CHAPTER 3

New Iteration for Solving Non-Linear Equations

In this thesis we introduce a new iteration method for finding a fixed point of a continuous function on a closed interval and give neccesary and sufficient conditions for convergence of the proposed method.

3.1 W - iteration

Definition 3.1.1. Let E be a closed interval on real line, $f : E \to E$ be a function.The W-iteration is defined by $x_1 \in E$, and

$$z_n = (1-\gamma_n)x_n + \gamma_n f(x_n)$$
$$y_n = (1-\beta_n)f(x_n) + \beta_n f(z_n)$$
$$x_{n+1} = (1-\alpha_n)f(z_n) + \alpha_n f(y_n)$$

for all $n \geq 1$ and $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0,1]. We will denote this iteration method by $W(x_1, \alpha_n, \beta_n \gamma_n, f)$ and it is called W-iteration.

Next, we will prove the convergence theorems.To prove this, we need the following lemmas.

Lemma 3.1.1. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 \in E$, let $\{x_n\}$ be a sequence defined by W-iteration. Then the following hold:*

i) If $f(x_1) < x_1$ then $f(x_n) \leq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-increasing.

ii) If $f(x_1) > x_1$ then $f(x_n) \geq x_n$, for all $n \geq 1$ and $\{x_n\}$ is non-decreasing.

Proof. i) Let $\{x_n\}$ be sequence defined by $W-iteration$ and $f(x_1) < x_1$. From Definition 4.1.1, we have $z_1 = (1-\gamma_1)x_1 + \gamma_1 f(x_1)$. Since $f(x_1) < x_1$,then $f(x_1) \leq z_1 \leq x_1$. From f is non-decreasing function, then $f^2(x_1) \leq f(z_1) \leq f(x_1)$. So we have $f^2(x_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1$. From Definition 4.1.1, $y_1 = (1-\beta_1)f(x_1) + \beta_1 f(z_1)$, we have $f(z_1) \leq y_1 \leq f(x_1)$. From f is non-decreasing, we get $f^2(z_1) \leq f(y_1) \leq f^2(x_1)$. Form $f^2(x_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1$, we have,

$$f^2(z_1) \leq f(y_1) \leq f^2(x_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1. \tag{3.1}$$

From Definition 4.1.1, $x_2 = (1-\alpha_1)f(z_1) + \alpha_1 f(y_1)$, we have,

$$f(y_1) \leq x_2 \leq f(z_1) \leq x_1. \tag{3.2}$$

From f is non-decreasing, we obtain,

$$f^2(y_1) \leq f(x_2) \leq f^2(z_1). \tag{3.3}$$

Form (3.1), (3.3) and (3.3), we have $f(x_2) \leq x_2$ and $x_2 \leq x_1$.

Next, we assume that $f(x_k) \leq x_k$. From Definition 4.1.1, we have $z_k = (1-\gamma_k)x_k + \gamma_k f(x_k)$. Since $f(x_k) \leq x_k$,we get $f(x_k) \leq z_k \leq x_k$. From f is non-decreasing, we have $f^2(x_k) \leq f(z_k) \leq f(x_k)$, so we get $f^2(x_k) \leq f(z_k) \leq f(x_k) \leq z_k \leq x_k$. From Definition 4.1.1, $y_k = (1-\beta_k)f(x_k) + \beta_1 f(z_k)$. So, we have $f(z_k) \leq y_k \leq f(x_k)$. From f is non-decreasing, we get $f^2(z_k) \leq f(y_k) \leq f^2(x_k)$ and

$$f^2(z_k) \leq f(y_k) \leq f^2(x_k) \leq f(z_k) \leq f(x_k) \leq z_k \leq x_k. \tag{3.4}$$

From Definition 4.1.1, $x_{k+1} = (1-\alpha_k)f(z_k) + \alpha_k f(y_k)$, so,we have

$$f(y_k) \leq x_{k+1} \leq f(z_k) \leq x_k. \tag{3.5}$$

From f is non-decreasing, we obtain $f^2(y_k) \leq f(x_{k+1}) \leq f^2(z_k)$. This together with (3.4) and (3.5), we have $f^2(y_k) \leq f(x_{k+1}) \leq f^2(z_k) \leq f(y_k) \leq x_{k+1} \leq f(z_k) \leq y_k \leq z_k \leq x_k$. Thus $f(x_{k+1}) \leq x_{k+1}$ and $x_{k+1} \leq x_k$. By induction, we can conclude that $f(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}$ is non-increasing.

ii) By using tha same argument as in i) we obtain the desired result. □

Theorem 3.1.2. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $lim_{n\to\infty}\alpha_n = 0$, $lim_{n\to\infty}\beta_n = 0$, $lim_{n\to\infty}\gamma_n = 0$. For $x_1 \in E$, let $\{x_n\}$ be a sequence defined by W-iteration. Then $\{x_n\}$ is bounded if and only if $\{x_n\}$ converges to a fixed point of f.*

Proof. It is easy to see that if $\{x_n\}$ converges to a fixed point of f, then $\{x_n\}$ is bounded. Next, suppose that $\{x_n\}$ is bounded.

Case 1. $f(x_1) = x_1$ From Definition 3.1.1 we obtain that $x_n = x_1$ for all $n \geq 1$. Thus $\{x_n\}$ converges to $x_1 \in F(f)$.

Case 2. $f(x_1) \neq x_1$

If $f(x_1) < x_1$, by Lemma 2(i), we have $\{x_n\}$ is non-increasing. It follows that $\{x_n\}$ is convergent.

If $f(x_1) > x_1$, by Lemma 2(ii), we have $\{x_n\}$ is non-decreasing. It follows that $\{x_n\}$ is convergent.

Next, we will show $\{x_n\}$ converges to a fixed point of f. Let $p = lim_{n\to\infty} x_n$. By continuity of f, we have that $f(x_n)$ converges to $f(p)$. From $z_n = (1-\gamma_n)x_n + \gamma_n f(x_n)$ and $lim_{n\to\infty}\gamma_n = 0$, we obtain that $lim_{n\to\infty} z_n = lim_{n\to\infty} x_n = p$. By continuity of f, we have that $f(z_n)$ converges to $f(p)$. From $y_n = (1-\beta_n)f(x_n) + \beta_n f(z_n)$ and $lim_{n\to\infty}\beta_n = 0$, we get $lim_{n\to\infty} y_n = lim_{n\to\infty} f(x_n) = f(p)$. From $x_{n+1} = (1-\alpha_n)f(z_n) + \alpha_n f(y_n)$ and $lim_{n\to\infty}\alpha_n = 0$, we have $lim_{n\to\infty} x_{n+1} = lim_{n\to\infty} f(z_n) = f(p)$. So we get $p = f(p)$. Thus $\{x_n\}$ converges to a fixed point of f. □

Next, we compare the rate of convergence between W-iteration and P-iteration. To do this, we need the following definition and lemmas.

Lemma 3.1.3. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence defined by W-iteration. Then we have the following :*

i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.

ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Proof. i) Let $p \in F(f)$ and $x_1 > p$. Since f is non-decreasing, $f(x_1) \geq f(p)$. From $z_1 = (1-\gamma_1)x_1 + \gamma_1 f(x_1)$ and $x_1 > p$, we have $z_1 \geq (1-\gamma_1)p + \gamma_1 f(p) = p$. Since f is non-decreasing, we have $f(z_1) \geq p$. From $y_1 = (1-\beta_1)f(x_1) + \beta_1 f(z_1)$, $x_1 > p$ and $z_1 \geq p$, we get $y_1 \geq (1-\beta_1)f(p) + \beta_1 f(p) = p$. It follows that $f(y_1) \geq p$. From $x_2 = (1-\alpha_1)f(z_1) + \alpha_1 f(y_1)$ and $f(z_1) \geq p, f(y_1) \geq p$. we get $x_2 \geq p$.

Next, we assume $x_k \geq p$, we will show $x_{k+1} \geq p$. From $z_k = (1-\gamma_k)x_k + \gamma_k f(x_k)$ and $x_k > p$, we get $z_k \geq (1-\gamma_k)p + \gamma_k f(p) = p$. Since of f is non-decreasing, we have $f(z_k) \geq p$. From $y_k = (1-\beta_k)f(x_k) + \beta_k f(z_k)$, $f(x_k) > p$ and $f(z_k) \geq p$, we have $y_k \geq (1-\beta_k)p + \beta_k p = p$. It follows that $f(y_k) \geq p$. From $x_{k+1} = (1-\alpha_k)f(z_k) + \alpha_k f(y_k)$ and $f(z_k) \geq p, f(y_k) \geq p$, we get $x_{k+1} \geq p$. By induction, we conclude that $x_n \geq p$ for all $n \geq 1$.

ii) By using the same argument as i), we obtain the desired result. □

Lemma 3.1.4. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 = q_1 \in E$, let $\{q_n\}$ be a sequence defined by P - Iteration and $\{x_n\}$ be a sequence defined by W-iteration. Then we have the following results:*

i) If $f(q_1) < q_1$, then $x_n \leq q_n$ for all $n \geq 1$.

ii) If $f(q_1) > q_1$, then $x_n \geq q_n$ for all $n \geq 1$.

