

CHAPTER 1

Introduction

Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. A point $p \in E$ is a fixed point of f if $f(p) = p$. We denote by $F(f)$ the set of fixed point of f . It is known that if E is also bounded, then $F(f)$ is nonempty. The Mann iteration (see [9]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \quad (1.1)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$, and will denote by $M(u_1, \alpha_n, f)$. The Ishikawa iteration (see [6]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - \beta_n)s_n + \beta_n f(s_n) \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \quad (1.2)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $I(s_1, \alpha_n, \beta_n, f)$. The Noor iteration (see [4]) is defined by $a_1 \in E$ and

$$\begin{cases} c_n = (1 - \gamma_n)a_n + \gamma_n f(a_n) \\ b_n = (1 - \beta_n)a_n + \beta_n f(c_n) \\ a_{n+1} = (1 - \alpha_n)a_n + \alpha_n f(b_n) \end{cases} \quad (1.3)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $N(a_1, \alpha_n, \beta_n, \gamma_n, f)$. Clearly the Mann and Ishikawa iterations are special cases of Noor iteration. The SP-iteration (see [8]) is defined by $q_1^* \in E$ and

$$\begin{cases} r_n^* = (1 - \gamma_n)q_n^* + \gamma_n f(q_n^*) \\ t_n^* = (1 - \beta_n)r_n^* + \beta_n f(r_n^*) \\ q_{n+1}^* = (1 - \alpha_n)t_n^* + \alpha_n f(t_n^*) \end{cases} \quad (1.4)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $SP(q_1^*, \alpha_n, \beta_n, \gamma_n, f)$. The S-iteration (see [7]) is defined by $s_1^* \in E$ and

$$\begin{cases} t_n^* = (1 - \beta_n)s_n^* + \beta_n f(s_n^*) \\ s_{n+1}^* = (1 - \alpha_n)f(s_n^*) + \alpha_n f(t_n^*) \end{cases} \quad (1.5)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $S(s_1^*, \alpha_n, \beta_n, f)$. The P-iteration (see [5]) is defined by $q_1 \in E$ and

$$\begin{cases} r_n = (1 - \gamma_n)q_n + \gamma_n f(q_n) \\ t_n = (1 - \beta_n)r_n + \beta_n f(r_n) \\ q_{n+1} = (1 - \alpha_n)f(r_n) + \alpha_n f(t_n) \end{cases} \quad (1.6)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $P(q, \alpha_n, \beta_n, \gamma_n, f)$.

In 1976, Rhoades [1] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and non-decreasing functions on unit closed interval. After that in 1991 Borwein and Borwein [2] proved the convergence of the Mann iteration of the continuous functions on a bounded closed interval. It was shown in [8] that the SP-iteration converges faster than the Noor iteration, the Ishikawa iteration and the Mann iteration. In 2013, Kosol [7] showed that the S-iteration converges faster than the Ishikawa iteration on an arbitrary interval, after that, Sainuan [5] showed that the P-iteration converges faster than S-iteration on an arbitrary interval.

Motivated by the above results, we propose a new iteration as follows: The W-iteration is defined by $x_1 \in E$ and

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n) \\ y_n = (1 - \beta_n)f(x_n) + \beta_n f(z_n) \\ x_{n+1} = (1 - \alpha_n)f(z_n) + \alpha_n f(y_n) \end{cases} \quad (1.7)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequence in $[0, 1]$, and will denote by $W(x_1, \alpha_n, \beta_n, \gamma_n, f)$. Moreover, we also introduce the ST-iteration as follow: The ST-iteration is defined by $x_1^* \in E$ and

$$\begin{cases} z_n^* = (1 - \gamma_n)x_n^* + \gamma_n f(x_n^*) \\ y_n^* = (1 - \beta_n)f(x_n^*) + \beta_n f^2(z_n^*) \\ x_{n+1}^* = (1 - \alpha_n)f(z_n^*) + \alpha_n f^2(y_n^*) \end{cases} \quad (1.8)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will denote by $ST(x_1^*, \alpha_n, \beta_n, \gamma_n, f)$.

In this study, we give a necessary and sufficient condition for the convergence of the W-iteration and ST-iteration of continuous functions on an arbitrary interval. We also prove that if the P-iteration converges, then the W-iteration and ST-iteration converge and converge faster than the P-iteration for the class of continuous and non-decreasing function. Moreover, we present numerical examples for comparing the rate of convergence of W-iteration, ST-iteration and P-iteration.