# CHAPTER 2

## Basic Concepts and Preliminaries

In this chapter, we recall and give some useful definitions, and results which will be used in the later chapter.

### 2.1 Functions

In this part, we recall some definitions, examples and results of functions which are the background knowlegdes and be used for next chapter.

A set is a collection of objects, called the elements or members of the set. The objects could be anything but for us they will be mathematical objects such as numbers, set of numbers, points or functions. We write $x \in X$ if x is an element of the set $X$ and $x \notin X$ if x is not an element of $X$.

**Definition 2.1.1.** [3] The Cartesian product $X \times Y$ of sets $X$ and $Y$, is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$ . If $X = Y$ , we often write $X \times X = X^2$. Two ordered pairs $(x_1, y_1), (x_2, y_2) \in X \times Y$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. Thus, $(x, y) \neq (y, x)$ unless $x = y$. This contrasts with sets where $\{x, y\} = \{y, x\}$.

**Example 2.1.1.** Let $X = \{3, 5, 7\}$ and $Y = \{4, 5\}$, then

$$X \times Y = \{(3, 4), (3, 5), (5, 4), (5, 5), (7, 4), (7, 5)\}$$
$$Y \times X = \{(4, 3), (4, 5), (4, 7), (5, 3), (5, 5), (5, 7)\}$$

**Definition 2.1.2.** [3] Let $X$ and $Y$ be nonempty sets. $f = \{(x, y) \in X \times Y\}$ is a function when if $(x_1, y_1)$ and $(x_2, y_2) \in f$ which $x_1 = x_2$, then $y_1 = y_2$. Functions are also called maps, mappings, or transformations. The set $X$ on which $f$ is defined is called the domain of $f$ and the set $Y$ in which it takes its values is called the range of $f$.

**Example 2.1.2.** The identity function $f : X \to X$ is the function that maps every element of $X$ to itself such that $f(x) = x$ for all $x \in X$. (See Figure 2.1)

**Example 2.1.3.** The square function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. (See Figure 2.2)

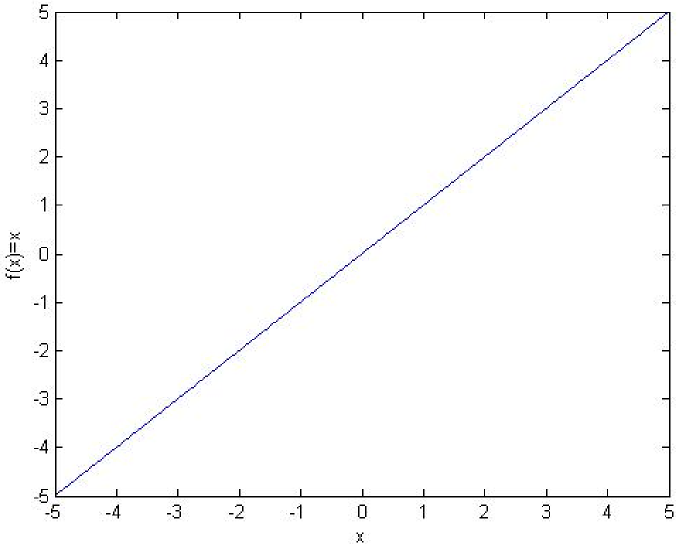


Figure 2.1: Graph of identity function.