Proof. Suppose that $f(q_1) < q_1$. Since $x_1 = q_1$, we get $f(x_1) < x_1$ and $f(x_1) = f(q_1)$. From $z_1 = (1 - \gamma_1)x_1 + \gamma_1 f(x_1)$, $f(x_1) \leq z_1 \leq x_1$. We note that $z_1 - r_1 = (1 - \gamma_1)(x_1 - q_1) + \gamma_1(f(x_1) - f(q_1)) = 0$, so $z_1 = r_1$ and $f(z_1) = f(r_1)$. It follows that $y_1 - t_1 = (1 - \beta_1)(f(x_1) - r_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$, so $y_1 \leq t_1$. Since f is non-decreasing, $f(y_1) \leq f(t_1)$. From $x_2 - q_2 = (1 - \alpha_1)[f(z_1) - f(r_1)] + \alpha_1(f(y_1) - f(t_1)$, we get $x_2 - q_2 \leq 0$, so $x_2 \leq q_2$.

Next, we assume that $x_k \leq q_k$. Then $f(x_k) \leq f(q_k)$. It follows that $z_k - r_k = (1 - \gamma_k)(x_k - q_k) + \gamma_k(f(x_k) - f(q_k)) \leq 0$, so $z_k \leq r_k$. Since f is non-decreasing, $f(z_k) \leq f(r_k)$. By Lemma 2.3.1(i), $f(q_k) \leq q_k$. From $r_k = (1 - \gamma_k)q_k + \gamma_k f(q_k)$, we have $f(q_k) \leq r_k \leq q_k$. From $y_k - t_k = (1 - \beta_k)(f(x_k) - r_k) + (\beta_k)(f(z_k) - f(r_k)) \leq 0$, it follows that $y_k \leq t_k$. Since f is non-decreasing, $f(y_k) \leq f(t_k)$. From $x_{k+1} - q_{k+1} = (1 - \alpha_k)[f(z_k) - f(r_k)] + \alpha_k(f(y_k) - f(t_k) \leq 0$, we get that $x_{k+1} - q_{k+1} \leq 0$, so $x_{k+1} \leq q_{k+1}$. By induction, we obtain $x_n \leq q_n$, for all $n \geq 1$.

ii) By using same argument as (i) we obtain $x_n \geq q_n$, for all $n \geq 1$. □

Proposition 3.1.5. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 < inf\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ defined by W-iteration dose not converges to a fixed point of f.*

Proof. Suppose $f(x_1) < x_1$. By Lemma 2.3.1(i), $\{x_n\}$ is non-increasing. Since the initial point $x_1 < inf\{p \in E; f(p) = p\}$, it follows that $\{x_n\}$ dose not converges to a fixed point of f. □

Proposition 3.1.6. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 > sup\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$,$\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ defined by W-iteration dose not converges to a fixed point of f.*

Proof. Suppose $f(x_1) > x_1$. By Lemma 2.3.1(ii), $\{x_n\}$ is non-decreasing. Since the initial point $x_1 > sup\{p \in E; f(p) = p\}$, it follows that $\{x_n\}$ dose not converges to a fixed point of f. □

We next compare the rate of convergence between P-iteration and W-iteration.

Theorem 3.1.7. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded. For $x_1 = q_1 \in E$, let $\{q_n\}$ and $\{x_n\}$ be the sequences defined by P-iteration and W-iteration, respectively. If $\{q_n\}$ converges to a fixed point p of f, then $\{x_n\}$ converges to p. Moreover $\{x_n\}$ converges faster than $\{q_n\}$.*

Proof. Suppose that $\{q_n\}$ converges to a fixed point p of f. Let $l = \inf \mathrm{F(f)}$ and $u = \sup \mathrm{F(f)}$. We divide our proof into three cases.

Case 1: $q_1 = x_1 > u$. Since $\{q_n\}$ converges to p, by Proposition 2.3.5, we get $f(q_1) < q_1$. By Lemma 3.1.4(i), we get $x_n \leq q_n$ for all $n \geq 1$. Since f is non-decreasing, we have $u = f(u) \leq f(x_1) < x_1$, it follows that $f(x_1) \leq z_1 \leq x_1$, so $u \leq z_1 \leq x_1$. Because of f is non-decreasing, we get $u = f(u) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1$. From $y_1 = (1 - \beta_1)f(x_1) + \beta_1 f(z_1)$, we get $u = f(u) \leq f(z_1) \leq y_1 \leq f(x_1)$, since f is non-decreasing, $u = f(u) \leq f^2(z_1) \leq f(y_1) \leq f^2(x_1) \leq f(z_1) \leq f(x_1)$. From definition of x_2, we have $f(y_1) \leq x_2 \leq f(z_1)$, so we get $u = f(u) \leq f(y_1) \leq x_2$. Hence $u \leq x_2$.

Next, we assume $u \leq x_k$. Then $u = f(u) \leq f(x_k)$. By Lemma 4.1.1(i) and definition of z_k, we get $f(x_k) \leq z_k \leq x_k$, so $u \leq z_k \leq x_k$. Since f is non-decreasing, we get $u = f(u) \leq f(z_k) \leq f(x_k) \leq z_k \leq x_k$. From $y_k = (1 - \beta_k)f(x_k) + \beta_k f(z_k)$, we get $u = f(u) \leq f(z_k) \leq y_k \leq f(x_k)$ and $u = f(u) \leq f^2(z_k) \leq f(y_k) \leq f^2(x_k) \leq f(z_k) \leq f(x_k)$. From definition of x_{n+1}, we have $f(y_k) \leq x_{k+1} \leq f(z_k)$, so we get $u = f(u) \leq f(y_k) \leq x_{k+1}$. By induction, we can conclude that $u \leq x_n$ for all $n \geq 1$. By Lemma 3.1.4(i) we have $p \leq x_n \leq q_n$ for all $n \geq 1$. It implies that $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n \to p$ faster than $q_n \to p$.

Case 2: $q_1 = x_1 < l$. Since $\{q_n\}$ converges to p, by Proposition 2.3.4, we get $f(q_1) > q_1$. By Lemma 2.3.1(ii), we get $x_n \geq q_n$ for all $n \geq 1$. By using the same proof as above, we can show that $x_n \leq l$ for all $n \geq 1$. So $q_n \leq x_n \leq p$ for all $n \geq 1$. It implies that $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n \to p$ faster than $q_n \to p$.

Caase 3: $l < q_1 = x_1 < u$

If $f(x_1) = x_1$, it follows by definitions of $\{x_n\}$ and $\{q_n\}$ that $x_n = q_n = x_1$ for all $n \geq 1$. So $p = x_1$ and $|x_n - p| = |q_n - p|$ for all $n \geq 1$.

If $f(x_1) < x_1$, by Lemma 2.3.1(i), we get $\{q_n\}$ is non-increasing. It follws that $p \leq q_n$ for all $n \geq 1$. By Lemma 3.1.3(i), we get $p \leq x_n \leq q_n$ for all $n \geq 1$. This implies that $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n \to p$ faster than $q_n \to p$.

If $f(x_1) > x_1$, by Lemma 2.3.1(ii), we get $\{q_n\}$ is non-decreasing. It follws that $q_n \leq p$ for all $n \geq 1$. By Lemma 3.1.3(ii), we get $q_n \leq x_n \leq p$ for all $n \geq 1$. This implies that $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n \to p$ faster than $q_n \to p$. $\square$

Example 3.1.8. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \frac{x^2+3}{4}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of P-iteration and W-iteration to a fixed point $p = 1$ of f are given in Table 1 with the initial point $x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$ (see figure 3.1).

