

## CHAPTER 4

### Conclusion

In this thesis, we introduce The new iteration to solving non-linear equation. We have main results as the following:

#### 4.1 W-iteration

**Definition 4.1.1.** Let  $E$  be a closed interval on real line,  $f : E \rightarrow E$  be a function. The W-iteration is defined by  $x_1 \in E$ , and

$$\begin{aligned}z_n &= (1 - \gamma_n)x_n + \gamma_n f(x_n) \\y_n &= (1 - \beta_n)f(x_n) + \beta_n f(z_n) \\x_{n+1} &= (1 - \alpha_n)f(z_n) + \alpha_n f(y_n)\end{aligned}$$

for all  $n \geq 1$  and  $\{\beta_n\}_{n=1}^{\infty}$ ,  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\gamma_n\}_{n=1}^{\infty}$  are sequences in  $[0,1]$ . We will denote this iteration method by  $W(x_1, \alpha_n, \beta_n, \gamma_n, f)$  and it is called W-iteration.

Next, we will prove the convergence theorems. To prove this, we need the following lemmas.

**Lemma 4.1.1.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{\gamma_n\}_{n=1}^{\infty}$  be sequences in  $[0, 1]$ . For  $x_1 \in E$ , let  $\{x_n\}$  be a sequence defined by W-iteration. Then the following hold:

- i) If  $f(x_1) < x_1$  then  $f(x_n) \leq x_n$ , for all  $n \geq 1$  and  $\{x_n\}$  is non-increasing.
- ii) If  $f(x_1) > x_1$  then  $f(x_n) \geq x_n$ , for all  $n \geq 1$  and  $\{x_n\}$  is non-decreasing.

**Theorem 4.1.2.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{\gamma_n\}_{n=1}^{\infty}$  be sequences in  $[0, 1]$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . For  $x_1 \in E$ , let  $\{x_n\}$  be a sequence defined by W-iteration. Then  $\{x_n\}$  is bounded if and only if  $\{x_n\}$  converges to a fixed point of  $f$ .

Next, we compare the rate of convergence between W-iteration and P-iteration. To do this, we need the following definition and lemmas.

**Lemma 4.1.3.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{\gamma_n\}_{n=1}^{\infty}$  be sequences in  $[0, 1]$ . Let  $\{x_n\}$  be a sequence defined by W-iteration. Then we have the following :

i) If  $p \in F(f)$  with  $x_1 > p$ , then  $x_n \geq p$  for all  $n \geq 1$ .

ii) If  $p \in F(f)$  with  $x_1 < p$ , then  $x_n \leq p$  for all  $n \geq 1$ .

**Lemma 4.1.4.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{\gamma_n\}_{n=1}^{\infty}$  be sequences in  $[0, 1]$ . For  $x_1 = q_1 \in E$ , let  $\{q_n\}$  be a sequence defined by P - Iteration and  $\{x_n\}$  be a sequence defined by W-iteration. Then we have the following results:

i) If  $f(q_1) < q_1$ , then  $x_n \leq q_n$  for all  $n \geq 1$ .

ii) If  $f(q_1) > q_1$ , then  $x_n \geq q_n$  for all  $n \geq 1$ .

**Proposition 4.1.5.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is nonempty and bounded with  $x_1 < \inf\{p \in E; f(p) = p\}$ . Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1) < x_1$ , then the sequence  $\{x_n\}$  defined by W-iteration dose not converges to a fixed point of  $f$ .

**Proposition 4.1.6.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is nonempty and bounded with  $x_1 > \sup\{p \in E; f(p) = p\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1) > x_1$ , then the sequence  $\{x_n\}$  defined by W-iteration dose not converges to a fixed point of  $f$ .

We next compare the rate of convergence between P-iteration and W-iteration.

**Theorem 4.1.7.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is nonempty and bounded. For  $x_1 = q_1 \in E$ , let  $\{q_n\}$  and  $\{x_n\}$  be the sequences defined by P-iteration and W-iteration, respectively. If  $\{q_n\}$  converges to a fixed point  $p$  of  $f$ , then  $\{x_n\}$  converges to  $p$ . Moreover  $\{x_n\}$  converges faster than  $\{q_n\}$ .

## 4.2 ST-iteration

In this section, we introduced a new iteration method for a continuous function on a closed interval on a real line.

**Definition 4.2.1.** Let  $E$  be a closed interval on real line,  $f : E \rightarrow E$  be a continuous function. The ST-iteration is defined by  $x_1^* \in E$ , and

$$z_n^* = (1 - \gamma_n)x_n^* + \gamma_n f(x_n^*)$$

$$y_n^* = (1 - \beta_n)f(x_n^*) + \beta_n f^2(z_n^*)$$

$$x_{n+1}^* = (1 - \alpha_n)f(z_n^*) + \alpha_n f^2(y_n^*), \quad \text{for all } n \geq 1, \text{ where } \{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$$

are sequences in  $[0,1]$ . We will denote this iteration method by  $ST(x_1^*, \alpha_n, \beta_n, \gamma_n, f)$  and it is called ST-iteration.

