CHAPTER 1
Introduction
CHAPTER 2
Machining Dynamics

CHAPTER 3

Robust Control Approach

This chapter presents the relevant theory and fundamental aspects for the controller design problem. For model-based controller design there is always error in the system mathematical model relative to the actual system. A robust control approach is one that guarantees that a synthesized controller will work well on the actual system despite the fact that there are errors in the model. Firstly, a review of relevant robust control theory is given. Then a novel controller formulation is introduced based on a delay-dependent Lyapunov-Krasovskii functional (LKF). This leads to a robust stability condition in linear matrix inequality (LMI) form that can be solved to obtain optimized controller solutions.

3.1 Robust Control Approach

3.1.1 Basic concepts [84]

The section presents the basic theory for robust control of linear time invariant (LTI) finite-dimensional systems, described in the state-space as

$$\begin{aligned} \dot{x} &= Ax + Bu \quad , \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

where x is the system state vector, u is the input signals, y is the output measurement signals and the matrices A, B, C, D of suitable size. The signals of x, y and u are the functions of time $t \in [0, \infty)$.

The system responds to the input $u(.)$ with the output $y(.)$ which can be computed according to the relation

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \quad \text{for } t \geq 0 .$$

To determine if a system can be controlled using feedback some basic properties are needed:

- The system (3.1) or (A, B) is said to be *stabilizable* if there exists a feedback matrix F such that $A+BF$ has all its eigenvalues in the open left-half plane $\mathbb{C}^{-}$. The *Hautus test for stabilizability* is (A, B) is stabilizable if and only if the matrix

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix}$$

has full row rank for all $\lambda \in \mathbb{C}^{0} \cup \mathbb{C}^{+}$.

- The system (3.1) or (A, C) is said to be *detectable* if there exists an L such that $A+LC$ has all its eigenvalues in the open left-half plane $\mathbb{C}^{-}$. According to the Hautus test for detectability (A, C) is detectable if and only if the matrix

$$\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}$$

has full row rank for all $\lambda \in \mathbb{C}^{0} \cup \mathbb{C}^{+}$.

The transfer function matrix of the system (3.1) is

$$G(s) = C(sI - A)^{-1} B + D$$

and is a matrix whose elements consist of real-rational and proper functions.

Suppose the input signal $u(.)$ has the Laplace transform

$$\hat{u}(s) = \int_0^\infty s^{-st} u(t)dt \, .$$

Then the output $y(.)$ of (3.1) does also have a Laplace transform that can be calculated as

$$\hat{y}(s) = C(sI - A)^{-1} x_0 + \left[C(sI - A)^{-1} B + D \right] \hat{u}(s).$$

For $x_0 = 0$ (such that the system starts at time 0 at rest), the relation between the Laplace transform of the input and the output signals is given by the transfer matrix as follows:

$$\hat{y}(s) = G(s)\hat{u}(s).$$

The fundamental relation between the state-space and frequency domain representation is investigated in the so-called realization theory. Going from the state-space to the frequency domain just requires to calculate the transfer matrix $G(s)$.

Conversely, suppose $H(s)$ is an arbitrary matrix whose elements are real-rational proper functions. Then there always exist matrices A_H, B_H, C_H, D_H, such that

$$H(s) = C_H \left(sI - A\right)^{-1} B_H + D_H$$

holds true. This representation of the transfer matrix is called a *realization*. Realizations are not unique. Even more importantly, the size of the matrix A_H can vary for various realizations. However, there are realizations where A_H is of minimal size, the so-called minimal realization. There is a simple answer to the question of whether a realization is minimal: This happens if and only if $\left(A_H, B_H\right)$ is controllable and $\left(A_H, C_H\right)$ is observable.

Pictorially, the system representations in the time-domain and frequency domain and the interpretation as a mapping of signals can be shown as follows:

The stability of LTI systems will be described as follows: recall the any matrix $H(s)$ whose elements are rational functions is *stable* if