Table 1

	P - iteration	W - iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n \rvert$
2	1.32009974598799	1.28190234803510	0.12108394056062300
3	1.16873763795310	1.13769360359780	0.06406380186114350
4	1.08520700524722	1.06509357013648	0.03148749184996150
5	1.04199952353343	1.03043841621111	0.01498758381019450
6	1.02047765541418	1.01422024459309	0.00705956845747302
7	1.00994902108379	1.00666780796783	0.00332278906813865
8	1.00483360389513	1.00314247074816	0.00156876659348026
9	1.00235171798544	1.00148867353262	0.000743782729087039
10	1.0011463768199	1.00070856136599	0.000354155168191594

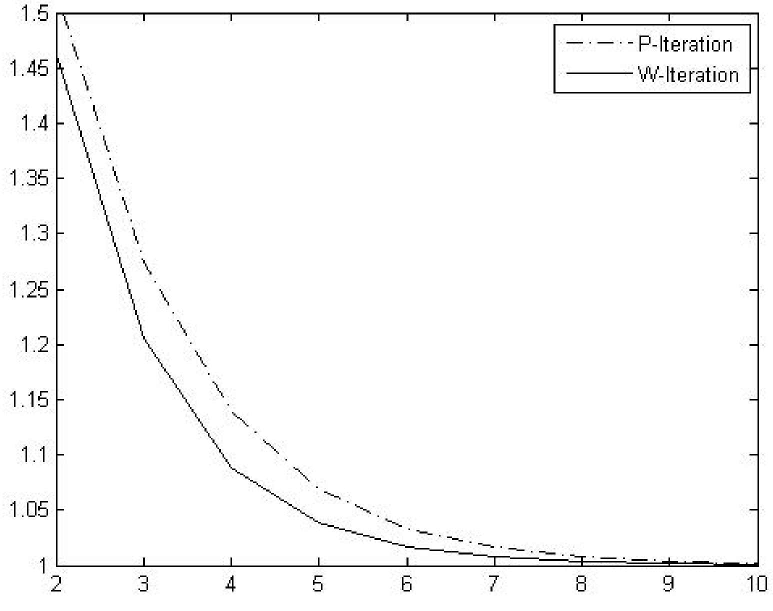

Figure 3.1: The comparision graph of the convergence between P-iteration and W-iteration to the fixed point $p = 1$ of f.

Example 3.1.9. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of P-iteration and W-iteration to a fixed point $p = 1$ of f are given in Table 2 with the initial point $x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$ (see figure 3.2).

Table 2

	P - iteration	W - iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n \rvert$
2	1.01485426853655	1.00535475326824	4.2809352763E-02
3	1.00247640304860	1.00069030245038	5.5219430855E-03
4	1.00043509997004	1.00010004020841	8.0031165922E-05
5	1.00007879342814	1.00001552057883	1.2416438973E-05
6	1.00001454522412	1.00000251722673	2.0137807479E-06
7	1.00000272044222	1.00000042119199	3.3695357304E-07
8	1.00000051365417	1.00000007212256	5.7698049005E-08
9	1.00000009768028	1.00000001257169	1.7780588112E-09
10	1.00000001867960	1.00000000222257	3.1798252919E-10

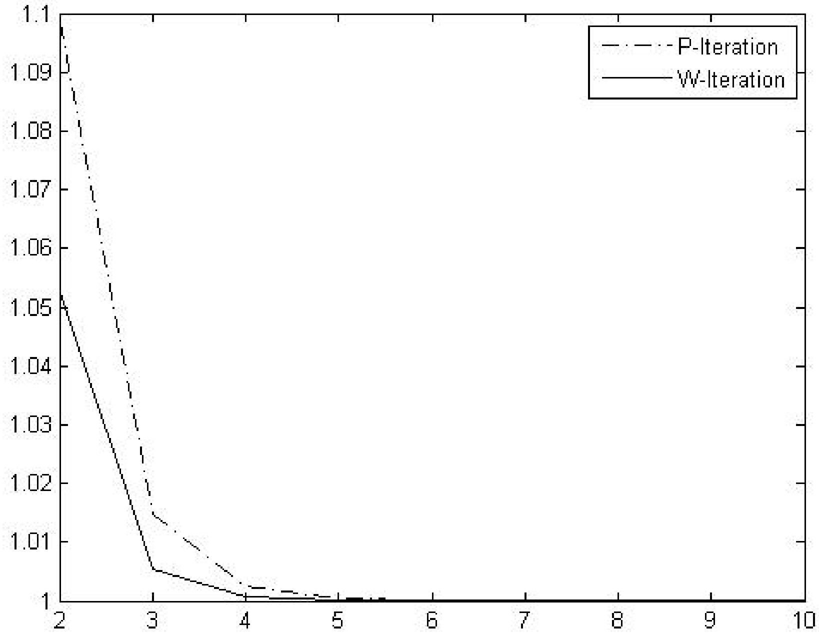

Figure 3.2: The comparision graph of the convergence between P-iteration and W-iteration to the fixed point $p = 1$ of f.

Example 3.1.10. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x + 6}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of P-iteration and W-iteration to a fixed point $p = 3$ of f are given in Table 3 with the initial point $x_1 = q_1 = 4$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$

Table 3

	P - iteration	W - iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n \rvert$
2	3.04244943718771	3.01032880295576	8.6078294214E-02
3	3.00580431124035	3.00108138900311	9.0116291616E-03
4	3.00084392502112	3.00012850130324	1.0708449581E-04
5	3.00012678136091	3.00001641360650	1.3678006661E-05
6	3.00001943520651	3.00000219690554	1.8307546350E-06
7	3.00000302048526	3.00000030384327	2.5320272900E-07
8	3.00000047408795	3.00000004305422	3.5878513937E-08
9	3.00000007496893	3.00000000621565	5.1797064415E-09
10	3.00000001192431	3.00000000091073	7.5894046602E-10

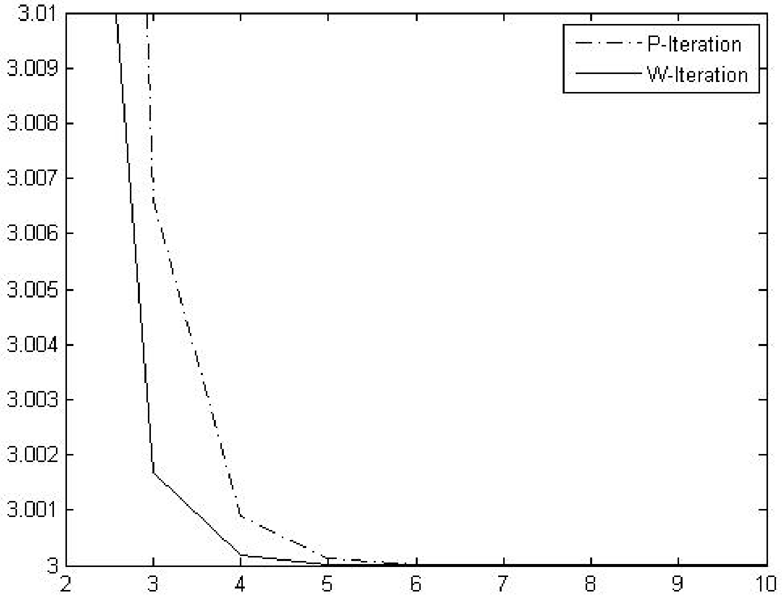

Figure 3.3: The comparision graph of the convergence between P-iteration and W-iteration to the fixed point $p = 3$ of f.

3.2 ST-iteration

In this section, we introduced a new iteration method for a continuous function on a closed interval on a real line.

Definition 3.2.1. Let E be a closed interval on real line, $f : E \to E$ be a continuous function.The ST-iteration is defined by $x_1^* \in E$, and

$$z_n^* = (1-\gamma_n)x_n^* + \gamma_n f(x_n^*)$$
$$y_n^* = (1-\beta_n)f(x_n^*) + \beta_n f^2(z_n^*)$$
$$x_{n+1}^* = (1-\alpha_n)f(z_n^*) + \alpha_n f^2(y_n^*)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in [0,1]. We will denote this iteration method by $ST(x_1^*, \alpha_n, \beta_n, \gamma_n, f)$ and it is called ST-iteration.

Next, we compare the rate of convergence between ST-iteration and P-iteration. To do this we need the following lemmas.

Lemma 3.2.1. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. Let $\{x_n^*\}$ be a sequence defined by ST-iteration. Then we have the following hold:*

i) If $p \in F(f)$ with $x_1^* > p$, then $x_n^* \geq p$, for all $n \geq 1$.

ii) If $p \in F(f)$ with $x_1^* < p$, then $x_n^* \leq p$, for all $n \geq 1$.