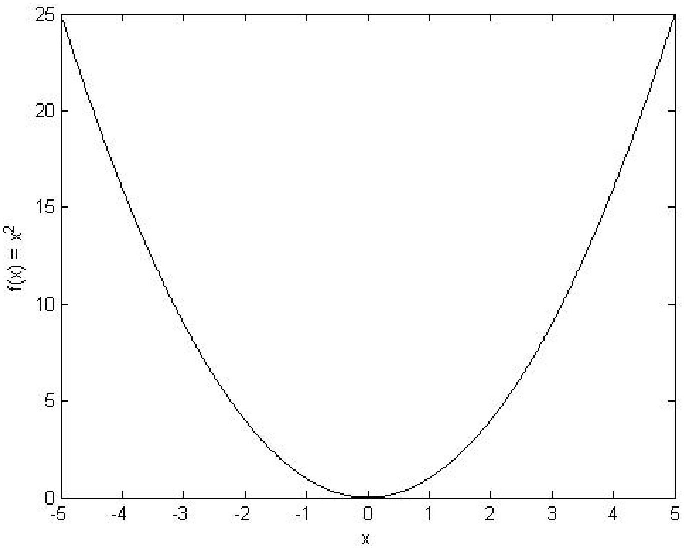


Figure 2.2: Graph of $f(x) = x^2$ from -5 to 5

From example 2.1.2 we have if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$ and if $x_1 \geq x_2$ then $f(x_1) \geq f(x_2)$. It leads to the important definition of function which will be used for our results.

**Definition 2.1.3.** [3] Let $X$ and $Y$ be nonempty sets and $f : X \to Y$ be a function. For every $x_1, x_2 \in X$

i) $f$ is called increasing if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$.

ii) $f$ is called strictly increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

iii) $f$ is called decreasing if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$.

iv) $f$ is called strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

**Example 2.1.4.** Let $E = [0, 1]$ be a closed interval on real line and $f : E \to E$ be defined by $f(x) = sin(x)$, we obtain $f$ is increasing.

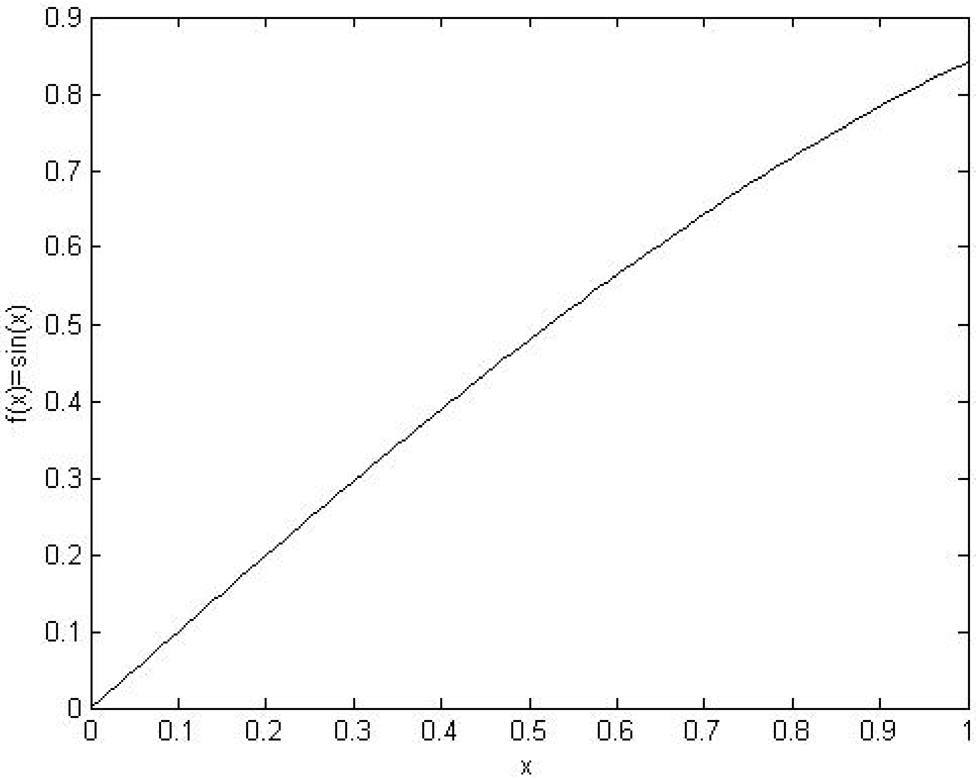




Figure 2.3: Graph of f(x) = sin(x) from 0 to 1.

**Example 2.1.5.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -x$, we obtain $f$ is decreasing. (See figure 2.4)

Next we will recall definition of limit function and continuity of function.