Next, we compare the rate of convergence between ST-iteration and P-iteration. To do this we need the following lemmas.

**Lemma 4.2.1.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$  and  $\{\gamma_n\}_{n=1}^\infty$  be sequences in  $[0, 1]$ . Let  $\{x_n^*\}$  be a sequence defined by ST-iteration. Then we have the following hold:

i) If  $p \in F(f)$  with  $x_1^* > p$ , then  $x_n^* \geq p$ , for all  $n \geq 1$ .

ii) If  $p \in F(f)$  with  $x_1^* < p$ , then  $x_n^* \leq p$ , for all  $n \geq 1$ .

**Lemma 4.2.2.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$  and  $\{\gamma_n\}_{n=1}^\infty$  be sequences in  $[0, 1]$ . For  $x_1^* = q_1 \in E$ , let  $\{q_n\}$  be a sequence defined by P-Iteration and  $\{x_n^*\}$  be a sequence defined by ST-iteration. Then we have the following results:

i) If  $f(q_1) < q_1$ , then  $x_n^* \leq q_n$  for all  $n \geq 1$ .

ii) If  $f(q_1) > q_1$ , then  $x_n^* \geq q_n$  for all  $n \geq 1$ .

**Proposition 4.2.3.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is nonempty and bounded with  $x_1^* < \inf\{p \in E; f(p) = p\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1^*) < x_1^*$ , then the sequence  $\{x_n^*\}$  defined by ST-iteration dose not converges to a fixed point of  $f$ .

**Proposition 4.2.4.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is nonempty and bounded with  $x_1^* > \sup\{p \in E; f(p) = p\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1^*) > x_1^*$ , then the sequence  $\{x_n^*\}$  defined by ST-iteration dose not converges to a fixed point of  $f$ .

We next compare the rate of convergence between P-iteration and ST-iteration.

**Theorem 4.2.5.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is non empty set and bounded. For  $x_1^* = q_1 \in E$ , let  $\{q_n\}$  and  $\{x_n^*\}$  be the sequences defined by  $P$ -iteration and  $ST$ -iteration, respectively. If  $\{q_n\}$  converges to fixed point  $p$  of  $f$  then  $\{x_n^*\}$  converges to  $p$ . Moreover  $\{x_n^*\}$  converges faster than  $\{q_n\}$ .

**Corollary 4.2.6.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^\infty$ ,  $\{\beta_n\}_{n=1}^\infty$  and  $\{\gamma_n\}_{n=1}^\infty$  be sequences in  $[0, 1]$  and  $\lim_{n \rightarrow \infty} \beta_n = 0$  and  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . Let  $\{x_n^*\}$  be a sequence defined by  $ST$ -iteration. Then  $\{x_n^*\}$  is bounded if and only if  $\{x_n^*\}$  converges to a fixed point of  $f$ .

Next we will prove and compare the rate of convergence of  $ST$ -iteration and  $W$ -iteration, to do this we need the following lemma.

**Lemma 4.2.7.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^\infty$ ,  $\{\beta_n\}_{n=1}^\infty$  and  $\{\gamma_n\}_{n=1}^\infty$  be sequences in  $[0, 1]$ . For  $x_1^* = x_1 \in E$ , let  $\{x_n\}$  be a sequence defined by  $W$ -Iteration and  $\{x_n^*\}$  be a sequence defined by  $ST$ -iteration. Then we have the following results:

- i) If  $f(x_1) < x_1$ , then  $x_n^* \leq x_n$  for all  $n \geq 1$ .
- ii) If  $f(x_1) > x_1$ , then  $x_n^* \geq x_n$  for all  $n \geq 1$ .

**Theorem 4.2.8.** Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous non-decreasing function such that  $F(f)$  is non empty set and bounded. For  $x_1^* = x_1 \in E$ , let  $\{x_n\}$  and  $\{x_n^*\}$  be the sequences defined by  $W$ -iteration and  $ST$ -iteration, respectively. If  $\{x_n\}$  converges to fixed point  $p$  of  $f$  then  $\{x_n^*\}$  converges to  $p$ . Moreover  $\{x_n^*\}$  converges faster than  $\{x_n\}$ .

**Corollary 4.2.9.** Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  be a continuous and non-decreasing function. Let  $\{\alpha_n\}_{n=1}^\infty$ ,  $\{\beta_n\}_{n=1}^\infty$  and  $\{\gamma_n\}_{n=1}^\infty$  be sequences in  $[0, 1]$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . For  $x_1 \in E$ , let  $\{x_n^*\}$  be a sequence defined by  $SP$ -iteration. Then  $\{x_n^*\}$  is bounded if and only if  $\{x_n^*\}$  converges to a fixed point of  $f$ .