Proof. i) Let $p \in F(f)$ and $x_1^* > p$. Since f is non-decreasing, $f(x_1^*) \geq f(p)$. From $z_1^* = (1-\gamma_1)x_1^* + \gamma_1 f(x_1^*)$ and $x_1^* > p$, we have $z_1^* \geq (1-\gamma_1)p + \gamma_1 f(p) = p$. Since of f is non-decreasing, we have $f(z_1^*) \geq p$. From $y_1^* = (1-\beta_1)f(x_1^*) + \beta_1 f^2(z_1^*)$, $x_1^* > p$ and $z_1^* \geq p$, we get $y_1^* \geq (1-\beta_1)p + \beta_1 p = p$. It follows that $f(y_1^*) \geq p$ and $f^2(y_1^*) \geq f(p) = p$. From $x_2^* = (1-\alpha_1)f(z_1^*) + \alpha_1 f^2(y_1^*)$ and $f(z_1^*) \geq p$, $f^2(y_1^*) \geq p$, we get $x_2^* \geq p$.

Next, we assume $x_k^* \geq p$, we will show $x_{k+1}^* \geq p$. From $z_k^* = (1-\gamma_k)x_k^* + \gamma_k f(x_k^*)$ and $x_k^* \geq p$, we get $z_k^* \geq (1-\gamma_k)p + \gamma_k p = p$. Since of f is non-decreasing, we have $f(z_k^*) \geq p$. From $y_k^* = (1-\beta_k)f(x_k^*) + \beta_k f^2(z_k^*)$, $f(x_k^*) \geq p$ and $f^2(z_k^*) \geq p$, we have $y_k^* \geq (1-\beta_k)p + \beta_k p = p$. It follows that $f(y_k^*) \geq p$. From $x_{k+1}^* = (1-\alpha_k)f(z_k^*) + \alpha_k f^2(y_k^*)$ and $f(z_k^*) \geq p$, $f^2(y_k^*) \geq p$, we get $x_{k+1}^* \geq p$. By induction, we conclude that $x_n^* \geq p$ for all $n \geq 1$.

ii) By using the same argument as i) we obtain the desired result. □

Lemma 3.2.2. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$.*

For $x_1^* = q_1 \in E$, let $\{q_n\}$ be a sequence defined by P-Iteration and $\{x_n^*\}$ be a sequence defined by ST-iteration.Then we have the followwing results:

i) If $f(q_1) < q_1$, then $x_n^* \leq q_n$ for all $n \geq 1$.

ii) If $f(q_1) > q_1$, then $x_n^* \geq q_n$ for all $n \geq 1$.

Proof. i) Suppose that $f(q_1) < q_1$. Since $x_1^* = q_1$,we get $f(x_1^*) < x_1^*$ and $f(x_1^*) = f(q_1)$. We note that $z_1^* - r_1 = (1-\gamma_1)(x_1^* - q_1) + \gamma_1(f(x_1^*) - f(q_1)) = 0$, so $z_1^* = r_1$ and $f(z_1^*) = f(r_1)$. Since f is non-decreasing and $f(x_1^*) \leq z_1^* \leq x_1^*$, we get $f^2(x_1^*) \leq f(z_1^*) \leq f(x_1^*) \leq z_1^* \leq x_1^*$. Since f is non-decreasing , we have $f^3(x_1^*) \leq f^2(z_1^*) \leq f^2(x_1^*) \leq f(z_1^*) \leq f(x_1^*) \leq z_1^* \leq x_1^*$. From definition of y_1^*,we get $f^2(z_1^*) \leq y_1^* \leq f(x_1^*)$. We note that $y_1^* - t_1 = (1-\beta_1)(f(x_1^*) - r_1) + \beta_1(f^2(z_1^*) - f(r_1))$, From $f(x_1^*) = f(q_1) \leq r_1$ and $f^2(z_1^*) \leq f(x_1^*) = f(r_1)$, we have $y_1^* - t_1 \leq 0$, so $y_1^* \leq t_1$. Since f is non-decreasing, $f(y_1^*) \leq f(t_1)$ and $f^2(y_1^*) \leq f^2(t_1)$. Because f is non-decreasing and $f(r_1) \leq t_1 \leq r_1$, we get $f^2(r_1) \leq f(t_1) \leq f(r_1) \leq t_1 \leq r_1$. So $f(t_1) \leq t_1$, since f is non-decreasing, $f^2(t_1) \leq f(t_1)$, so we obtain $f^2(y_1^*) \leq f^2(t_1) \leq f(t_1)$. From $x_2^* - q_2 = (1-\alpha_1)(f(z_1^*) - f(r_1)) + \alpha_1(f^2(y_1^*) - f(t_1))$ and $f^2(y_1^*) \leq f^2(t_1) \leq f(t_1)$, we get $x_2^* - q_2 \leq 0$, so $x_2^* \leq q_2$.

Next we assume that $x_k^* \leq q_k$. Then $f(x_k^*) \leq f(q_k)$. It follows that $z_k^* - r_k = (1-\gamma_k)(x_k^* - q_k) + \gamma_k(f(x_k^*) - f(q_k)) \leq 0$, so $z_k^* \leq r_k$. Since f is non-decreasing, $f(z_k^*) \leq f(r_k)$. By Lemma 2.3.1(i), $f(q_k) \leq q_k$. From $r_k = (1-\gamma_k)q_k + \gamma_k f(q_k)$, we have $f(q_k) \leq r_k \leq q_k$. From $y_k^* - t_k = (1-\beta_k)(f(x_k^*) - r_k) + (\beta_k)(f^2(z_k^*) - f(r_k))$. Since $f^3(x_k^*) \leq f^2(z_k^*) \leq f^2(x_k^*) \leq f^2(q_k) \leq f(r_k)$, $f^2(z_k^*) \leq f(r_k)$ and $f(x_k^*) \leq f(q_k) \leq r_k$, it follows that $y_k^* - t_k \leq 0$, $y_k^* \leq t_k$. Since f is non-decreasing, $f(y_k^*) \leq f(t_k)$. Because f is non-decreasing and $f(r_k) \leq t_k \leq r_k$, we get $f^2(r_k) \leq f(t_k) \leq f(r_k) \leq t_k \leq r_k$. So $f(t_k) \leq t_k$, since f is non-decreasing, $f^2(t_k) \leq f(t_k)$, so we obtain $f^2(y_k^*) \leq f^2(t_k) \leq f(t_k)$. From $x_{k+1}^* - q_{k+1} = (1-\alpha_k)(f(z_k^*) - f(r_k)) + \alpha_k(f^2(y_k^*) - f(t_k))$, we get that $x_{k+1}^* - q_{k+1} \leq 0$, so $x_{k+1}^* \leq q_{k+1}$. By induction, we obtain $x_n^* \leq q_n$, for all $n \geq 1$.

ii) By using same argument as (i) we obtain $x_n^* \geq q_n$, for all $n \geq 1$. □

Proposition 3.2.3. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1^* < inf\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1^*) < x_1^*$, then the sequence $\{x_n^*\}$ defined by ST-iteration dose not converges to a fixed point of f.*

Proof. Suppose $f(x_1^*) < x_1^*$. By Lemma 2.3.1(i), we have that $\{x_n^*\}$ is non - increasing. Since the initial point $x_1^* < inf\{p \in E; f(p) = p\}$, it follows that $\{x_n^*\}$ dose not converges to a fixed point of f. □

Proposition 3.2.4. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1^* > sup\{p \in E; f(p) = p\}$. Let $\{\alpha_n\}$,$\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$. If $f(x_1^*) > x_1^*$, then the sequence $\{x_n^*\}$ defined by ST-iteration dose not converges to a fixed point of f.*

Proof. Suppose $f(x_1^*) > x_1^*$. By Lemma 2.3.1(ii), we have that $\{x_n^*\}$ is non-decreasing. Since the initial point $x_1^* > sup\{p \in E; f(p) = p\}$, it follows that $\{x_n^*\}$ dose not converges to a fixed point of f. □

We next compare the rate of convergence between P-iteration and ST-iteration.

Theorem 3.2.5. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is non empty set and bounded. For $x_1^* = q_1 \in E$, let $\{q_n\}$ and $\{x_n^*\}$ be the sequences defined by P-iteration and ST-iteration, respectively. If $\{q_n\}$ converges to fixed point p of f then $\{x_n^*\}$ converges to p. Moreover $\{x_n^*\}$ converges faster than $\{q_n\}$.*

Proof. Suppose that $\{q_n\}$ converges to a fixed point p of f. Let $l = infF(f)$ and $u = supF(f)$. We divide our proof into three cases.