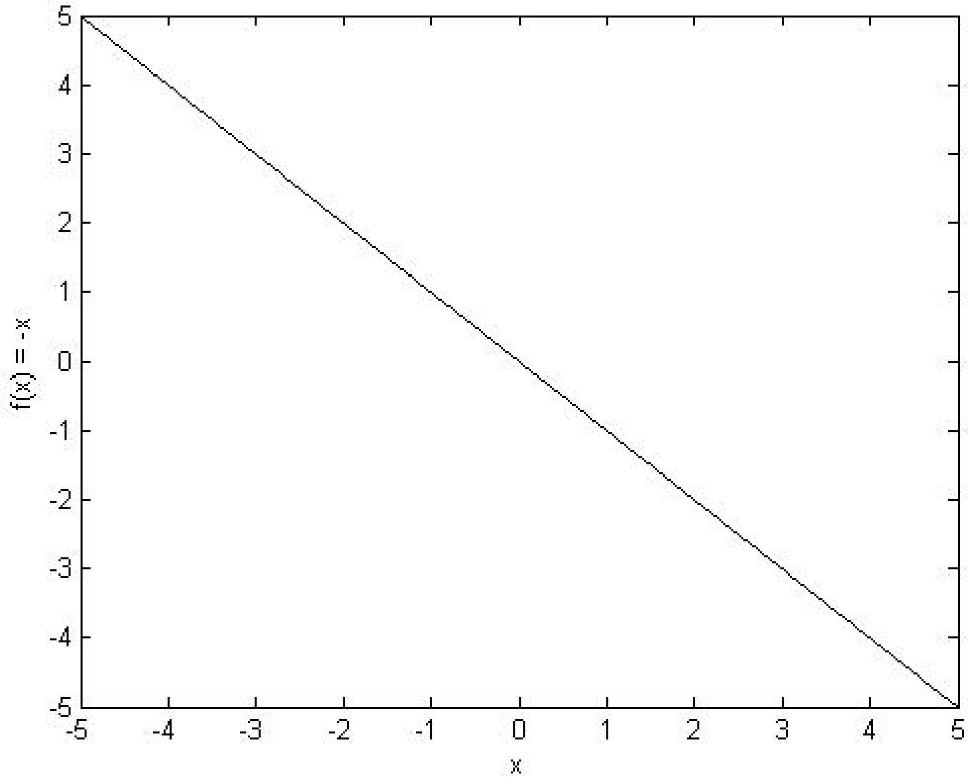


Figure 2.4: Graph of f(x) = -x from -5 to 5

**Definition 2.1.4.** [3] Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of $A$. Then $lim_{x\to c}f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in A$ implies that $|f(x) - L| < \epsilon$. We also denote limits by the arrow notation $f(x) \to L$ as $x \to c$, and often leave it to be implicitly understood that $x \in A$ is restricted to the domain of $f$.

**Example 2.1.6.** Let $A = [0, \infty) - \{9\}$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x-9}{\sqrt{x}-3}.$$

We claim that

$$lim_{x\to 9}f(x) = 6.$$

To prove this, let $\epsilon > 0$ be given. If $x \in A$, then $\sqrt{x} - 3 \neq 0$ , and dividing this factor into the numerator we get $f(x) = \sqrt{x} - 3$ It follows that

$$|f(x) - 6| = |\sqrt{x} - 3| = |\frac{x-9}{\sqrt{x}-3}| \leq \frac{1}{3}|x - 9|.$$

Thus, if $\delta = 3\epsilon$, then $x \in A$ and $|x - 9| < \sigma$ implies that $|f(x) - 6| < \epsilon$.

**Definition 2.1.5.** [3] Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in A$. Then $f$ is continuous at $c$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies that $|f(x) - f(c)| < \epsilon$. A function $f : A \to \mathbb{R}$ is continuous if it is continuous at every point of $A$, and it is continuous on $B \subset A$ if it is continuous at every point in $B$.

**Example 2.1.7.** If $f : (a, b) \to \mathbb{R}$ is defined on an open interval, then $f$ is continuous on $(a, b)$ if and only if

$$lim_{x\to c}f(x) = f(c) \quad \text{for every} \quad a < c < b$$

since every point of $(a, b)$ is an accumulation point.

