2|$ and $X \setminus (Y_1 \cup Y_2) = \emptyset$. In this case*

$$S(X,Y_1,Y_2) = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\},$$

*where $Y_1 = \{a\}$ and $Y_2 = \{b\}$.*

**Theorem 3.1.2.** *$S(X,Y_1,Y_2)$ has the zero element if and only if $|X| = 2$.*

*Proof.* Assume that $|X| = 2$. Then $|S(X,Y_1,Y_2)| = 1$ which implies that $S(X,Y_1,Y_2)$ has the zero element.

Conversely, assume that $S(X,Y_1,Y_2)$ has the zero element, say $\alpha$. Suppose that $|X| > 2$. Let $a \in Y_1$, $b \in Y_2$ and $c \in X \setminus \{a,b\}$. We consider in three cases:

**Case 1**: $c \in X \setminus (Y_1 \cup Y_2)$. Define

$$\beta = \begin{pmatrix} Y_1 & X \setminus Y_1 \\ a & b \end{pmatrix},\ \gamma = \begin{pmatrix} X \setminus Y_2 & Y_2 \\ a & b \end{pmatrix}.$$

Then $\beta, \gamma \in S(X,Y_1,Y_2)$ and $\beta \neq \gamma$. Since $\alpha$ is the zero element, we have $\beta\alpha = \alpha = \gamma\alpha$. But

$$\beta\alpha = \begin{pmatrix} Y_1 & X \setminus Y_1 \\ a\alpha & b\alpha \end{pmatrix},\ \gamma\alpha = \begin{pmatrix} X \setminus Y_2 & Y_2 \\ a\alpha & b\alpha \end{pmatrix}.$$

Since $a\alpha \in Y_1, b\alpha \in Y_2$ and $Y_1 \cap Y_2 = \emptyset$, we have $a\alpha \neq b\alpha$ and hence $\beta\alpha \neq \gamma\alpha$ which is a contradiction.

**Case 2**: $c \in Y_1$. Define

$$\beta = \begin{pmatrix} Y_1 & X \setminus Y_1 \\ a & b \end{pmatrix}, \ \gamma = \begin{pmatrix} Y_1 & X \setminus Y_1 \\ c & b \end{pmatrix}.$$

Then $\beta, \gamma \in S(X, Y_1, Y_2)$ and $\beta \neq \gamma$. Since $\alpha$ is the zero element, we have $\alpha\beta = \alpha = \alpha\gamma$. But $Y_1\alpha\beta = (Y_1\alpha)\beta = \{a\}$ and $Y_1\alpha\gamma = (Y_1\alpha)\gamma = \{c\}$. So $Y_1\alpha = Y_1\alpha\beta = \{a\} \neq \{c\} = Y_1\alpha\gamma = Y_1\alpha$ which is a contradiction.

**Case 3**: $c \in Y_2$. Define

$$\beta = \begin{pmatrix} X \setminus Y_2 & Y_2 \\ a & b \end{pmatrix}, \ \gamma = \begin{pmatrix} X \setminus Y_2 & Y_2 \\ a & c \end{pmatrix}.$$

Then $\beta, \gamma \in S(X, Y_1, Y_2)$ and $\beta \neq \gamma$. Since $\alpha$ is the zero element, we have $\alpha\beta = \alpha = \alpha\gamma$. But $Y_2\alpha\beta = (Y_2\alpha)\beta = \{b\}$ and $Y_2\alpha\gamma = (Y_2\alpha)\gamma = \{c\}$. So $Y_2\alpha = Y_2\alpha\beta = \{b\} \neq \{c\} = Y_2\alpha\gamma = Y_2\alpha$ which is a contradiction.

Therefore, $|X| = 2$. □

## 3.2 Green's Relations on $S(X, Y_1, Y_2)$

In this section, we describe Green's relations $\mathcal{L}$ and $\mathcal{R}$ for $S(X, Y_1, Y_2)$ and apply these results to obtain its group $\mathcal{H}-$classes. Since the identity map $id_X \in S(X, Y_1, Y_2)$, it follows that $S(X, Y_1, Y_2)^1 = S(X, Y_1, Y_2)$.

Note that for each $\alpha$, $\beta \in S(X, Y_1, Y_2)$, if $\alpha\mathcal{L}\beta$ on $S(X, Y_1, Y_2)$, then $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ for some $\gamma, \delta \in S(X, Y_1, Y_2) = S(X, Y_1) \cap S(X, Y_2)$ and so $\alpha\mathcal{L}\beta$ on $S(X, Y_1)$ and $\alpha\mathcal{L}\beta$ on $S(X, Y_2)$. Also, if $\alpha\mathcal{R}\beta$ on $S(X, Y_1, Y_2)$, then $\alpha\mathcal{R}\beta$ on $S(X, Y_1)$ and $\alpha\mathcal{R}\beta$ on $S(X, Y_2)$.

**Theorem 3.2.1.** *Let $\alpha$, $\beta \in S(X, Y_1, Y_2)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(X, Y_1, Y_2)$ if and only if $X\alpha \subseteq X\beta$, $Y_1\alpha \subseteq Y_1\beta$ and $Y_2\alpha \subseteq Y_2\beta$. Consequently, $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$, $Y_1\alpha = Y_1\beta$ and $Y_2\alpha = Y_2\beta$.*

*Proof.* Assume that $\alpha = \gamma\beta$ for some $\gamma \in S(X, Y_1, Y_2)$. Since $S(X, Y_1, Y_2) = S(X, Y_1) \cap S(X, Y_2)$, we have $\gamma \in S(X, Y_1)$ and $\gamma \in S(X, Y_2)$ such that $\alpha = \gamma\beta$. By Theorem 2.4.7, we have $X\alpha \subseteq X\beta$, $Y_1\alpha \subseteq Y_1\beta$ and $Y_2\alpha \subseteq Y_2\beta$.

Conversely, suppose that $X\alpha \subseteq X\beta$, $Y_1\alpha \subseteq Y_1\beta$ and $Y_2\alpha \subseteq Y_2\beta$. So

$Y_1\alpha|_{Y_1} \subseteq Y_1\beta|_{Y_1}$ and $Y_2\alpha|_{Y_2} \subseteq Y_2\beta|_{Y_2}$ where $\alpha|_{Y_1}, \beta|_{Y_1} \in T(Y_1)$ and $\alpha|_{Y_2}, \beta|_{Y_2} \in T(Y_2)$. Then there exist $\delta_1 \in T(Y_1)$ and $\delta_2 \in T(Y_2)$ such that $\alpha|_{Y_1} = \delta_1(\beta|_{Y_1})$ and $\alpha|_{Y_2} = \delta_2(\beta|_{Y_2})$, that is, $y_1\alpha = (y_1\delta_1)\beta$ for each $y_1 \in Y_1$ and $y_2\alpha = (y_2\delta_2)\beta$ for each $y_2 \in Y_2$. Now, for each $x \notin Y_1 \cup Y_2$, there exists $x' \in X$ such that $x\alpha = x'\beta$ since $X\alpha \subseteq X\beta$. Then for each $x \in X \setminus (Y_1 \cup Y_2)$, choose such an $x'$ and extend $\delta_1 \in T(Y_1)$ and $\delta_2 \in T(Y_2)$ to $\gamma \in T(X)$ by

$$x\gamma = \begin{cases} x\delta_1, & \text{if } x \in Y_1, \\ x\delta_2, & \text{if } x \in Y_2, \\ x', & \text{if } x \in X \setminus (Y_1 \cup Y_2). \end{cases}$$

For each $y_1 \in Y_1$, we have $y_1\gamma = y_1\delta_1 \in Y_1$, that means $Y_1\gamma \subseteq Y_1$. For each $y_2 \in Y_2$, we get $y_2\gamma = y_2\delta_2 \in Y_2$ and hence $Y_2\gamma \subseteq Y_2$. Thus $\gamma \in S(X, Y_1, Y_2)$. For each $x \in X$, if $x \in Y_1$, then $x(\gamma\beta) = (x\gamma)\beta = (x\delta_1)\beta = x(\delta_1\beta) = x\alpha$, if $x \in Y_2$, then $x(\gamma\beta) = (x\gamma)\beta = (x\delta_2)\beta = x(\delta_2\beta) = x\alpha$, and if $x \in X \setminus (Y_1 \cup Y_2)$, then $x(\gamma\beta) = (x\gamma)\beta = x'\beta = x\alpha$. So $\alpha = \gamma\beta$ as required. □

**Example 2.** Let $X = \{1, 2, 3, 4, 5, 6\}$, $Y_1 = \{1, 2, 3\}$ and $Y_2 = \{4, 5\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 & 2 & 3 & \{4,5\} & 6 \\ 2 & 3 & 1 & 4 & 6 \end{pmatrix}.$$

Then $\alpha, \beta, \gamma \in S(X, Y_1, Y_2)$ and $X\alpha = \{1, 2, 4, 5, 6\} = X\beta$, $Y_1\alpha = \{1, 2\} = Y_1\beta$, $Y_2\alpha = \{4, 5\} = Y_2\beta$. So $\alpha\mathcal{L}\beta$. Since $Y_2\alpha = \{4, 5\} \neq \{4\} = Y_2\gamma$, we have $\alpha$ and $\gamma$ are not $\mathcal{L}-$related on $S(X, Y_1, Y_2)$.