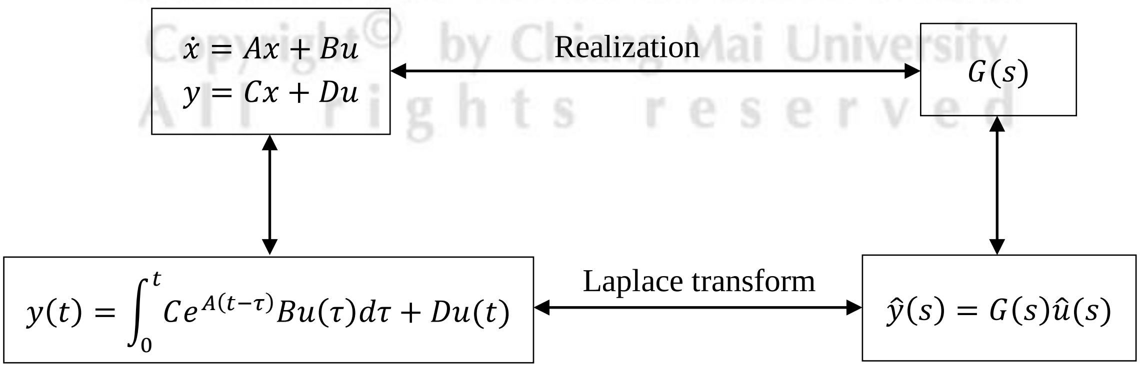


Figure 3.1 The mapping representations of the time and frequency domain for LTI system

i) $H(s)$ is proper (there is no pole at infinity)

ii) $H(s)$ has only poles in the open left half $\mathbb{C}^-$ plane (there is no pole in the closed right half plane $\mathbb{C} \setminus \mathbb{C}^-$).

On the other hand, the system (3.1) is said to be stable if A has all its eigenvalues in the open left-half plane $\mathbb{C}^-$. We may denote the set of eigenvalues of A by $\lambda(A)$, the spectrum of A. Then stability of (3.1) is simply expressed as

$$\lambda(\mathrm{A}) \subset \mathbb{C}^-.$$

We say as well that the matrix A is stable if it has this property. The relationship between the system stability in (3.1) and the corresponding stability of the transfer matrix $G(s)$ is that if (3.1) (or A) is stable, then $G(s)$ is stable.

Note that all these definitions are given in terms of properties of the representation. Nevertheless, these concepts are closely related to the so called *bounded-input bounded-output stability properties*.

A vector valued signal $u(.)$ is bounded if the maximal amplitude or peak

$$\|u\|_\infty = \sup_{t \geq 0} \|u(t)\|$$

is finite. Note that $\|u(t)\|$ just equals the Euclidean norm $\sqrt{u(t)^T u(t)}$ of the vector $u(t)$. The symbol $\|u\|_\infty$ for the peak indicates that the peak is, in fact, a norm on the vector space of all bounded signals; it is called the $L_\infty -$norm.

The system (3.1) is said to be bounded-input bounded-output (BIBO) stable if it maps all arbitrary bounded inputs $u(.)$ into outputs that are bounded as well. In short, $\|u\|_\infty < \infty$ implies $\|y\|_\infty < \infty$. For LTI systems, BIBO stability is equivalent to the stability of the corresponding transfer matrix as defined earlier.

Theorem 3.1

The system in (3.1) maps bounded inputs $u(.)$ into bounded output $y(.)$ if and only if the corresponding transfer matrix $G(s)$ is stable.

To summarize, for a stabilizable and detectable realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of an LTI system, the following notions are equivalent: stability of the system (3.1), stability of the corresponding $G(s)$, and BIBO stability of the system (3.1) viewed as an input-output mapping.

Stability is a qualitative property. Another important aspect of system behaviour is to quantify how far signals are amplified or attenuated by a system. If we look at one input $u(.)$, and if we take $\|u\|_\infty$ and $\|y\|_\infty$ as a measure of size for the input and the output of the system (3.1), the amplification for this specific input signal is

$$\frac{\|y\|_\infty}{\|u\|_\infty}.$$

The worst possible amplification is obtained by finding the largest of these quotients if varying $u(.)$ over all bounded signals:

$$\gamma_{peak} = \sup_{0 < \|u\|_\infty < \infty} \frac{\|y\|_\infty}{\|u\|_\infty}. \tag{3.2}$$

This is the so-called *peak-to-peak gain* of the system (3.1). Then it just follows from the definition that

$$\|y\|_\infty = \gamma_{peak} \|u\|_\infty$$

holds for all bounded input signals $u(.)$: Hence γ_{peak} quantifies how the amplitudes of the bounded input signals are amplified or attenuated by the system. Since γ_{peak} is, in fact, the smallest number such that this inequality is satisfied, there does exist an input signal such that the peak amplification is actually arbitrarily close to γ_{peak}. (The supremum in (3.2) is not necessarily attained by some input signal. Hence we cannot say that γ_{peak} is attained, but we can come arbitrarily close.)

Besides the peak, we could also consider the energy of a signal $x(.)$ defined as

$$\|x\|_2 = \sqrt{\int_0^\infty \|x(t)\|^2 \, dt},$$

to measure its size. This norm is called the L_2 norm of a signal. Note that a signal with a large energy can have a small peak and vice versa and so it is important to think about different physical motivations if deciding for $\|.\|_\infty$ or for $\|.\|_2$ as a measure of size.