**Example 2.1.8.** If $f : [a, b] \to \mathbb{R}$ is defined on an open interval, then $f$ is continuous on $[a, b]$ if and only if

$$lim_{x\to c}f(x) = f(c) \quad \text{for every} \quad a < c < b$$
$$lim_{x\to a^+}f(x) = f(a) \quad lim_{x\to b^-}f(x) = f(b)$$

**Definition 2.1.6.** Let $E$ be a closed interval on the real line and $f : E \to \mathbb{R}$ be a function. A point $p \in E$ is a fixed point of $f$ if $f(p) = p$. We denote by $F(f)$ the set of fixed point of $f$.

**Example 2.1.9.** Let $E = [-1, 1]$ and $f(x) = sin(x)$. We have $0 \in E$ such that $0 = sin(0)$, 0 is a fixed point of $f$. (See figure 2.5)

In this thesis we interest to find a fixed point of continuous function by using a new iterative method. The fixed points can be used to solve nonlinear equation such as from example 2.1.9 we have 0 is an answer of $sin(x) - x = 0$, where $x \in [0, 1]$.



## 2.2 Sequences

In this section, we recall the definitions, examples and some theorems about the sequences which will be used in main results.

**Definition 2.2.1.** [3] A sequence $\{x_n\}$ of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$, where $x_n = f(n)$, for all $n \in \mathbb{N}$.

We can consider the sequences of many differrent types of objects, but now we only consider the sequences of real number.

**Example 2.2.1.** (Fibonacci sequence)
A well-known example of a recursive sequence is the sequence

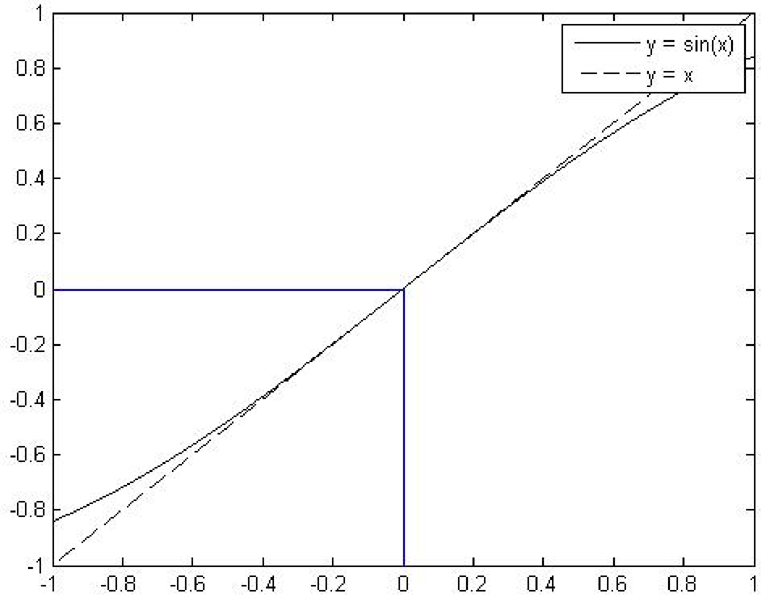


Figure 2.5: Graph of f(x) = sin(x), $-1 \leq x \leq 1$.

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

is called Fibonacci sequence $(F_n)$ or we said $F_1 = F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 1. \quad \text{(See figure 2.6)}$$

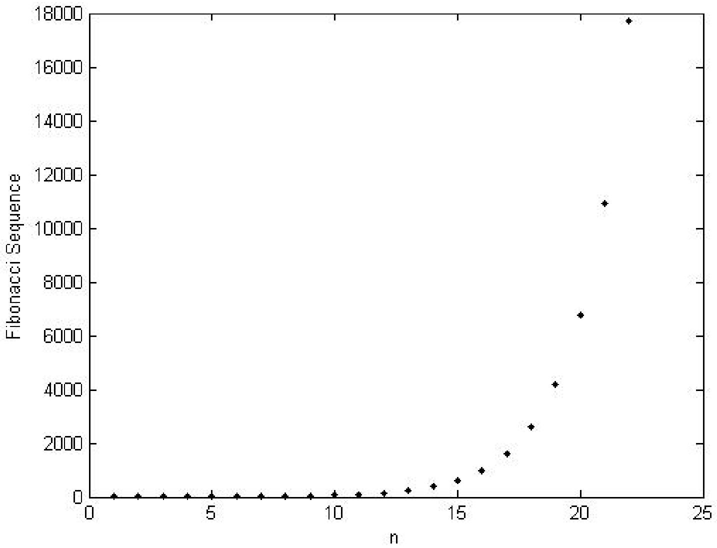


Figure 2.6: Graph of fibonacci sequence.