We note that for any $\alpha \in S(X, Y_1, Y_2)$, the symbol $\pi_\alpha$ denotes the decomposition of $X$ induced by the map $\alpha$, namely

$$\pi_\alpha = \{x\alpha^{-1} \mid x \in X\alpha\}.$$

For a nonempty subset $Z$ of $X$, we denote $\pi_\alpha(Z)$ by

$$\pi_\alpha(Z) = \{x\alpha^{-1} \mid x \in X\alpha \cap Z\}.$$

Thus $\pi_\alpha(Y_1) = \{x\alpha^{-1} \mid x \in X\alpha \cap Y_1\}$ and $\pi_\alpha(Y_2) = \{x\alpha^{-1} \mid x \in X\alpha \cap Y_2\}$. For $\alpha, \beta \in S(X, Y_1, Y_2)$, $\mathcal{A} \subseteq \pi_\alpha$ and $\mathcal{B} \subseteq \pi_\beta$, we say that $\mathcal{A}$ refines $\mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

**Theorem 3.2.2.** *Let $\alpha,\ \beta \in S(X, Y_1, Y_2)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(X, Y_1, Y_2)$ if and only if $\pi_\beta$ refines $\pi_\alpha$, $\pi_\beta(Y_1)$ refines $\pi_\alpha(Y_1)$ and $\pi_\beta(Y_2)$ refines $\pi_\alpha(Y_2)$. Consequently, $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$, $\pi_\alpha(Y_1) = \pi_\beta(Y_1)$ and $\pi_\alpha(Y_2) = \pi_\beta(Y_2)$.*

*Proof.* Assume that $\alpha = \beta\gamma$ for some $\gamma \in S(X, Y_1, Y_2)$. Since $S(X, Y_1, Y_2) = S(X, Y_1) \cap S(X, Y_2)$, we obtain $\gamma \in S(X, Y_1)$ and $\gamma \in S(X, Y_2)$ such that $\alpha = \beta\gamma$. Then by Theorem 2.4.8, we have $\pi_\beta$ refines $\pi_\alpha$, $\pi_\beta(Y_1)$ refines $\pi_\alpha(Y_1)$ and $\pi_\beta(Y_2)$ refines $\pi_\alpha(Y_2)$.

Conversely, assume that the conditions hold. For each $x \in X\beta$, there exists $z \in X$ such that $x = z\beta$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} z\alpha, & \text{if } x \in X\beta, \\ x\beta, & \text{if } x \in X \setminus X\beta. \end{cases}$$

If $z_1\beta = x = z_2\beta$, then $z_1, z_2 \in x\beta^{-1}$. Since $\pi_\beta$ refines $\pi_\alpha$, we have $x\beta^{-1} \subseteq x'\alpha^{-1}$ for some $x' \in X\alpha$. Thus $z_1, z_2 \in x'\alpha^{-1}$ and hence $z_1\alpha = z_2\alpha = x\gamma$. So $\gamma$ is well-defined. Now, we prove that $\gamma \in S(X, Y_1, Y_2)$. For each $y \in Y_i$ $(i = 1, 2)$, we have $y \in X \setminus X\beta$ or $y \in X\beta \cap Y_i$. If $y \in X \setminus X\beta$, then $y\gamma = y\beta \in Y_i$ since $\beta \in S(X, Y_1, Y_2)$. If $y \in X\beta \cap Y_i$, then there exists $x \in X$ such that $y = x\beta$. Since $\pi_\beta(Y_i)$ refines $\pi_\alpha(Y_i)$, we have $x \in y\beta^{-1} \subseteq y'\alpha^{-1}$ for some $y' \in X\alpha \cap Y_i$. Thus $y\gamma = x\beta\gamma = x\alpha = y' \in Y_i$. So $Y_1\gamma \subseteq Y_1$ and $Y_2\gamma \subseteq Y_2$ and hence $\gamma \in S(X, Y_1, Y_2)$. Also, we have $x(\beta\gamma) = (x\beta)\gamma = x\alpha$ for all $x \in X$ by the definition of $\gamma$. □

**Example 3.** Let $X = \{1, 2, 3, 4, 5, 6\}$, $Y_1 = \{1, 2, 3\}$ and $Y_2 = \{4, 5\}$. Define

$$\alpha = \begin{pmatrix} \{1,3\} & 2 & \{4,5\} & 6 \\ 1 & 2 & 5 & 6 \end{pmatrix},\ \beta = \begin{pmatrix} \{1,3\} & 2 & \{4,5\} & 6 \\ 3 & 1 & 4 & 6 \end{pmatrix},\ \gamma = \begin{pmatrix} \{1,3\} & 2 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}.$$

Then $\alpha, \beta, \gamma \in S(X, Y_1, Y_2)$ and $\pi_\alpha = \{\{1,3\}, \{2\}, \{4,5\}, \{6\}\} = \pi_\beta$, $\pi_\alpha(Y_1) = \{\{1,3\}, \{2\}\} = \pi_\beta(Y_1)$, $\pi_\alpha(Y_2) = \{\{4,5\}\} = \pi_\beta(Y_2)$. So $\alpha\mathcal{R}\beta$. Since $\pi_\alpha(Y_2) = \{\{4,5\}\} \neq \{\{4\}, \{5\}\} = \pi_\gamma(Y_2)$, we have $\alpha$ and $\gamma$ are not $\mathcal{R}$−related on $S(X, Y_1, Y_2)$.



**Corollary 3.2.3.** *Let $\alpha,\ \beta \in S(X, Y_1, Y_2)$. Then $\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta$, $Y_1\alpha = Y_1\beta$, $Y_2\alpha = Y_2\beta$ and $\pi_\alpha = \pi_\beta$, $\pi_\alpha(Y_1) = \pi_\beta(Y_1)$, $\pi_\alpha(Y_2) = \pi_\beta(Y_2)$.*

**Example 4.** Let $X = \{1, 2, 3, 4, 5, 6\}$, $Y_1 = \{1, 2, 3\}$ and $Y_2 = \{4, 5\}$. Define

$$\epsilon = \begin{pmatrix} \{1,2\} & 3 & \{4,5\} & 6 \\ 1 & 3 & 5 & 6 \end{pmatrix}.$$

Thus $H_\epsilon = \{\alpha \in S(X, Y_1, Y_2) \mid X\alpha = X\epsilon,\ Y_1\alpha = Y_1\epsilon,\ Y_2\alpha = Y_2\epsilon$ and $\pi_\alpha = \pi_\epsilon,\ \pi_\alpha(Y_1) = \pi_\epsilon(Y_1),\ \pi_\alpha(Y_2) = \pi_\epsilon(Y_2)\}$ and then

$$H_\epsilon = \left\{ \begin{pmatrix} \{1,2\} & 3 & \{4,5\} & 6 \\ 1 & 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 & \{4,5\} & 6 \\ 3 & 1 & 5 & 6 \end{pmatrix} \right\}.$$

To describe group $\mathcal{H}-$classes of $S(X, Y_1, Y_2)$, we let

$$G(A, B, C) = \{\rho \in G(A) \mid \rho|_B \in G(B) \text{ and } \rho|_C \in G(C)\},$$

where $B, C \subseteq A$ for some $A \subseteq X$, $B \subseteq Y_1$ and $C \subseteq Y_2$. Then $G(A, B, C)$ is a subgroup of the permutation group $G(A)$.