Case 1: $q_1 = x_1^* > u$. Since $\{q_n\}$ converges to p, by Proposition 2.3.5 , we get $f(q_1) < q_1$. By Lemma 3.2.2(i), we get $x_n^* \le q_n$ for all $n \ge 1$. Since f is non-decreasing and $x_1^* = q_1$, we have $u = f(u) \le f(x_1^*) < x_1^*$. It follows that $f(x_1^*) \le z_1^* \le x_1^*$, so $u \le z_1^* \le x_1^*$. Because of f is non-decreasing, we get $u = f(u) \le f(z_1^*) \le f(x_1^*) \le z_1^* \le x_1^*$ and $u = f(u) \le f^2(z_1^*) \le f^2(x_1^*) \le f(z_1^*) \le f(x_1^*) \le z_1 \le x_1$. From $y_1^* = (1-\beta_1)f(x_1^*) + \beta_1 f^2(z_1^*)$, we get $u = f(u) \le f^2(z_1^*) \le y_1^* \le f(x_1^*)$, since f is non-decreasing, $u = f(u) \le f^3(z_1^*) \le f(y_1^*) \le f^2(x_1^*) \le f(z_1^*) \le f(x_1^*)$ and $u = f(u) \le f^4(z_1^*) \le f^2(y_1^*) \le f^3(x_1^*) \le f^2(z_1^*) \le f^2(x_1^*) \le f(z_1)$. From definition of x_2^*, we have $f^2(y_1^*) \le x_2^* \le f(z_1^*)$, so we get $u = f(u) \le f^2(y_1^*) \le x_2^*$. Hence $u \le x_2^*$.

Next, we assume $u \le x_k^*$. Then $u = f(u) \le f(x_k^*)$. Since $z_k^* = (1-\gamma_k)x_k^* + \gamma_k f(x_k^*)$, we get $u \le z_k^*$. Since f is non-decreasing, we get $u = f(u) \le f(z_k^*)$ and $u = f(u) = f^2(u) \le f^2(z_k^*)$. From $y_k^* = (1-\beta_k)f(x_k^*) + \beta_k f^2(z_k^*)$, $u \le f(x_k^*)$ and $u \le f^2(z_k^*)$, we get $y_k^* \ge u$. Since f is non-decreasing, we have $f(y_k^*) \ge f(u) = u$ and $f^2(y_k^*) \ge f^2(u) = f(u) = u$. From $x_{k+1}^* = (1-\alpha_k)(f(z_k^*)) + \alpha_k f^2(y_k^*)$, $u \le f^2(y_k^*)$ and $u \le f(z_k^*)$, so we get $u \le x_{k+1}^*$.

By induction, we can conclude that $u \le x_n^*$ for all $n \ge 1$. By Lemma 3.2.2(i) we have $p \le x_n^* \le q_n$ for all $n \ge 1$. It implies that $|x_n^* - p| \le |q_n - p|$ for all $n \ge 1$. Thus $x_n^* \to p$ faster than $q_n \to p$.

Case 2: $q_1 = x_1^* < l$. Since $\{q_n\}$ converges to p, by Proposition 2.3.4, we get $f(q_1) > q_1$. By Lemma 2.3.1(ii), we get $x_n^* \geq q_n$ for all $n \geq 1$. By using the same proof as above, we can show that $x_n^* \leq l$ for all $n \geq 1$. So $q_n \leq x_n^* \leq p$ for all $n \geq 1$. It implies that $|x_n^* - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n^* \to p$ faster than $q_n \to p$.

Caase 3: $l < q_1 = x_1^* < u$

If $f(x_1^*) = x_1^*$, It follows by definitions of $\{x_n^*\}$ and $\{q_n\}$ that $x_n^* = q_n = x_1^*$ for all $n \geq 1$. So $p = x_1^*$ and $|x_n^* - p| = |q_n^* - p|$ for all $n \geq 1$.

If $f(x_1^*) < x_1^*$, by Lemma 2.3.1(i), we get $\{q_n\}$ is non-increasing. It follws that $p \leq q_n$ for all $n \geq 1$. By Lemma 3.2.1, we get $p \leq x_n^* \leq q_n$ for all $n \geq 1$. It implies that $|x_n^* - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n^* \to p$ faster than $q_n \to p$.

If $f(x_1^*) > x_1^*$, by Lemma 2.3.1(ii), we get $\{q_n\}$ is non-decreasing. It follws that $q_n \leq p$ for all $n \geq 1$. By Lemma 3.2.1, we get $q_n \leq x_n^* \leq p$ for all $n \geq 1$. It implies that $|x_n^* - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n^* \to p$ faster than $q_n \to p$. □

Corollary 3.2.6. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $lim_{n\to\infty}\beta_n = 0$ and $lim_{n\to\infty}\gamma_n = 0$. Let $\{x_n^*\}$ be a sequence defined by ST-iteration. Then $\{x_n^*\}$ is bounded if and only if $\{x_n^*\}$ converges to a fixed point of f.*

Proof. By Theorem 2.3.2[5], we have sequnece $\{q_n\}$ is defined by P-itareation converges and from Theorem 3.2.5 we can conclude $\{x_n^*\}$ converges to a fixed point of f. □

Example 3.2.7. Let $f : [0,\infty) \to [0,\infty)$ be defined by $f(x) = \frac{x^2+3}{4}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration and P-iteration to the fixed point $p = 1$ of f are given in Table 4 with the initial point $x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$.

Table 4

	P-iteration	ST- iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n \rvert$
2	1.32009974598799	1.18833667821118	0.0853006630156865
3	1.16873763795310	1.08307369201437	0.0398115364309595
4	1.08520700524722	1.03680439940034	0.0180635587463636
5	1.04199952353343	1.01647478994560	0.0081695402968629
6	1.02047765541418	1.00746261307804	0.0037173838905315
7	1.00994902108379	1.00341811247173	0.0017061353626461
8	1.00483360389513	1.00158074637176	0.0007897484961059
9	1.00235171798544	1.00073698124456	0.0003683548369419
10	1.0011463768199	1.00034593127729	0.0001729357215312

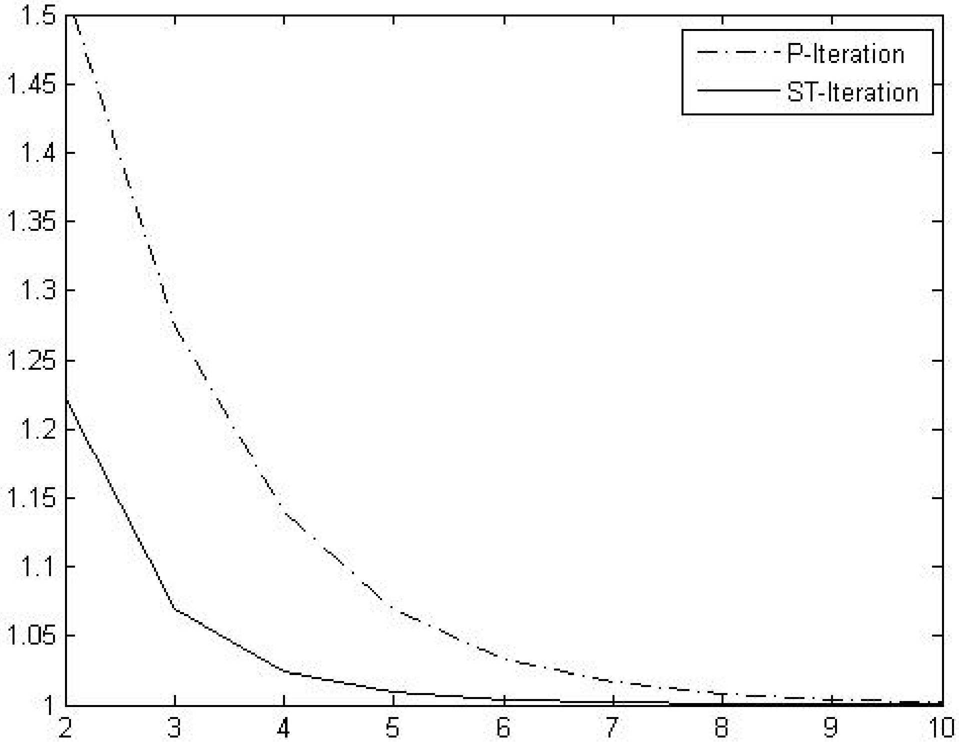

Figure 3.4: The comparision graph of the convergence between P-iteration and ST-iteration to a fixed point $p = 1$ of f.