Now the question arises when a system maps any signal of finite energy again into a signal of finite energy; in short:

$$\|u\|_2 < \infty \text{ implies } \|y\|_2 < \infty .$$

For the system (3.1), this property is again equivalent to the stability of the corresponding transfer matrix $G(s)$. Hence the qualitative property of BIBO stability does not depend on whether one chooses the peak $\|.\|_\infty$ or the energy $\|.\|_2$ to characterize boundedness of a signal.

Although the qualitative property of stability does not depend on the chosen measure of size for the signals, the quantitative measure for the system amplification, the system gain, is highly dependent on the chosen norm. The energy (peak-RMS) gain of (3.1) is analogously defined as for the peak-to-peak gain defined by

$$\gamma_{energy} = \sup_{0 < \|u\|_2 < \infty} \frac{\|y\|_2}{\|u\|_2}.$$

Contrary to the peak-to-peak gain, one can relate the energy gain of the system (3.1) to the transfer matrix of the system. In fact, one can prove that γ_{energy} is equal to the maximal value that is taken by

$$\sigma_{\max}\left(G(j\omega)\right) = \|G(j\omega)\|$$

over the frequency $\omega \in \mathbb{R}$. Let us hence introduce the abbreviation

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(G(j\omega)\right) = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|.$$

As indicated by the symbol, this formula defines a norm on the vector space of all real rational proper and stable matrices $RH_\infty^{k \times l}$; it is called the H_∞-norm. We can conclude that the peak energy gain of the stable LTI system (3.1) is just equal to the H_∞-norm of the corresponding transfer matrix: $\gamma_{energy} = \|G\|_\infty$ □

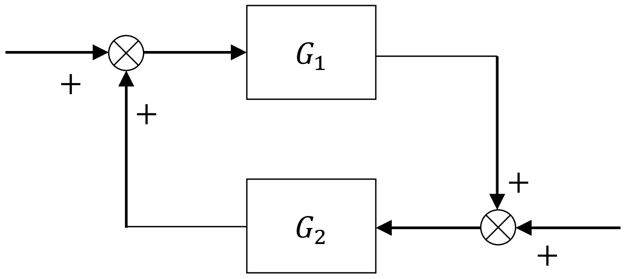


Figure 3.2 Feedback interconnection configuration

3.1.2 Small gain theorem

Consider a system with two subsystems with feedback interconnection as shown in Figure 3.2. Then the *small gain theorem* can be applied to establish that the overall system is stable, as follows:

Small gain theorem [85]: Suppose that two systems G_1 and G_2 are stable, i.e. $G_1 \in H_\infty$, $G_2 \in H_\infty$, then the closed-loop system is *internally stable* if

$$\|G_1G_2\|_\infty < 1 \text{ and } \|G_2G_1\|_\infty < 1 .$$ □

Note that, the small gain theorem provides a sufficient condition for stability but not necessary condition. This theorem is has many useful applications. In this thesis we will use it to treat uncertain systems with control feedback that require consideration of model error when trying to establish stability. When used in this way the stability condition is referred to as a *robust stability* condition.

3.1.3 Uncertainty

Robustness is particular important aspect to design the controller to use for the actual engineering system. The difference between the mathematical models used to design the controller and the actual system dynamics always can't be avoided. A robust control strategy is one for which the performance and/or stability of the system is less sensitive to the likely errors in a system model.

In reality, it is impossible to identify the mathematical model for any actual dynamic system with total accuracy. The possible errors between the actual dynamics of the system and the identified model can be called *uncertainty*. Generally, we can describe two versions of a dynamic system in Laplace domain: $G^{act}(s)$ is the dynamics of the actual system and $G^{m}(s)$ is the nominal/identified model. In our case, the nominal model will be represented by the finite-dimension active structure model given by (2.2). Thus:

$$G^{m}(s) = \left(sI - A\right)^{-1} B_u \,.$$

The identified model can be obtained by the numerical modelling approach such as finite element method or by using the system identification methods (eg. using toolbox in MATLAB® software program). The actual dynamics can be measured directly by testing the real system such as by tap test or frequency sweep. However, the measured response functions may still be subject to measurement error or the effects of system nonlinearity. Consequently, for model-based stability analysis or controller design it is important to consider potential errors, both known and unknown, in the system model. A key aspect is how best to represent the model uncertainty and, in this sense, there are a number of possible approaches to achieve robust control [86]. For the present situation, a key issue is dealing with neglected/unmodelled high frequency dynamics that tend to occur with finite order descriptions of continuous structures. For this reason, a frequency domain non-parametric uncertainty model is introduced. To this end, the absolute error (difference) in the system model may be defined as:

$$\Delta_a(s) = G^{act}(s) - G^{m}(s) \tag{3.3}$$

where $\Delta_a(s)$ is the additive uncertainty which is the difference in the transfer functions. Thus