**Example 2.2.2.** Let $\{x_n\}$ be a sequence defined by $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. (See figure 2.7)

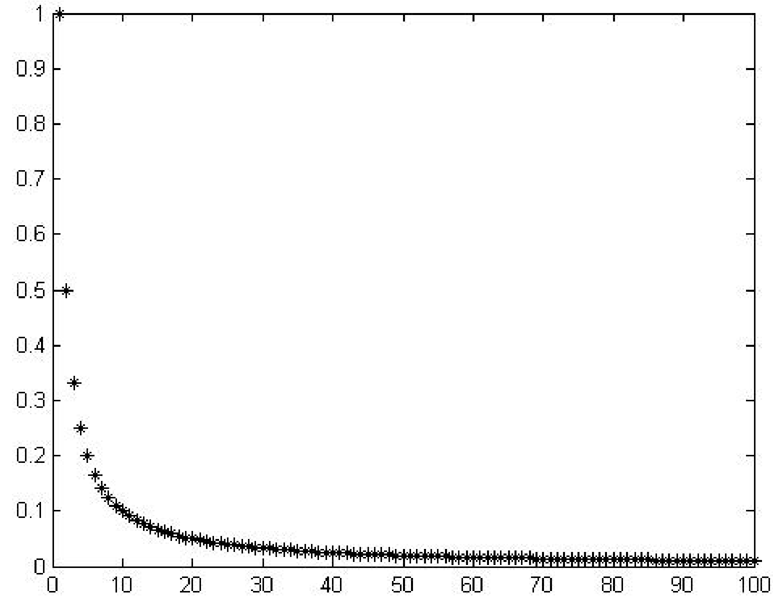


Figure 2.7: Figure of sequence $a_n = \frac{1}{n}$ for all $n \geq 1$.

From example 2.2.1, we have if $n_1 < n_2$ then $F_{n_1} \leq F_{n_2}$ and from example 2.2.2, we have if $n_1 < n_2$ then $a_{n_1} \geq a_{n_2}$. Next, we present definition of monotone sequences which will be use in next section.

**Definition 2.2.2.** [3] A sequence of real numbers $\{x_n\}$ is increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$, decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$ and monotone if it is increasing or decreasing. A sequence is strictly increasing if $x_{n+1} > x_n$, strictly decreasing if $x_{n+1} < x_n$, and strictly monotone if it is strictly increasing or strictly decreasing.

From example 2.2.1, we have $F_n$ is increasing, but not strictly increasing becausen $F_1 = F_2$ and example 2.2.2, $\{x_n\}$ is decreasing and strictly decreasing.

We said $\{x_n\}$ converges to a limit $x$ if its terms $x_n$ get arbitrarily close to $x$ for all sufciently large $n$.

**Definition 2.2.3.** [3] A sequence of real numbers $\{x_n\}$ converges to a limit $x \in \mathbb{R}$, written $x = lim_{n\to\infty} x_n$ or $x_n \to x$ as $n \to \infty$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n > N$. A sequence converges if it converges to some limit $x \in$ R, otherwise it diverges.

Note that if a sequence converges, then its limit is unique.

**Example 2.2.3.** The sequence $F_n$ in Example 2.2.1 is divergent and sequence $\{x_n\}$ in Example 2.2.2 converges to 0.

**Example 2.2.4.** The terms in the sequence $1, 8, 27, 64, ...\quad x_n = n^3$ eventually exceed any real number, so $n^3 \to \infty$ as $n \to \infty$ and this sequence does not converge. Explicitly, let $M \in \mathbb{R}$ be given, and choose $N \in \mathbb{N}$ such that $N > M^{\frac{1}{3}}$. (If $-\infty < M < 1$, we can choose $N = 1$.) Then for all $n > N$, we have $n^3 > N^3 > M$, which proves the result.