**Theorem 3.2.4.** *Let $\epsilon$ be an idempotent in $S(X, Y_1, Y_2)$. Then the group $\mathcal{H}-$class $H_\epsilon$ is isomorphic to $G(A, B, C)$ for some $A \subseteq X, B \subseteq Y_1 \cap A, C \subseteq Y_2 \cap A$. In this case, $A$ is a cross$-$section of $\pi_\epsilon$.*

*Proof.* Since $\epsilon$ is an idempotent, we can write

$$\epsilon = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $a_i \in A_i \cap Y_1$, $b_j \in B_j \cap Y_2$ and $c_k \in C_k \subseteq X \setminus (Y_1 \cup Y_2)$ for all $i \in I$, $j \in J$, $k \in K$. Let $B = \{a_i\} \subseteq Y_1$, $C = \{b_j\} \subseteq Y_2$ and $A = \{a_i\} \cup \{b_j\} \cup \{c_k\} = X\epsilon \subseteq X$. Since $H_\epsilon = L_\epsilon \cap R_\epsilon$, we have by Corollary 3.2.3 that

$$H_\epsilon = \left\{ \begin{pmatrix} A_i & B_j & C_k \\ a_i\sigma & b_j\gamma & c_k\delta \end{pmatrix} \mid \sigma \in G(B),\ \gamma \in G(C),\ \delta \in G(A \setminus (B \cup C)) \right\}.$$

Let $\rho = \sigma \cup \gamma \cup \delta$. Then $\rho \in G(A, B, C)$ and

$$H_\epsilon = \left\{ \begin{pmatrix} A_i & B_j & C_k \\ a_i\rho & b_j\rho & c_k\rho \end{pmatrix} \mid \rho \in G(A, B, C) \right\}.$$

Define $\psi : H_\epsilon \to G(A, B, C)$ by

$$\begin{pmatrix} A_i & B_j & C_k \\ a_i\rho & b_j\rho & c_k\rho \end{pmatrix} \mapsto \rho.$$

It is clear that $\psi$ is bijective. Let $\rho, \delta \in G(A, B, C)$ and

$$\begin{pmatrix} A_i & B_j & C_k \\ a_i\rho & b_j\rho & c_k\rho \end{pmatrix}, \begin{pmatrix} A_i & B_j & C_k \\ a_i\delta & b_j\delta & c_k\delta \end{pmatrix} \in H_\epsilon.$$

Then

$$\begin{pmatrix} A_i & B_j & C_k \\ a_i\rho & b_j\rho & c_k\rho \end{pmatrix}\begin{pmatrix} A_i & B_j & C_k \\ a_i\delta & b_j\delta & c_k\delta \end{pmatrix} = \begin{pmatrix} A_i & B_j & C_k \\ a_i(\rho\delta) & b_j(\rho\delta) & c_k(\rho\delta) \end{pmatrix}$$

and

$$\begin{pmatrix} A_i & B_j & C_k \\ a_i(\rho\delta) & b_j(\rho\delta) & c_k(\rho\delta) \end{pmatrix}\psi = \rho\delta = \begin{pmatrix} A_i & B_j & C_k \\ a_i\rho & b_j\rho & c_k\rho \end{pmatrix}\psi\begin{pmatrix} A_i & B_j & C_k \\ a_i\delta & b_j\delta & c_k\delta \end{pmatrix}\psi.$$

Thus $H_\epsilon$ is isomorphic to $G(A, B, C)$. Since $A \cap A_i = \{a_i\}$, $A \cap B_j = \{b_j\}$ and $A \cap C_k = \{c_k\}$ for all $i \in I$, $j \in J$, $k \in K$, we have $A$ is a cross−section of $\pi_\epsilon$. □

**Remark 3.2.5.** *We note that when*

$$\epsilon = id_X = \begin{pmatrix} a_i & b_j & c_k \\ a_i & b_j & c_k \end{pmatrix},$$

*where $Y_1 = \{a_i\}$, $Y_2 = \{b_j\}$ and $X \setminus (Y_1 \cup Y_2) = \{c_k\}$, then the group $\mathcal{H}-$class $H_\epsilon$ is the group of units of $S(X, Y_1, Y_2)$ by Lemma 2.3.3. In this case*

$$H_\epsilon = \left\{ \begin{pmatrix} a_i & b_j & c_k \\ a_i\sigma & b_j\sigma & c_k\sigma \end{pmatrix} \mid \sigma \in G(X, Y_1, Y_2) \right\}$$

*is isomorphic to $G(X, Y_1, Y_2)$.*

## 3.3 Regularity of $S(X, Y_1, Y_2)$

In this section, we give necessary and sufficient conditions for elements in $S(X, Y_1, Y_2)$ to be regular, left regular, right regular and completely regular.

**Theorem 3.3.1.** *Let $\alpha \in S(X, Y_1, Y_2)$. Then $\alpha$ is a regular element if and only if $X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha$.*

*Proof.* Assume that $\alpha$ is regular. Since $S(X, Y_1, Y_2) = S(X, Y_1) \cap S(X, Y_2)$, we have $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$. By Theorem 2.4.3 (1), we have $X\alpha \cap Y_1 = Y_1\alpha$ and $X\alpha \cap Y_2 = Y_2\alpha$. Thus $X\alpha \cap (Y_1 \cup Y_2) = (X\alpha \cap Y_1) \cup (X\alpha \cap Y_2) = Y_1\alpha \cup Y_2\alpha = (Y_1 \cup Y_2)\alpha$.

Conversely, assume that $X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha$. Then we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y_1 \neq \emptyset \neq B_j \cap Y_2$, $C_k \subseteq X \backslash (Y_1 \cup Y_2)$, $\{a_i\} \subseteq Y_1$, $\{b_j\} \subseteq Y_2$ and $\{c_k\} \subseteq X \setminus (Y_1 \cup Y_2)$. Choose $i_0 \in I$ and $j_0 \in J$ and let $I' = I \setminus \{i_0\}$ and $J' = J \setminus \{j_0\}$. Define

$$\beta = \begin{pmatrix} a_{i'} & b_{j'} & c_k & Y_1 \setminus \{a_{i'}\} & X \setminus (Y_1 \cup \{b_{j'}\} \cup \{c_k\}) \\ x_{i'} & y_{j'} & z_k & x_{i_0} & y_{j_0} \end{pmatrix},$$

where $x_i \in A_i \cap Y_1$, $y_j \in B_j \cap Y_2$ and $z_k \in C_k$ for all $i \in I$, $j \in J$, $k \in K$. Then $\beta \in S(X, Y_1, Y_2)$ and $\alpha = \alpha\beta\alpha$. Therefore, $\alpha$ is regular. □

**Example 5.** Let $X = \{1, 2, 3, 4\}$, $Y_1 = \{1, 2\}$ and $Y_2 = \{3\}$. Define $\alpha, \beta \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix}, \ \beta = \begin{pmatrix} \{1,2\} & 3 & 4 \\ 1 & 3 & 2 \end{pmatrix}.$$

Then $X\alpha \cap (Y_1 \cup Y_2) = \{1, 2, 3\} = (Y_1 \cup Y_2)\alpha$. So $\alpha$ is a regular element. Since

$$X\beta \cap (Y_1 \cup Y_2) = \{1, 2, 3\} \neq \{1, 3\} = (Y_1 \cup Y_2)\beta,$$

we have $\beta$ is not a regular element.