Example 3.2.8. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration and P-iteration to a fixed point $p = 1$ of f are given in Table 5 with the initial point $x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$.

Table 5

	P-iteration	ST-iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n \rvert$
2	1.01485426853655	1.00062168435326	0.0004973088334681
3	1.00247640304860	1.00007328924436	5.863085835411E-05
4	1.00043509997004	1.00001002594443	8.020745494663E-06
5	1.00007879342814	1.00000149055860	1.192446659414E-06
6	1.00001454522412	1.00000023371755	1.869740318127E-07
7	1.00000272044222	1.00000003802802	3.042241258910E-08
8	1.00000051365417	1.00000000635826	5.086608467763E-09

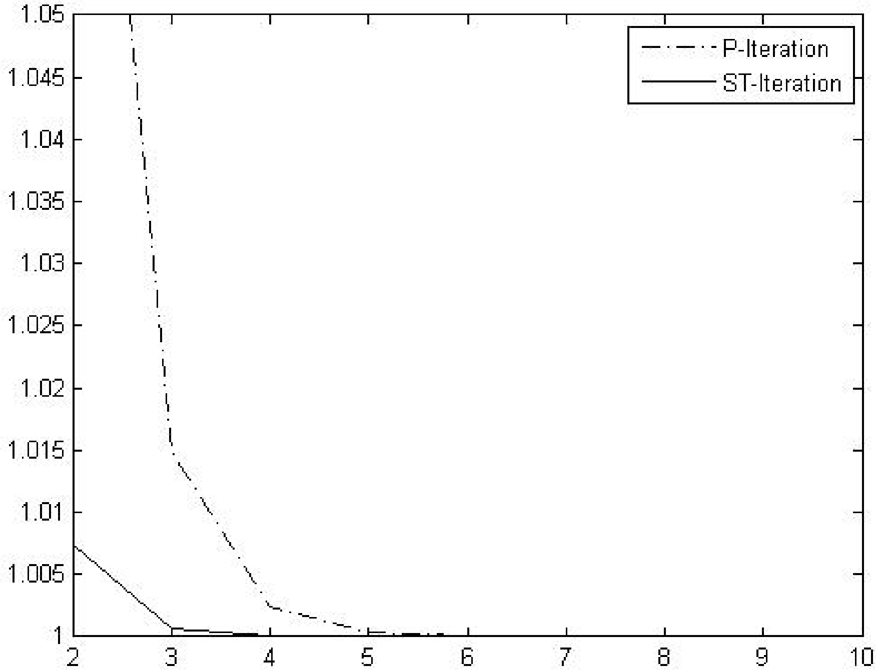

Figure 3.5: The comparision graph of the convergence between P-iteration and ST-iteration to a fixed point $p = 1$ of f.

Example 3.2.9. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x + 6}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration and P-iteration to the fixed point $p = 3$ of f are given in Table 6 with the initial point $x_1 = q_1 = 4$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$.

Table 6

	P-iteration	ST-iteration	
n	q_n	x_n	$\lvert f(x_n) - x_n\rvert$
2	3.04244943718771	3.00101668323559	0.0008472408147834
3	3.00580431124035	3.00009861774142	8.218149621219E-05
4	3.00084392502112	3.00001114510790	9.287590489126E-06
5	3.00012678136091	3.00000137162471	1.143020597282E-06
6	3.00001943520651	3.00000017825785	1.485482123975E-07
7	3.00000302048526	3.00000002406040	2.005033650220E-08
8	3.00000047408795	3.00000000333925	2.782708286019E-09
9	3.00000007496893	3.00000000047344	3.945319626552E-10

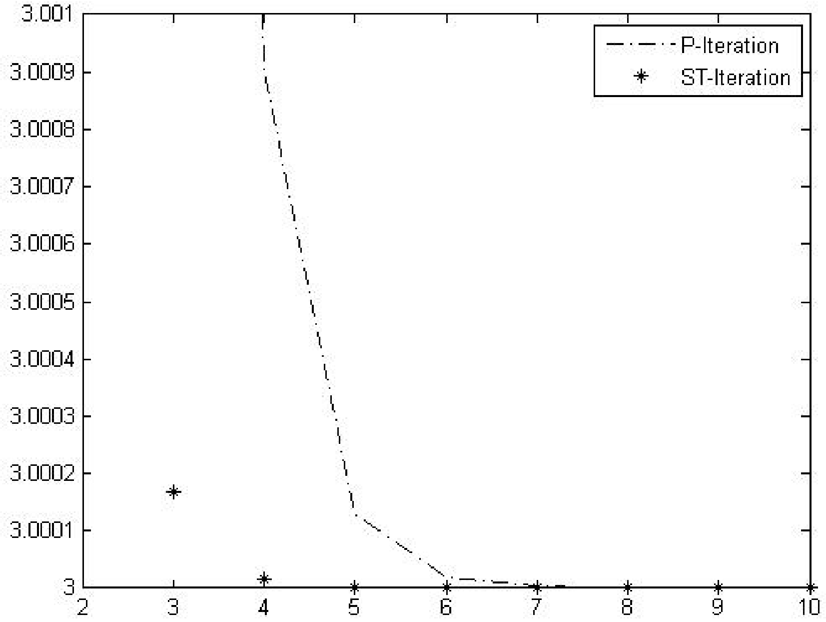

Figure 3.6: The comparision graph of the convergence between P-iteration and ST-iteration to a fixed point $p = 3$ of f

Next we will prove and compare the rate of convergence of ST-iteration and W-iteration, to do this we need the followwing lemma.

Lemma 3.2.10. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$. For $x_1^* = x_1 \in E$, let $\{x_n\}$ be a sequence defined by W-Iteration and $\{x_n^*\}$ be a sequence defined by ST-iteration.Then we have the following results:*

i) If $f(x_1) < x_1$, then $x_n^* \leq x_n$ for all $n \geq 1$.

ii) If $f(x_1) > x_1$, then $x_n^* \geq x_n$ for all $n \geq 1$.

Proof. (I) Suppose that $f(x_1) < x_1$. Since $x_1^* = x_1$,we get $f(x_1^*) < x_1^*$ and $f(x_1^*) = f(x_1)$. We note that $z_1^* - z_1 = (1-\gamma_1)(x_1^* - x_1) + \gamma_1(f(x_1^*) - f(x_1)) = 0$, so $z_1^* = z_1$ and $f(z_1^*) = f(z_1)$. Since $f(x_1) \leq z_1 \leq x_1$, f is non-decreasing, we get $f(z_1) \leq z_1$ and $f^2(z_1) \leq f(z_1)$. It follows that $y_1^* - y_1 = (1-\beta_1)(f(x_1^*) - f(x_1)) + \beta_1(f^2(z_1^*) - f(z_1)) \leq 0$, we obtain $y_1^* \leq y_1$, since f is non-decreasing, we get $f(y_1^*) \leq f(y_1)$ and $f^2(y_1^*) \leq f^2(y_1)$. Since $f(z_1) \leq y_1 \leq f(x_1)$ and f is non-decreasing, we have $f^2(z_1) \leq f(y_1) \leq f^2(x_1) \leq f(z_1) \leq y_1$, so $f(y_1) \leq y_1$ and $f^2(y_1^*) \leq f^2(y_1) \leq f(y_1)$. From $x_2^* - x_2 = (1-\alpha_1)(f(z_1^*) - f(z_1)) + \alpha_1(f^2(y_1^*) - f(y_1))$, we get $x_2^* - x_2 \leq 0$, $x_2^* \leq x_2$.