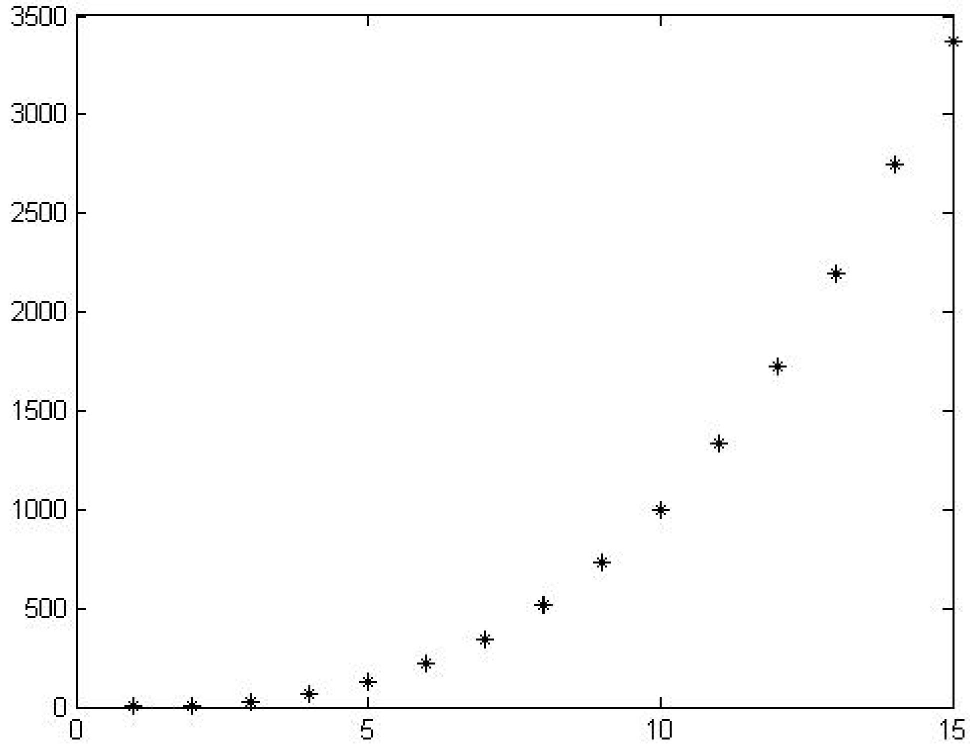


Figure 2.8: Figure of $a_n = n^3$ for all $n \geq 1$.

**Example 2.2.5.** The terms in the sequence

$$1, -1, 1, -1, 1, -1, ... \quad \text{or } x_n = (-1)^{n+1}$$

oscillate back and forth infinitely often between 1 and $-1$, but they do not approach any fixed limit, so the sequence does not converge. To show this explicitly, note that for every $x \in R$ we have either $|x - 1| \geq 1$ or $|x + 1| \geq 1$. It follows that there is no $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n > N$. Thus, definition 2.2.3 fails if $\epsilon = 1$ however we choose $x \in \mathbb{R}$, and the sequence does not converge.

We then have the following properties of a limit of sequences.

**Theorem 2.2.6.** *[3] Let* $\{x_n\}$ *and* $\{y_n\}$ *be convergent real sequences and* $c \in \mathbb{R}$*. Then the sequences* $\{cx_n\}$*,* $\{x_n + y_n\}$ *and* $\{x_n y_n\}$ *converge, and*

*i)* $lim_{n\to\infty} cx_n = c lim_{n\to\infty} x_n$*.*

*ii)* $lim_{n\to\infty}(x_n + y_n) = (lim_{n\to\infty} x_n) + (lim_{n\to\infty} y_n)$*.*

*iii)* $lim_{n\to\infty} x_n y_n = (lim_{n\to\infty} x_n)(lim_{n\to\infty} y_n)$*.*

Now, we will give an important property of a sequence is whether or not it is bounded.