**Theorem 3.3.2.** *$S(X, Y_1, Y_2)$ is a regular semigroup if and only if $X = Y_1 \cup Y_2$ or $|Y_1| = 1 = |Y_2|$.*

*Proof.* Suppose that ($X \neq Y_1 \cup Y_2$) and ($|Y_1| > 1$ or $|Y_2| > 1$). If $|Y_1| > 1$, then there are $a$, $b \in Y_1$ such that $a \neq b$. Let $c \in Y_2$. Then we define

$$\alpha = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ a & c & b \end{pmatrix}.$$

Hence $\alpha \in S(X, Y_1, Y_2)$. Since $X\alpha \cap (Y_1 \cup Y_2) = \{a, b, c\} \neq \{a, c\} = (Y_1 \cup Y_2)\alpha$, we have by Theorem 3.3.1 that $\alpha$ is not a regular element. If $|Y_2| > 1$, then there are $d$, $e \in Y_2$ such that $d \neq e$. Let $f \in Y_1$. Define

$$\beta = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ f & d & e \end{pmatrix}.$$

Then $\beta \in S(X, Y_1, Y_2)$. Since $X\beta \cap (Y_1 \cup Y_2) = \{d, e, f\} \neq \{d, f\} = (Y_1 \cup Y_2)\beta$, we have $\beta$ is not a regular element. Therefore, $S(X, Y_1, Y_2)$ is not a regular semigroup.

Conversely, assume that $X = Y_1 \cup Y_2$ or $|Y_1| = 1 = |Y_2|$. Let $\alpha \in S(X, Y_1, Y_2)$. If $X = Y_1 \cup Y_2$, then $X\alpha \cap (Y_1 \cup Y_2) = X\alpha \cap X = X\alpha = (Y_1 \cup Y_2)\alpha$. If $|Y_1| = 1 = |Y_2|$, say $Y_1 = \{a\}$ and $Y_2 = \{b\}$, then $X\alpha \cap (Y_1 \cup Y_2) = X\alpha \cap \{a, b\} = \{a, b\} = (Y_1 \cup Y_2)\alpha$. Thus by Theorem 3.3.1, we have $\alpha$ is a regular element and hence $S(X, Y_1, Y_2)$ is a regular semigroup. □

Let

$$\text{Reg } S(X,Y_1,Y_2) = \{\alpha \in S(X,Y_1,Y_2) \mid X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha\}.$$

Then Reg $S(X,Y_1,Y_2)$ is the set of all regular elements of $S(X,Y_1,Y_2)$.

**Example 6.** (a) Let $X = \{1, 2, 3, 4\}$, $Y_1 = \{1, 2\}$ and $Y_2 = \{3\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & \{2,4\} & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Then we have $\alpha, \beta \in$ Reg $S(X,Y_1,Y_2)$, but

$$\alpha\beta = \begin{pmatrix} \{1,2\} & 3 & 4 \\ 2 & 3 & 1 \end{pmatrix} \notin \text{Reg } S(X,Y_1,Y_2).$$

(b) Let $X = \mathbb{N}$ denote the set of positive integers, $Y_1 = \{1,2\}$ and $Y_2 = \{3,4\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & X \setminus \{1,2,3,4\} \\ 1 & 4 & 5 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & \{3,4\} & X \setminus \{1,3,4\} \\ 2 & 3 & 1 \end{pmatrix}.$$

Thus $\alpha, \beta \in$ Reg $S(X,Y_1,Y_2)$, but

$$\alpha\beta = \begin{pmatrix} \{1,2\} & \{3,4\} & X \setminus \{1,2,3,4\} \\ 2 & 3 & 1 \end{pmatrix} \notin \text{Reg } S(X,Y_1,Y_2).$$

Therefore, Reg $S(X,Y_1,Y_2)$ in (a) and (b) are not subsemigroups of $S(X,Y_1,Y_2)$.

We give a necessary and sufficient condition for Reg $S(X,Y_1,Y_2)$ to be a regular subsemigroup as follows.

**Theorem 3.3.3.** *The following statements are equivalent;*

(1) Reg $S(X,Y_1,Y_2)$ *is a regular subsemigroup of* $S(X,Y_1,Y_2)$.

(2) $S(X,Y_1,Y_2)$ *is a regular semigroup.*

(3) $X = Y_1 \cup Y_2$ *or* $|Y_1| = 1 = |Y_2|$.

*Proof.* (2) $\Leftrightarrow$ (3) By Theorem 3.3.2, we have $S(X,Y_1,Y_2)$ is a regular semigroup if and only if $X = Y_1 \cup Y_2$ or $|Y_1| = 1 = |Y_2|$.

(1) $\Rightarrow$ (3) Assume that ($Y_1 \cup Y_2 \subsetneq X$) and ($|Y_1| > 1$ or $|Y_2| > 1$). If $|Y_1| > 1$,

then there exist $a, b \in Y_1$ such that $a \neq b$. Let $c \in Y_2$ and $e \in X \setminus (Y_1 \cup Y_2)$. Define $\alpha, \beta \in \text{Reg } S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ a & c & e \end{pmatrix}, \ \beta = \begin{pmatrix} a & Y_2 & X \setminus (\{a\} \cup Y_2) \\ a & c & b \end{pmatrix}.$$

Then

$$\alpha\beta = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ a & c & b \end{pmatrix}.$$

So $X\alpha\beta \cap (Y_1 \cup Y_2) = \{a, b, c\} \neq \{a, c\} = (Y_1 \cup Y_2)\alpha\beta$ and hence $\alpha\beta \notin \text{Reg } S(X, Y_1, Y_2)$. If $|Y_2| > 1$, then there exist $c, d \in Y_2$ such that $c \neq d$. Let $a \in Y_1$ and $e \in X \setminus (Y_1 \cup Y_2)$. Define $\alpha, \ \beta \in \text{Reg } S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ a & c & e \end{pmatrix}, \ \beta = \begin{pmatrix} Y_1 & c & X \setminus (Y_1 \cup \{c\}) \\ a & c & d \end{pmatrix}.$$

Thus

$$\alpha\beta = \begin{pmatrix} Y_1 & Y_2 & X \setminus (Y_1 \cup Y_2) \\ a & c & d \end{pmatrix} \notin \text{Reg } S(X, Y_1, Y_2).$$

Therefore, $\text{Reg } S(X, Y_1, Y_2)$ is not a regular semigroup.

(3) $\Rightarrow$ (1) Assume $X = Y_1 \cup Y_2$ or $|Y_1| = 1 = |Y_2|$. By Theorem 3.3.2, we have $\text{Reg } S(X, Y_1, Y_2) = S(X, Y_1, Y_2)$ and thus $\text{Reg } S(X, Y_1, Y_2)$ is a regular subsemigroup. □

**Remark 3.3.4.** *Let $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. Then*

$$|S(X, Y_1, Y_2)| = m_1^{m_1} m_2^{m_2} n^{n-m_1-m_2}.$$

**Theorem 3.3.5.** *Let $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. Then the number of regular elements in $S(X, Y_1, Y_2)$ is*

$$\sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} \binom{m_1}{r_1} \binom{m_2}{r_2} r_1! S(m_1, r_1) r_2! S(m_2, r_2) (n - m_1 - m_2 + r_1 + r_2)^{n-m_1-m_2}.$$

*Proof.* Assume that $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. Let $Y_1' \subseteq Y_1$, $Y_2' \subseteq Y_2$ be such that $|Y_1'| = r_1$, $|Y_2'| = r_2$. Then the number of maps from $Y_1$ onto $Y_1'$ is $r_1! S(m_1, r_1)$ and the number of maps from $Y_2$ onto $Y_2'$ is $r_2! S(m_2, r_2)$. Then the number of maps from $\alpha : Y_1 \cup Y_2 \to Y_1 \cup Y_2$ such that $Y_1\alpha = Y_1'$ and $Y_2\alpha = Y_2'$ is $r_1! S(m_1, r_1) r_2! S(m_2, r_2)$. We see that

$$|(X \setminus (Y_1 \cup Y_2)) \cup (Y_1' \cup Y_2')| = |X \setminus (Y_1 \cup Y_2)| + |Y_1'| + |Y_2'| = n - m_1 - m_2 + r_1 + r_2.$$