Next we assume that $x_k^* \leq x_k$. Then $f(x_k^*) \leq f(x_k)$. It follows that $z_k^* - z_k = (1-\gamma_k)(x_k^* - x_k) + \gamma_k(f(x_k^*) - f(x_k)) \leq 0$, so $z_k^* \leq z_k$ and $f(z_k^*) \leq f(z_k)$. From definition of z_k we have $f(z_k) \leq z_k$. Since f is non-decreasing, we get $f^2(z_k^*) \leq f^2(z_k) \leq f(z_k)$. It follows that $y_k^* - y_k = (1-\beta_k)(f(x_k^*) - f(x_k)) + \beta_k(f^2(z_k^*) - f(z_k)) \leq 0$, we obtain $y_k^* \leq y_k$ since f is non-decreasing, we get $f(y_k^*) \leq f(y_k)$ and $f^2(y_k^*) \leq f^2(y_k)$. Since $f(z_k) \leq y_k \leq f(x_k)$ and f is non-decreasing, we have $f^2(z_k) \leq f(y_k) \leq f^2(x_k) \leq f(z_k) \leq y_k$, so $f(y_k) \leq y_k$ and $f^2(y_k^*) \leq f^2(y_k) \leq f(y_k)$. From $x_{k+1}^* - x_{k+1} = (1-\alpha_k)(f(z_k^*) - f(z_k)) + \alpha_k(f^2(y_k^*) - f(y_k))$, we get $x_{k+1}^* - x_{k+1} \leq 0$, $x_{k+1}^* \leq x_{k+1}$. By induction, we obtain $x_n^* \leq x_n$, for all $n \geq 1$.

ii) By using same argument as (i) we obtain $x_n^* \geq x_n$, for all $n \geq 1$. □

Theorem 3.2.11. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous non-decreasing function such that $F(f)$ is non empty set and bounded. For $x_1^* = x_1 \in E$, let $\{x_n\}$ and $\{x_n^*\}$ be the sequences defined by W-iteration and ST-iteration, respectively. If $\{x_n\}$ converges to fixed point p of f then $\{x_n^*\}$ converges to p. Moreover $\{x_n^*\}$ converges faster than $\{x_n\}$.*

Proof. Suppose that $\{x_n\}$ converges to a fixed point p of f. Let $l = inf$ Ff and $u = sup$ F(f). We divide our proof into three cases.

Case 1: $x_1 = x_1^* > u$. Since $\{x_n\}$ converges to p, by Proposition 3.2.4, we get $f(x_1) < x_1$. By Lemma 3.2.10(i), we get $x_n^* \leq x_n$ for all $n \geq 1$. Since f is non-decreasing, we have $u = f(u) \leq f(x_1^*) < x_1^*$. It follows that $f(x_1^*) \leq z_1^* \leq x_1^*$, so $u \leq z_1^* \leq x_1^*$. Because f is non-decreasing, we get $u = f(u) \leq f(z_1^*) \leq f(x_1^*) \leq z_1^* \leq x_1^*$ and $u = f(u) = f^2(u) \leq f^2(z_1^*) \leq f^2(x_1^*) \leq f(z_1^*) \leq f(x_1^*)$. From $y_1^* = (1-\beta_1)f(x_1^*) + \beta_1 f^2(z_1^*)$, we get $u = f(u) \leq f^2(z_1^*) \leq y_1^* \leq f(x_1^*)$, since f is non-decreasing, we have $u = f(u) \leq f^3(z_1^*) \leq f(y_1^*) \leq$

$f^2(x_1^*) \le f(z_1^*) \le f(x_1^*)$ and $u = f(u) \le f^2(y_1^*) \le f^3(x_1^*) \le f^2(z_1^*) \le f^2(x_1^*) \le f(z_1^*)$. From $x_2^* = (1-\alpha_1)f(z_1^*)+\alpha_1 f^2(y_1^*)$ and we have $u \le f(z_1^*)$, $u \le f^2(y_1^*)$, so we get $u \le x_2^*$.

Next, we assume $u \le x_k^*$. Then $u = f(u) \le f(x_k^*)$. By definition of z_k^* we get $f(x_k^*) \le z_k^* \le x_k^*$, so $u \le z_k^* \le x_k^*$. Since f is non-decreasing, we get $u = f(u) \le f(z_k^*) \le f(x_k^*) \le z_k^* \le x_k^*$. From $y_k^* = (1-\beta_k)f(x_k^*) + \beta_k f^2(z_k^*)$, we get $u = f(u) \le f^2(z_k^*) \le y_k^* \le f(x_k^*)$ and $u = f(u) \le f^3(z_k^*) \le f(y_k^*) \le f^2(x_k^*) \le f(z_k^*) \le f(x_k^*)$. From definition of x_{n+1}^*, we have $f^2(y_k^*) \le x_{k+1}^* \le f(z_k^*)$, so we get $u = f(u) \le f^2(y_k^*) \le x_{k+1}^*$.Hence $u \le x_{k+1}^*$.

By induction, we can conclude that $u \le x_n^*$ for all $n \ge 1$. By Lemma 3.2.10(i) we have $p \le x_n^* \le x_n$ for all $n \ge 1$. It implies that $|x_n^* - p| \le |x_n - p|$ for all $n \ge 1$. Thus $x_n^* \to p$ faster than $x_n \to p$.

Case 2: $x_1 = x_1^* < l$. Since $\{x_n\}$ converges to p, by Proposition 3.2.3, we get $f(x_1) > x_1$. By Lemma 3.2.10(ii), we get $x_n^* \ge x_n$ for all $n \ge 1$. By using the same proof as above, we can show that $x_n^* \le l$ for all $n \ge 1$. So $x_n \le x_n^* \le p$ for all $n \ge 1$. It implies that $|x_n^* - p| \le |x_n - p|$ for all $n \ge 1$. Thus $x_n^* \to p$ faster than $x_n \to p$.

Caase 3: $l < x_1 = x_1^* < u$

If $f(x_1^*) = x_1^*$, It follows by definitions of $\{x_n^*\}$ and $\{x_n\}$ that $x_n^* = x_n = x_1^*$ for all $n \ge 1$. So $p = x_1^*$ and $|x_n^* - p| = |x_n - p|$ for all $n \ge 1$.

If $f(x_1^*) < x_1^*$, by Lemma 4.1.1(i), we get $\{x_n\}$ is non-increasing. It follws that $p \le x_n$ for all $n \ge 1$. By Lemma 3.2.10(i), we get $p \le x_n^* \le x_n$ for all $n \ge 1$. It implies that $|x_n^* - p| \le |x_n - p|$ for all $n \ge 1$. Thus $x_n^* \to p$ faster than $x_n \to p$.

If $f(x_1^*) > x_1^*$, by Lemma 4.1.1(ii), we get $\{x_n\}$ is non-decreasing. It follws that $x_n \le p$ for all $n \ge 1$. By Lemma 3.2.10(ii), we get $x_n \le x_n^* \le p$ for all $n \ge 1$. It implies that $|x_n^* - p| \le |x_n - p|$ for all $n \ge 1$. Thus $x_n^* \to p$ faster than $x_n \to p$. □

Corollary 3.2.12. *Let E be a closed interval on real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $lim_{n\to\infty}\alpha_n = 0$, $lim_{n\to\infty}\beta_n = 0$, $lim_{n\to\infty}\gamma_n = 0$. For $x_1 \in E$, let $\{x_n^*\}$ be a sequence defined by SP-iteration. Then $\{x_n^*\}$ is bounded if and only if $\{x_n^*\}$ converges to a fixed point of f.*

Proof. By Theorem 3.1.2, we have sequnece $\{x_n\}$ is defined by W-itareation converges and from Theorem 3.2.11 we can conclude $\{x_n^*\}$ converges to a fixed point of f. □

Example 3.2.13. Let $f : [0,\infty) \to [0,\infty)$ be defined by $f(x) = \frac{x^2+3}{4}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration, W-iteration and P-iteration to the fixed point $p = 1$ are given in Fig. 3.7 with the initial point $x_1^* = x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$. The graph of convergence of each iteration is showed in figure 3.7.

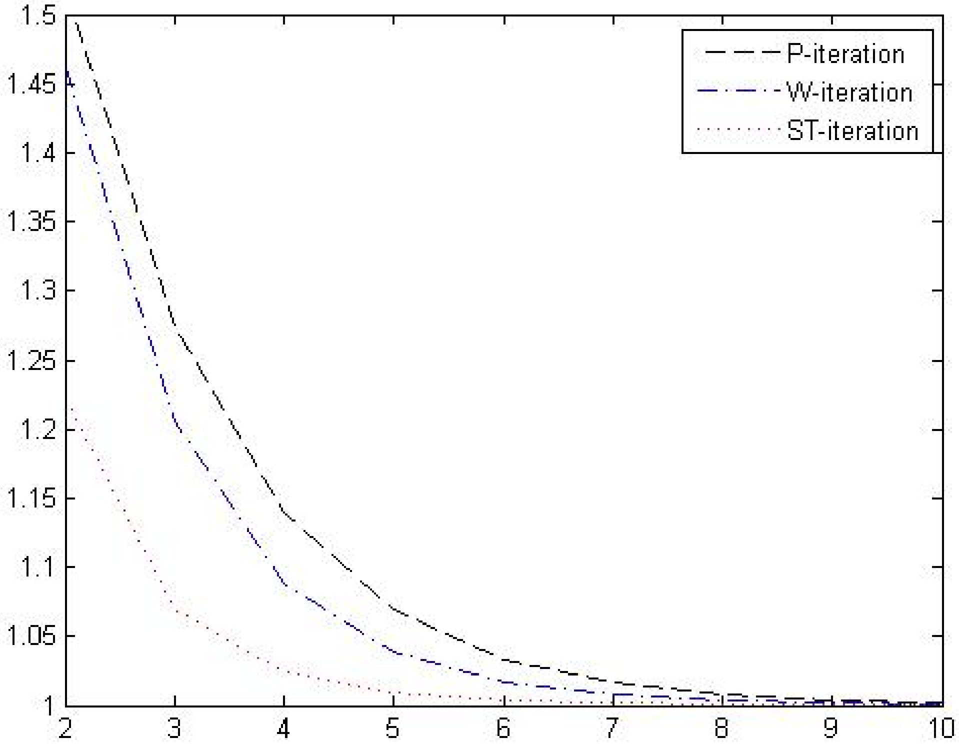

Figure 3.7: The comparision graph of the convergence between P-iteration, W-iteration and ST-iteration to the fixed point $p = 1$ of f.