**Definition 2.2.4.** [3] A sequence of real numbers $\{x_n\}$ is bounded from above if there exists $M \in \mathbb{R}$ such that $x_n \le M$ for all $n \in N$, and bounded from below if there exists $m \in \mathbb{R}$ such that $x_n \ge m$ for all $n \in N$. A sequence is bounded if it is bounded from above and below, otherwise it is unbounded.

An equivalent condition for a sequence $\{x_n\}$ to be bounded is that there exists $M \ge 0$ such that $|x_n| \le M$ for all $n \in N$.

**Example 2.2.7.** The sequence $F_n$ (in example 2.2.1) is bounded from below (choose $m$ = 0), but not bounded from above. A sequence $\{x_n\}$ (in example 2.2.2) is bounded from above and below (choose $M$ =2) .

We then have the following property of convergent sequences.

**Theorem 2.2.8.** *[3] A convergent sequence is bounded.*



**Theorem 2.2.9.** *[3] Let* $f : E \to E$ *is a continuous function and* $\{x_n\}$ *is a bounded sequence. Then* $f(\{x_n\})$ *is bounded.*

**Theorem 2.2.10.** *[3] Every monotone bounded sequence in* $\mathbb{R}$ *converges.*

## 2.3 P-Iteration

In this part, we present the definitions and useful lemmas and theorems which will be used for our main results.

**Definition 2.3.1.** [5] Let $E$ be closed interval on real line, $f : E \to E$ be a continuous function. The P-iteration is defined by $q_1 \in E$,and

$$r_n = (1-\gamma_n)q_n + \gamma_n f(q_n)$$
$$t_n = (1-\beta_n)r_n + \beta_n f(r_n)$$
$$q_{n+1} = (1-\alpha_n)f(r_n) + \alpha_n f(t_n)$$

for all $n \geq 1$ and $\{\beta_n\}_{n=1}^{\infty}$,$\{\alpha_n\}_{n=1}^{\infty}$,$\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0,1], and we will denote by $P(x_1, \alpha_n, \beta_n, \gamma_n, f)$.

**Lemma 2.3.1.** *[5] Let $E$ be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $q_1 \in E$, let $\{q_n\}$ be a sequence defined by P-iteration. Then the following hold:*

*i) If $f(q_1) < q_1$ then $f(q_n) \leq q_n$ , for all $n \geq 1$ and $\{q_n\}$ is non-increasing.*

*ii) If $f(q_1) > q_1$ then $f(q_n) \geq q_n$, for all $n \geq 1$ and $\{q_n\}$ is non-decreasing.*

**Theorem 2.3.2.** *[5] Let $E$ be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $lim_{n\to\infty}\beta_n = 0$ and $lim_{n\to\infty}\gamma_n = 0$. Let $\{q_n\}$ be a sequence defined by P-iteration. Then $\{q_n\}$ is bounded if and only if $\{q_n\}$ converges to a fixed point of $f$.*

**Lemma 2.3.3.** *[5] Let $E$ be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $q_1 \in E$, let $\{q_n\}$ be a sequence defined by P-iteration. Then the following hold:*

*i) If $p \in F(f)$ with $q_1 > p$, then $q_n \geq p$ , for all $n \geq 1$.*

*ii) If $p \in F(f)$ with $q_1 < p$, then $q_n \leq p$ , for all $n \geq 1$.*



**Proposition 2.3.4.** *[5] Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 < inf\, F(f)$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of f.*

**Proposition 2.3.5.** *[5] Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 > sup\ F(f)$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of f.*

**Definition 2.3.2.** [8] Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous function. Suppose that $\{x_n\}$ and $\{y_n\}$ are two iterations with converge to the fixed point $p$ of $f$. Then $\{x_n\}$ is said to converge faster than $\{y_n\}$ if $|x_n - p| \leq |y_n - p|$ for all $n \geq 1$.

**Theorem 2.3.6.** *[5] Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded. For $q_1 = x_1 \in E$, let $\{q_n\}$ and $\{x_n\}$ be the sequences defined by S-iteration and P-iteration respectively. If $\{q_n\}$ converges to a fixed point $p$, then $\{x_n\}$ converges to $p$. Moreover, $\{x_n\}$ converges to $p$ faster than $\{q_n\}$.*