So it follows that the number of maps $\alpha : X \to X$ such that $Y_1\alpha = Y_1'$, $Y_2\alpha = Y_2'$ and $X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha$ is

$$r_1!S(m_1, r_1)r_2!S(m_2, r_2)(n - m_1 - m_2 + r_1 + r_2)^{n-m_1-m_2}.$$

Thus the number of maps $\alpha \in \text{Reg } S(X, Y_1, Y_2)$ such that $X\alpha \cap (Y_1 \cup Y_2) = Y_1' \cup Y_2'$ is

$$r_1!S(m_1, r_1)r_2!S(m_2, r_2)(n - m_1 - m_2 + r_1 + r_2)^{n-m_1-m_2}.$$

Consequently, for $1 \leq r_1 \leq m_1$ and $1 \leq r_2 \leq m_2$, the number of maps $\alpha \in \text{Reg } S(X, Y_1, Y_2)$ such that $|X\alpha \cap (Y_1 \cup Y_2)| = r_1 + r_2$ is

$$\binom{m_1}{r_1}\binom{m_2}{r_2} r_1!S(m_1, r_1)r_2!S(m_2, r_2)(n - m_1 - m_2 + r_1 + r_2)^{n-m_1-m_2}.$$

Therefore, the number of regular elements in $S(X, Y_1, Y_2)$ is

$$\sum_{r_1=1}^{m_1}\sum_{r_2=1}^{m_2}\binom{m_1}{r_1}\binom{m_2}{r_2} r_1!S(m_1, r_1)r_2!S(m_2, r_2)(n - m_1 - m_2 + r_1 + r_2)^{n-m_1-m_2}.$$

□

**Example 7.** Let $X = \{1, 2, 3, 4\}$, $Y_1 = \{1, 2\}$ and $Y_2 = \{3\}$. Then $|X| = 4$, $|Y_1| = 2$ and $|Y_2| = 1$. So

$$|S(X, Y_1, Y_2)| = 2^2 \cdot 1^1 \cdot 4^{4-2-1} = 4(4) = 16$$

and

$$\begin{aligned}
|\text{Reg } S(X, Y_1, Y_2)| &= \sum_{r_1=1}^{2}\sum_{r_2=1}^{1}\binom{2}{r_1}\binom{1}{r_2} r_1!S(2, r_1)r_2!S(1, r_2)(4 - 2 - 1 + r_1 + r_2)^{4-2-1} \\
&= \sum_{r_1=1}^{2}\binom{2}{r_1}\binom{1}{1} r_1!S(2, r_1)1!S(1, 1)(1 + r_1 + 1) \\
&= \left[\binom{2}{1}\binom{1}{1} 1!S(2, 1)1!S(1, 1)(1 + 1 + 1)\right] \\
&\quad + \left[\binom{2}{2}\binom{1}{1} 2!S(2, 2)1!S(1, 1)(1 + 2 + 1)\right] \\
&= 6 + 8 \\
&= 14.
\end{aligned}$$

Then we have

$$\text{Reg } S(X, Y_1, Y_2) = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}, \right.$$

$$\begin{pmatrix} \{1,2\} & \{3,4\} \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2,4\} & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2,4\} & 3 \\ 2 & 3 \end{pmatrix},$$

$$\begin{pmatrix}1 & 2 & \{3,4\}\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & \{3,4\}\\ 2 & 1 & 3\end{pmatrix}, \begin{pmatrix}\{1,4\} & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}\{1,4\} & 2 & 3\\ 2 & 1 & 3\end{pmatrix},$$
$$\begin{pmatrix}1 & \{2,4\} & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & \{2,4\} & 3\\ 2 & 1 & 3\end{pmatrix}\Bigg\}.$$

**Theorem 3.3.6.** *Let $\alpha \in S(X, Y_1, Y_2)$. Then the following statements are equivalent:*

(1) *$\alpha$ is left regular;*

(2) *$X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$;*

(3) *$\alpha^2 \in L_\alpha$.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $\alpha$ is left regular. Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$ by Theorem 2.4.12.

(2) $\Rightarrow$ (3) Assume that $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$. Then by Theorem 3.2.1, we obtain that $\alpha\mathcal{L}\alpha^2$, that is $\alpha^2 \in L_\alpha$.

(3) $\Rightarrow$ (1) Assume that $\alpha^2 \in L_\alpha$. Then $\alpha = \beta\alpha^2$ for some $\beta \in S(X, Y_1, Y_2)^1 = S(X, Y_1, Y_2)$ since $S(X, Y_1, Y_2)$ contains an identity. So $\alpha$ is left regular. □

**Theorem 3.3.7.** *Let $\alpha \in S(X, Y_1, Y_2)$. Then the following statements are equivalent:*

(1) *$\alpha$ is right regular;*

(2) *$\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$;*

(3) *$\alpha^2 \in R_\alpha$.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $\alpha$ is right regular. Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$ by Theorem 2.4.13.

(2) $\Rightarrow$ (3) Assume that $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$. Then by Theorem 3.2.2, we obtain that $\alpha\mathcal{R}\alpha^2$, that is $\alpha^2 \in R_\alpha$.

(3) $\Rightarrow$ (1) Assume that $\alpha^2 \in R_\alpha$. Then $\alpha = \alpha^2\beta$ for some $\beta \in S(X, Y_1, Y_2)^1 = S(X, Y_1, Y_2)$. Thus $\alpha$ is right regular. □

**Example 8.** (a) Let $X = \{1, 2, 3, 4, 5\}$, $Y_1 = \{1, 2\}$ and $Y_2 = \{3\}$. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix}1 & \{2,5\} & \{3,4\}\\ 2 & 1 & 3\end{pmatrix}.$$

Then

$$\alpha^2 = \begin{pmatrix} 1 & \{2,5\} & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix}.$$

Thus $X\alpha = \{1,2,3\} = X\alpha^2$, $Y_1\alpha = \{1,2\} = Y_1\alpha^2$ and $Y_2\alpha = \{3\} = Y_2\alpha^2$ and $\pi_\alpha = \{\{1\},\{2,5\},\{3,4\}\} = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \{\{1\},\{2,5\}\} = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \{\{3,4\}\} = \pi_{\alpha^2}(Y_2)$. So $\alpha$ is left regular and right regular.

(b) Let $X = \mathbb{N}$, $Y_1 = \{1,2,3,4\}$, $Y_2 = \{5,6\}$. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \{4,7\} & \{5,6\} & n \\ 1 & 2 & 4 & 3 & 5 & n-1 \end{pmatrix}_{n\in\mathbb{N}\setminus\{1,\ 2,\ \ldots,\ 7\}}.$$

Then

$$\alpha^2 = \begin{pmatrix} 1 & 2 & \{3,8\} & \{4,7\} & \{5,6\} & n \\ 1 & 2 & 3 & 4 & 5 & n-2 \end{pmatrix}_{n\in\mathbb{N}\setminus\{1,\ 2,\ \ldots,\ 8\}}.$$

Thus $X\alpha = \mathbb{N}\setminus\{6\} = X\alpha^2$, $Y_1\alpha = \{1,2,3,4\} = Y_1\alpha^2$ and $Y_2\alpha = \{5\} = Y_2\alpha^2$. So $\alpha$ is left regular. But $\alpha$ is not right regular since

$$\pi_\alpha(Y_1) = \{\{1\},\{2\},\{3\},\{4,7\}\} \neq \{\{1\},\{2\},\{3,8\},\{4,7\}\} = \pi_{\alpha^2}(Y_1).$$

(c) Let $X = \mathbb{N}$, $Y_1$ the set of all positive even integers, and $Y_2$ the set of all positive odd integers. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} n \\ n+2 \end{pmatrix}_{n\in\mathbb{N}}.$$

Then

$$\alpha^2 = \begin{pmatrix} n \\ n+4 \end{pmatrix}_{n\in\mathbb{N}}.$$

Thus $\pi_\alpha = \{\{n\} \mid n \in \mathbb{N}\} = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \{\{2n\} \mid n \in \mathbb{N}\} = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \{\{2n-1\} \mid n \in \mathbb{N}\} = \pi_{\alpha^2}(Y_2)$. So $\alpha$ is right regular. But $\alpha$ is not left regular since $Y_1\alpha = \{2n+2 \mid n \in \mathbb{N}\} \neq \{2n+4 \mid n \in \mathbb{N}\} = Y_1\alpha^2$.