Example 3.2.14. Let $f : [0, \infty) \rightarrow [0, \infty)$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration and P-iteration to a fixed point $p = 1$ are given in Fig 3.8 with the initial point $x_1^* = x_1 = q_1 = 2$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$.The graph of convergence of each iteration is showed in figure 3.8.

Example 3.2.15. Let $f : [0, \infty) \rightarrow [0, \infty)$ be defined by $f(x) = \sqrt{x + 6}$. Then f is continuous and non-decreasing function. The comparisions of the convergence of ST-iteration, W-iteration and P-iteration to a fixed point $p = 3$ of f are given in Figure 3.9 with the initial point $x_1^* = x_1 = q_1 = 4$ and $\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n}$. The graph of convergence of each iteration is showed in figure 3.9.

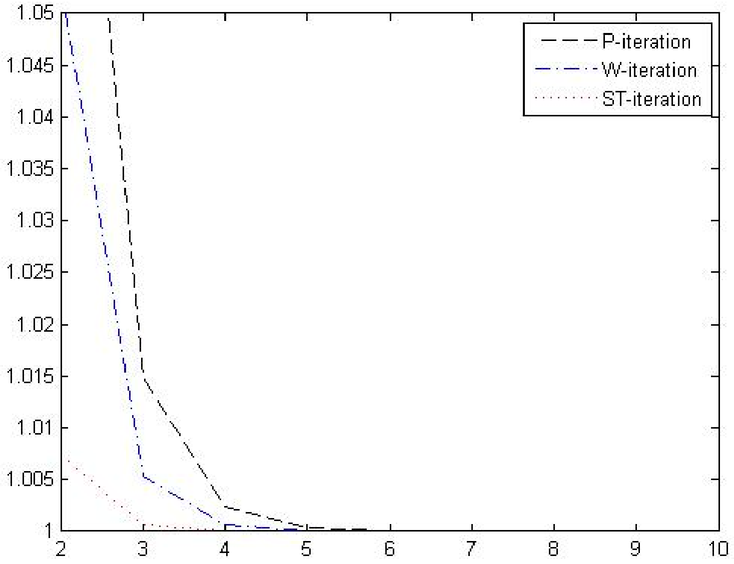

Figure 3.8: The comparision graph of the convergence between P-iteration, W-iteration and ST-iteration to the fixed point $p = 1$ of f.

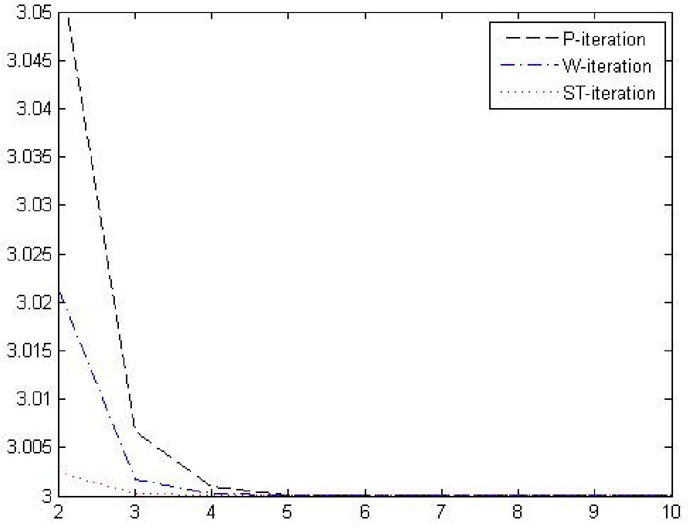

Figure 3.9: The comparision graph of the convergence between P-iteration, W-iteration and ST-iteration to a fixed point $p = 3$ of f.

3.3 Solving of One Variable Nonlinear Equations

In this section, we introduced the method to find the solution of the one variable nonlinear equations by using the W-iteration and the ST-iteration.

Example 3.3.1. Find the answer of $\frac{\sqrt{x^5+1}}{5} - x = 0$.

Solution from the equation we have $\frac{\sqrt{x^5+1}}{5} = x$. Let $f(x) = \frac{\sqrt{x^5+1}}{5}$, the answer of equation is the fixed point of f, we can find the fixed point of f by the W-iteration and the ST-iteration following:

i) W-iteration.

Use the initial point $x = 2$, $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{2n}$ and $\gamma_n = \frac{1}{3n}$, we get the table of W-iteration.

Table 7

	W-iteration	
n	x_n	$\lvert f(x_n) - x_n \rvert$
2	0.273685740860383	0.07353224586070
3	0.200077097489619	0.000045038333530
4	0.200032044462004	0.000000021381913
5	0.200032023074743	0.000000000011770
6	0.200032023062971	0.000000000000007
7	0.200032023062964	0.000000000000000
8	0.200032023062964	0.000000000000000
9	0.200032023062964	0.000000000000000

From the table, we have a fixed point of $f = 0.200032023062964$. So an answer of equation is 0.200032023062964.

ii) ST-iteration.

Use the initial point $x = 2$, $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{2n}$ and $\gamma_n = \frac{1}{3n}$, we get the table of ST-iteration.

Table 8

	ST-iteration	
n	x_n	$\lvert f(x_n) - x_n \rvert$
2	0.200045949631832	0.000013915420708
3	0.200032027708669	0.000000004641987
4	0.200032023065168	0.000000000002202
5	0.200032023062965	0.000000000000001
6	0.200032023062964	0.000000000000000
7	0.200032023062964	0.000000000000000
8	0.200032023062964	0.000000000000000
9	0.200032023062964	0.000000000000000

From the table, we have a fixed point of $f = 0.200032023062964$. So an answer of equation is 0.200032023062964.

Example 3.3.2. Find the answer of $\sqrt{0.9lnx+1}-x=0$.

Solution from the equation we have $\sqrt{0.9lnx+1}=x$. Let $f(x)=\sqrt{0.9lnx+1}$, the answer of equation is the fixed point of f, we can find the fixed point of f by the W-iteration and the ST-iteration following:

i) W-iteration.

Use the initial point $x=2$, $\alpha_n=\frac{1}{n}, \beta_n=\frac{1}{2n}$ and $\gamma_n=\frac{1}{3n}$, we get the table of W-iteration.

Table 9

	W-iteration	
n	x_n	$\lvert f(x_n)-x_n\rvert$
2	1.096188928574115	0.07353224586070
5	1.003416412592467	0.000045038333530
10	1.000035949054167	0.000000021381913
15	1.000000484998666	0.000000000011770
20	1.000000007191191	0.000000000000007
30	1.000000000001805	0.000000000000993
40	1.000000000000000	0.000000000000000

From the table, we have a fixed point of $f=1$. So an answer of equation is 1.

ii) ST-iteration.

Use the initial point $x=2$, $\alpha_n=\frac{1}{n}, \beta_n=\frac{1}{2n}$ and $\gamma_n=\frac{1}{3n}$, we get the table of ST-iteration.

Table 10

	ST-iteration	
n	x_n	$\lvert f(x_n)-x_n\rvert$
2	1.030971257674987	0.024896446899187
5	1.000281748340815	0.000617787379974
10	1.000006567918155	0.000005237923912
15	1.000000079000155	0.000000063002625
20	1.000000001083232	0.000000000863878
30	1.000000000000244	0.000000000000195
40	1.000000000000000	0.000000000000000

From the table, we have a fixed point of $f=1$. So an answer of equation is 1.