**Corollary 3.3.8.** *Let $\alpha \in S(X, Y_1, Y_2)$. Then the following statements are equivalent:*

(1) *$\alpha$ is completely regular;*

(2) *$\alpha$ is left regular and right regular;*

(3) *$X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$, $Y_2\alpha = Y_2\alpha^2$ and $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$, $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$;*

(4) $\alpha^2 \in H_\alpha$.

*Proof.* (1) $\Rightarrow$ (2) Assume that $\alpha$ is completely regular. By Theorem 2.2.1, we obtain that $\alpha$ is left regular and right regular.

(2) $\Rightarrow$ (3) Assume that $\alpha$ is left regular and right regular. Then by Theorem 3.3.7 and 3.3.8, we have that $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$, $Y_2\alpha = Y_2\alpha^2$ and $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$, $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$.

(3) $\Rightarrow$ (4) Assume that $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$, $Y_2\alpha = Y_2\alpha^2$ and $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$, $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$. Then by Theorem 3.2.1 and 3.2.2, we obtain $\alpha\mathcal{L}\alpha^2$ and $\alpha\mathcal{R}\alpha^2$, this is, $\alpha\mathcal{H}\alpha^2$. Hence $\alpha^2 \in H_\alpha$.

(4) $\Rightarrow$ (1) Assume that $\alpha^2 \in H_\alpha$. Then $\alpha\mathcal{H}\alpha^2$, this is, $\alpha\mathcal{L}\alpha^2$ and $\alpha\mathcal{R}\alpha^2$ and hence $\alpha = \beta\alpha^2$ and $\alpha = \alpha^2\gamma$ for some $\beta, \gamma \in S(X, Y_1, Y_2)^1 = S(X, Y_1, Y_2)$. Thus $\alpha$ is left regular and right regular. Hence by Theorem 2.2.1, we obtain $\alpha$ is completely regular. $\square$

To prove that $\alpha \in S(X, Y_1, Y_2)$ is left regular if and only if $\alpha$ is right regular when $X\alpha$ is finite, we begin with the following two lemmas.

**Lemma 3.3.9.** *Let $\alpha \in S(X, Y_1, Y_2)$. If $X\alpha = X\alpha^2$ is finite, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$, then*

(1) $(X \setminus (Y_1 \cup Y_2))\alpha \subseteq [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha$;

(2) $X\alpha \setminus (Y_1 \cup Y_2)\alpha \subseteq X \setminus ((Y_1 \cup Y_2)\alpha)\alpha^{-1}$;

(3) $|y\alpha^{-1} \cap (Y_1 \cup Y_2)\alpha| = 1$ *for all* $y \in (Y_1 \cup Y_2)\alpha$;

(4) $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| = 1$ *for all* $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$.

*Proof.* Assume that $X\alpha = X\alpha^2$ is finite, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$.

(1) Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $(X \setminus Y_1)\alpha \subseteq (X \setminus Y_1) \cup Y_1\alpha$ and $(X \setminus Y_2)\alpha \subseteq (X \setminus Y_2) \cup Y_2\alpha$ by Lemma 2.4.14 (1). To prove that $(X \setminus (Y_1 \cup Y_2))\alpha \subseteq [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha$, let $z \in (X \setminus (Y_1 \cup Y_2))\alpha$. Then $z = x\alpha$ for some $x \in X \setminus (Y_1 \cup Y_2)$. So $x \in (X \setminus Y_1) \cap (X \setminus Y_2)$, that means $x \in X \setminus Y_1$ and $x \in X \setminus Y_2$ such that $z = x\alpha$. Thus $z \in (X \setminus Y_1)\alpha$ and $z \in (X \setminus Y_2)\alpha$. Since $(X \setminus Y_1)\alpha \subseteq (X \setminus Y_1) \cup Y_1\alpha$ and $(X \setminus Y_2)\alpha \subseteq (X \setminus Y_2) \cup Y_2\alpha$, we have $z \in (X \setminus Y_1) \cup Y_1\alpha$ and $z \in (X \setminus Y_2) \cup Y_2\alpha$ and hence

$$\begin{aligned} z &\in [(X \setminus Y_1) \cup Y_1\alpha] \cap [(X \setminus Y_2) \cup Y_2\alpha] \\ &= [((X \setminus Y_1) \cup Y_1\alpha) \cap (X \setminus Y_2)] \cup [((X \setminus Y_1) \cup Y_1\alpha) \cap Y_2\alpha] \end{aligned}$$

$$= [(X \setminus Y_1) \cap (X \setminus Y_2)] \cup [Y_1\alpha \cap (X \setminus Y_2)] \cup [(X \setminus Y_1) \cap (Y_2\alpha)] \cup [Y_1\alpha \cap Y_2\alpha]$$

$$= [X \setminus (Y_1 \cup Y_2)] \cup Y_1\alpha \cup Y_2\alpha$$

$$= [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha.$$

So $(X \setminus (Y_1 \cup Y_2))\alpha \subseteq [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha$.

(2) Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $X\alpha \setminus Y_1\alpha \subseteq X \setminus (Y_1\alpha)\alpha^{-1}$ and $X\alpha \setminus Y_2\alpha \subseteq X \setminus (Y_2\alpha)\alpha^{-1}$ by Lemma 2.4.14 (2). Now we show that $X\alpha \setminus (Y_1 \cup Y_2)\alpha \subseteq X \setminus ((Y_1 \cup Y_2)\alpha)\alpha^{-1}$ as follows:

$$\begin{aligned} X\alpha \setminus (Y_1 \cup Y_2)\alpha &= X\alpha \setminus (Y_1\alpha \cup Y_2\alpha) \\ &= (X\alpha \setminus Y_1\alpha) \cap (X\alpha \setminus Y_2\alpha) \\ &\subseteq X \setminus (Y_1\alpha)\alpha^{-1} \cap X \setminus (Y_2\alpha)\alpha^{-1} \\ &= X \setminus ((Y_1\alpha)\alpha^{-1} \cup (Y_2\alpha)\alpha^{-1}) \\ &= X \setminus ((Y_1\alpha \cup Y_2\alpha))\alpha^{-1} \\ &= X \setminus ((Y_1 \cup Y_2)\alpha)\alpha^{-1}. \end{aligned}$$

Hence $X\alpha \setminus (Y_1 \cup Y_2)\alpha \subseteq X \setminus ((Y_1 \cup Y_2)\alpha)\alpha^{-1}$.

(3) Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, by Lemma 2.4.14 (3) we have $|y_1\alpha^{-1} \cap Y_1\alpha| = 1$ for all $y_1 \in Y_1\alpha$ and $|y_2\alpha^{-1} \cap Y_2\alpha| = 1$ for all $y_2 \in Y_2\alpha$. For each $y \in (Y_1 \cup Y_2)\alpha$, we have $y \in Y_1\alpha \cup Y_2\alpha$. If $y \in Y_1\alpha$, then

$$\begin{aligned} |y\alpha^{-1} \cap (Y_1 \cup Y_2)\alpha| &= |y\alpha^{-1} \cap (Y_1\alpha \cup Y_2\alpha)| \\ &= |(y\alpha^{-1} \cap Y_1\alpha) \cup (y\alpha^{-1} \cap Y_2\alpha)| \\ &= |y\alpha^{-1} \cap Y_1\alpha| \\ &= 1 \end{aligned}$$

and if $y \in Y_2\alpha$, then $|y\alpha^{-1} \cap (Y_1 \cup Y_2)\alpha| = |y\alpha^{-1} \cap Y_2\alpha| = 1$. Thus $|y\alpha^{-1} \cap (Y_1 \cup Y_2)\alpha| = 1$ for all $y \in (Y_1 \cup Y_2)\alpha$.

(4) Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $|x_1\alpha^{-1} \cap (X\alpha \setminus Y_1\alpha)| = 1$ for all $x_1 \in X\alpha \setminus Y_1\alpha$ and $|x_2\alpha^{-1} \cap (X\alpha \setminus Y_2\alpha)| = 1$ for all $x_2 \in X\alpha \setminus Y_2\alpha$ by Lemma 2.4.14 (4). Let $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Then $x \in X\alpha \setminus (Y_1\alpha \cup Y_2\alpha) = [(X\alpha \setminus Y_1\alpha) \cap (X\alpha \setminus Y_2\alpha)]$ and hence $x \in X\alpha \setminus Y_1\alpha$ and $x \in X\alpha \setminus Y_2\alpha$. Thus

$$\begin{aligned} |x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| &= |x\alpha^{-1} \cap (X\alpha \setminus Y_1\alpha \cap X\alpha \setminus Y_2\alpha)| \\ &\leq |x\alpha^{-1} \cap X\alpha \setminus Y_1| \end{aligned}$$

$$= 1.$$

To show that $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| \neq \emptyset$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Let

$$\{x\alpha^{-1} \mid x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha\} = \{E_1, E_2, ..., E_m\}$$

and $E_i\alpha = e_i$ for all $i = 1, 2, ..., m$. If there exists $E_k$ such that $[X\alpha \setminus (Y_1 \cup Y_2)\alpha] \cap E_k = \emptyset$, then there are $e_u, e_v \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$ such that $e_u \neq e_v$ and $e_u, e_v \in E_l$ for some $l \in \{1, 2, ..., m\}$. Then $E_u\alpha^2 = (E_u\alpha)\alpha = \{e_u\}\alpha = \{e_l\}$ and $E_v\alpha^2 = (E_v\alpha)\alpha = \{e_v\}\alpha = \{e_l\}$ and thus $E_u\alpha^2 = E_v\alpha^2$ which is a contradiction since $X\alpha^2 = X\alpha$ is finite. So $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| \neq \emptyset$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Therefore, $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| = 1$. □

**Lemma 3.3.10.** *Let $\alpha \in S(X, Y_1, Y_2)$. If $\pi_\alpha = \pi_{\alpha^2}$ is finite, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$, then*

(1) *$(X \setminus (Y_1 \cup Y_2))\alpha \subseteq [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha$;*

(2) *$X\alpha \setminus (Y_1 \cup Y_2)\alpha \subseteq X \setminus ((Y_1 \cup Y_2)\alpha)\alpha^{-1}$;*

(3) *$|y\alpha^{-1} \cap (Y_1 \cup Y_2)\alpha| = 1$ for all $y \in (Y_1 \cup Y_2)\alpha$;*

(4) *$|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| = 1$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$.*

*Proof.* Assume that $\pi_\alpha = \pi_{\alpha^2}$ is finite, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$. By using the same proof as given for Lemma 3.3.9 and Lemma 2.4.15, we obtain (1)−(3) as required.

(4) Since $\alpha \in S(X, Y_1)$ and $\alpha \in S(X, Y_2)$, we have $|x_1\alpha^{-1} \cap (X\alpha \setminus Y_1\alpha)| = 1$ for all $x_1 \in X\alpha \setminus Y_1\alpha$ and $|x_2\alpha^{-1} \cap (X\alpha \setminus Y_2\alpha)| = 1$ for all $x_2 \in X\alpha \setminus Y_2\alpha$ by Lemma 2.4.15 (4). Let $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Then $x \in X\alpha \setminus (Y_1\alpha \cup Y_2\alpha) = [(X\alpha \setminus Y_1\alpha) \cap (X\alpha \setminus Y_2\alpha)]$ and hence $x \in X\alpha \setminus Y_1\alpha$ and $x \in X\alpha \setminus Y_2\alpha$. Thus

$$\begin{aligned} |x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| &= |x\alpha^{-1} \cap (X\alpha \setminus Y_1\alpha \cap X\alpha \setminus Y_2\alpha)| \\ &\leq |x\alpha^{-1} \cap X\alpha \setminus Y_1| \\ &= 1. \end{aligned}$$

To show that $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| \neq \emptyset$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Let

$$\{x\alpha^{-1} \mid x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha\} = \{E_1, E_2, ..., E_m\}$$

and $E_i\alpha = e_i$ for all $i = 1, 2, ..., m$. If there exists $E_k$ such that $[X\alpha \setminus (Y_1 \cup Y_2)\alpha] \cap E_k = \emptyset$, then there are $e_u, e_v \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$ such that $e_u \neq e_v$ and $e_u, e_v \in E_l$ for some $l \in \{1, 2, ..., m\}$. Then $E_u\alpha^2 = (E_u\alpha)\alpha = \{e_u\}\alpha = \{e_l\}$ and $E_v\alpha^2 = (E_v\alpha)\alpha = \{e_v\}\alpha = \{e_l\}$. So $E_u \cup E_v \subseteq e_l(\alpha^2)^{-1}$. This implies that $|\pi^2_\alpha| < |\pi_\alpha|$ which is a contradiction since $\pi_\alpha = \pi_{\alpha^2}$. So $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| \neq \emptyset$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. Therefore, $|x\alpha^{-1} \cap (X\alpha \setminus (Y_1 \cup Y_2)\alpha)| = 1$. □

**Corollary 3.3.11.** *Let $\alpha \in S(X, Y_1, Y_2)$ be such that $X\alpha$ is finite. If $\alpha$ is left regular, then $\alpha$ is regular.*

*Proof.* Assume that $\alpha$ is left regular. Then $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$ by Theorem 3.3.6. Since $(Y_1 \cup Y_2)\alpha = Y_1\alpha \cup Y_2\alpha \subseteq Y_1 \cup Y_2$ and $(Y_1 \cup Y_2)\alpha \subseteq X\alpha$, we have $(Y_1 \cup Y_2)\alpha \subseteq X\alpha \cap (Y_1 \cup Y_2)$. To show that $X\alpha \cap (Y_1 \cup Y_2) \subseteq (Y_1 \cup Y_2)\alpha$, let $y \in X\alpha \cap (Y_1 \cup Y_2)$. If $y \notin (Y_1 \cup Y_2)\alpha$, then $y \in (X \setminus (Y_1 \cup Y_2))\alpha \subseteq [X \setminus (Y_1 \cup Y_2)] \cup (Y_1 \cup Y_2)\alpha$ by Lemma 3.3.9 (1). Thus $y \in X \setminus (Y_1 \cup Y_2)$ or $y \in (Y_1 \cup Y_2)\alpha$. Since $y \in Y_1 \cup Y_2$, we get $y \in (Y_1 \cup Y_2)\alpha$ which is a contradiction. So $y \in (Y_1 \cup Y_2)\alpha$ and that $X\alpha \cap (Y_1 \cup Y_2) \subseteq (Y_1 \cup Y_2)\alpha$. Thus $X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha$ which implies that $\alpha$ is regular by Theorem 3.3.1. □

**Corollary 3.3.12.** *Let $\alpha \in S(X, Y_1, Y_2)$ be such that $\pi_\alpha$ is finite. If $\alpha$ is right regular, then $\alpha$ is regular.*

*Proof.* Assume that $\alpha$ is right regular. Then $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$. By using the same proof as given for Corollary 3.3.11 and Lemma 3.3.10 (1), we get $X\alpha \cap (Y_1 \cup Y_2) = (Y_1 \cup Y_2)\alpha$ and hence $\alpha$ is regular as required. □

The converse of Corollary 3.3.11 and 3.3.12 does not hold as shown in the example below.

**Example 9.** Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $Y_1 = \{2, 4, 6\}$ and $Y_2 = \{1, 3, 5\}$. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} \{1,5\} & 2 & 3 & \{4,6\} & 7 \\ 1 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Then

$$\alpha^2 = \begin{pmatrix} \{1,3,5\} & \{2,4,6\} & 7 \\ 1 & 6 & 7 \end{pmatrix}.$$

Thus $X\alpha \cap (Y_1 \cup Y_2) = \{1, 4, 5, 6\} = (Y_1 \cup Y_2)\alpha$. So $\alpha$ is regular. But $\alpha$ is not left regular and right regular, since $Y_1\alpha = \{4, 6\} \neq \{6\} = Y_1\alpha^2$ and $\pi_\alpha(Y_1) = \{\{2\}, \{4, 6\}\} \neq \{\{2, 4, 6\}\} = \pi_{\alpha^2}(Y_1)$.

The following examples show that the finiteness of $X\alpha$ in Corollary 3.3.11 and 3.3.12 is necessary.

**Example 10.** (a) Let $X = \mathbb{N}$, $Y_1 = \{1, 2\}$ and $Y_2 = \{3, 4\}$. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & n \\ 2 & 3 & n-4 \end{pmatrix}_{n\in\mathbb{N}\setminus\{1,\ 2,\ 3,\ 4\}}.$$

Then

$$\alpha^2 = \begin{pmatrix} \{1,2,5,6\} & \{3,4,7,8\} & n \\ 2 & 3 & n-8 \end{pmatrix}_{n\in\mathbb{N}\setminus\{1,\ 2,\ ...,\ 8\}}.$$

Thus $X\alpha = \mathbb{N} = X\alpha^2$, $Y_1\alpha = \{2\} = Y_1\alpha^2$ and $Y_2\alpha = \{3\} = Y_2\alpha^2$. So $\alpha$ is left regular. But $\alpha$ is not regular since $X\alpha \cap (Y_1 \cup Y_2) = \{1, 2, 3, 4\} \neq \{2, 3\} = (Y_1 \cup Y_2)\alpha$.

(b) Let $X = \mathbb{N}$, $Y_1$ the set of all positive even integers, $Y_2 = \{1\}$. Define $\alpha \in S(X, Y_1, Y_2)$ by

$$\alpha = \begin{pmatrix} 1 & n \\ 1 & 2n \end{pmatrix}_{n\in\mathbb{N}\setminus\{1\}}.$$

Then

$$\alpha^2 = \begin{pmatrix} 1 & n \\ 1 & 4n \end{pmatrix}_{n\in\mathbb{N}\setminus\{1\}}.$$

Thus $\pi_\alpha = \{\{n\} \mid n \in \mathbb{N}\} = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \{\{n\} \mid n \in \mathbb{N} \setminus \{1\}\} = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \{\{1\}\} = \pi_{\alpha^2}(Y_2)$. So $\alpha$ is right regular. But $\alpha$ is not regular since $X\alpha \cap (Y_1 \cup Y_2) = \{1\} \cup \{2n \mid n \in \mathbb{N} \setminus \{1\}\} \neq \{1\} \cup \{4n \mid n \in \mathbb{N}\} = (Y_1 \cup Y_2)\alpha$.

**Theorem 3.3.13.** *Let $\alpha \in S(X, Y_1, Y_2)$ be such that $X\alpha$ is finite. Then $\alpha$ is left regular if and only if $\alpha$ is right regular.*

*Proof.* Assume that $\alpha$ is left regular. Then $X\alpha = X\alpha^2$, $Y_1\alpha = Y_1\alpha^2$ and $Y_2\alpha = Y_2\alpha^2$ by Theorem 3.3.6. Since $X\alpha$ is finite, we may write $X\alpha = \{a_1, ..., a_m, b_1, ..., b_n, e_1, ..., e_t\}$ where $Y_1\alpha = \{a_1, ..., a_m\}$ and $Y_2\alpha = \{b_1, ..., b_n\}$. By Lemma 3.3.9 (1), we can write

$$\alpha = \begin{pmatrix} A_1 & ... & A_m & B_1 & ... & B_n & E_1 & ... & E_t \\ a_1 & ... & a_m & b_1 & ... & b_n & e_1 & ... & e_t \end{pmatrix}$$

where $A_i \cap Y_1 \neq \emptyset \neq B_j \cap Y_2$, $E_k \subseteq X \setminus (Y_1 \cup Y_2)$; $a_i \in Y_1$, $b_j \in Y_2$, $e_k \in X \setminus (Y_1 \cup Y_2)$ for all $i = 1, ..., m$, $j = 1, ..., n$ and $k = 1, ..., t$. Since $|y\alpha^{-1} \cap Y_1\alpha| = |A_i \cap \{a_1, ..., a_m\}| = 1$ where $y \in Y_1\alpha$ for all $i = 1, ..., m$, there is a permutation $\delta$ on the set $\{1, ..., m\}$ such that $a_i \in A_{i\delta}$ for all $i$. So we obtain $A_i\alpha^2 = \{a_{i\delta}\}$. Since $|y\alpha^{-1} \cap Y_2\alpha| = |B_j \cap \{b_1, ..., b_n\}| = 1$

where $y \in Y_2\alpha$ for all $j = 1, ..., n$, there is a permutation $\sigma$ on the set $\{1, ..., n\}$ such that $b_j \in B_{j\sigma}$ for all $j$. So we obtain $B_j\alpha^2 = \{b_{j\sigma}\}$. Similarly, since $|x\alpha^{-1} \cap X\alpha \setminus (Y_1 \cup Y_2)\alpha| = |E_k \cap \{e_1, ..., e_t\}| = 1$ where $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$ for all $k = 1, ..., t$, there is a permutation $\gamma$ on the set $\{1, ..., t\}$ such that $e_k \in E_{k\gamma}$ for all $k$. Thus $E_k\alpha^2 = \{e_{k\gamma}\}$. So

$$\alpha^2 = \begin{pmatrix} A_1 & ... & A_m & B_1 & ... & B_n & E_1 & ... & E_t \\ a_{1\delta} & ... & a_{m\delta} & b_{1\sigma} & ... & b_{n\sigma} & e_{1\gamma} & ... & e_{t\gamma} \end{pmatrix}.$$

That is $\pi_\alpha(Y_1) = \{A_1, ..., A_m\} = \pi_{\alpha^2}(Y_1)$, $\pi_\alpha(Y_2) = \{B_1, ..., B_n\} = \pi_{\alpha^2}(Y_2)$ and $\pi_\alpha = \{A_1, ..., A_m, B_1, ..., B_n, E_1, ..., E_t\} = \pi_{\alpha^2}$. Hence $\alpha$ is right regular.

Conversely, assume that $\alpha$ is right regular. Then $\pi_\alpha = \pi_{\alpha^2}$, $\pi_\alpha(Y_1) = \pi_{\alpha^2}(Y_1)$ and $\pi_\alpha(Y_2) = \pi_{\alpha^2}(Y_2)$. Since $X\alpha$ is finite, we obtain $\pi_\alpha = \pi_{\alpha^2}$ is finite. Then by Lemma 3.3.10 (1), we can write

$$\alpha = \begin{pmatrix} A_1 & ... & A_m & B_1 & ... & B_n & E_1 & ... & E_t \\ a_1 & ... & a_m & b_1 & ... & b_n & e_1 & ... & e_t \end{pmatrix}$$

where $A_i \cap Y_1 \neq \emptyset \neq B_j \cap Y_2$, $E_k \subseteq X \setminus (Y_1 \cup Y_2)$; $a_i \in Y_1$, $b_j \in Y_2$, $e_k \in X \setminus (Y_1 \cup Y_2)$ for all $i = 1, ..., m$, $j = 1, ..., n$ and $k = 1, ..., t$. Since $|y\alpha^{-1} \cap Y_1\alpha| = |A_i \cap \{a_1, ..., a_m\}| = 1$ for all $y \in Y_1\alpha$, $|y\alpha^{-1} \cap Y_2\alpha| = |B_j \cap \{b_1, ..., b_n\}| = 1$ for all $y \in Y_2\alpha$ and $|x\alpha^{-1} \cap X\alpha(Y_1 \cup Y_2)\alpha| = |E_k \cap \{e_1, ..., e_t\}| = 1$ for all $x \in X\alpha \setminus (Y_1 \cup Y_2)\alpha$. So by the same proof as given above

$$\alpha^2 = \begin{pmatrix} A_1 & ... & A_m & B_1 & ... & B_n & E_1 & ... & E_t \\ a_{1\delta} & ... & a_{m\delta} & b_{1\sigma} & ... & b_{n\sigma} & e_{1\gamma} & ... & e_{t\gamma} \end{pmatrix}$$

where $\delta$ is a permutation on the set $\{1, ..., m\}$, $\sigma$ is a permutation on the set $\{1, ..., n\}$ and $\gamma$ is a permutation on the set $\{1, ..., t\}$. Thus $Y_1\alpha = \{a_1, ..., a_m\} = Y_1\alpha^2$, $Y_2\alpha = \{b_1, ..., b_n\} = Y_2\alpha^2$ and $X\alpha = \{a_1, ..., a_m, b_1, ..., b_n, e_1, ..., e_t\} = X\alpha^2$. Hence $\alpha$ is left regular. □