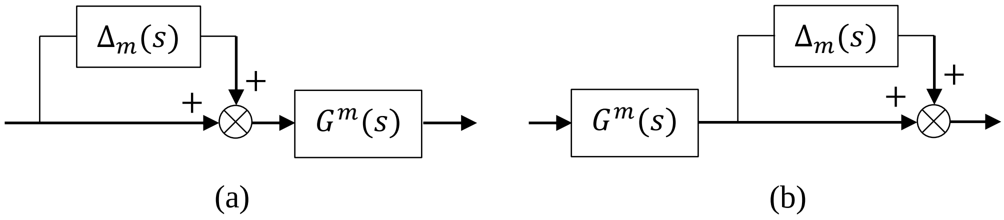


Figure 3.4 Multiplicative uncertainty representations (a) input multiplicative uncertainty (b) output multiplicative uncertainty

$$G^{act}(s) = G^m(s) + \Delta_a(s).$$

Then, another uncertainty representation can be described in the form of the relative error. The relative error is expressed as a multiplicative factor which may premultiply the transfer matrix (input multiplicative uncertainty) or post-multiply it (output multiplicative uncertainty).

$$G^{act}(s) = G^m(s)\left(I + \Delta_m^{in}(s)\right)$$
$$G^{act}(s) = \left(I + \Delta_m^{out}(s)\right)G^m(s).$$

The choice can be selected depending on the formulation of the controller design problem. The output and input multiplicative uncertainty can be defined as

$$\begin{aligned} \Delta_m^{in}(s) &= \left(G^m(s)\right)^{-1}\left(G^{act}(s) - G^m(s)\right) \\ \Delta_m^{out}(s) &= \left(G^{act}(s) - G^m(s)\right)\left(G^m(s)\right)^{-1}. \end{aligned} \tag{3.4}$$

This definition is only valid for square plant (G invertible) and an alternative definition must be used when the plant transfer matrix is non-square (see Section 3.1.4).

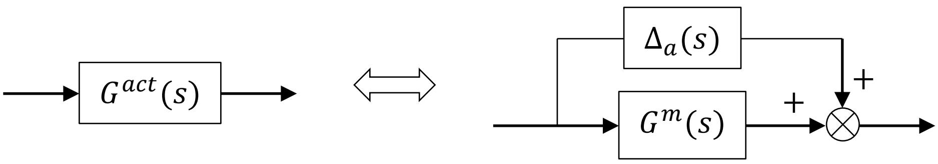


Figure 3.3 Additive uncertainty representation

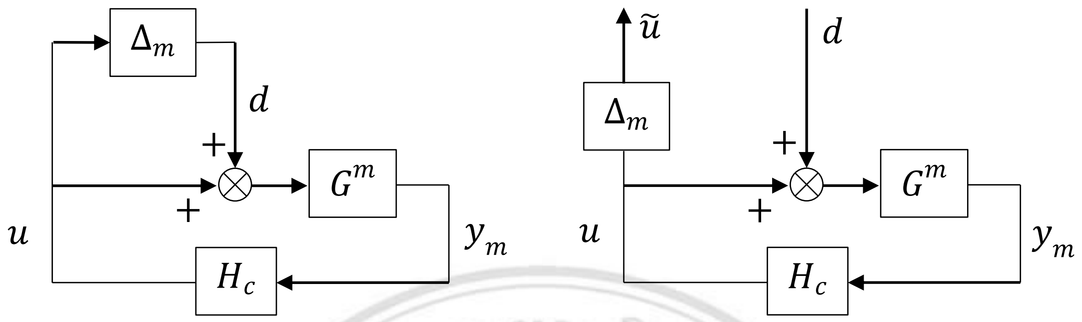
3.1.4 Robust stability conditions using small gain theorem

1) Basic condition for square plant

For given actual plant dynamics and approximate model we may define the input multiplicative error as in (3.4). The actual closed-loop system with controller $H_c(s)$ is shown in Figure 3.5a. We may assume that the closed-loop transfer function for the system model $T_{ud} = H_c G^m \left(I - H_c G^m\right)^{-1}$ is stable. To determine whether the actual system is stable we may apply the small gain theorem [86] for the interconnection from u to d. Thus we consider breaking the loop to obtain the system shown in Figure 3.5b. The sufficient condition for stability is that the L_2 gain from d to $\tilde{u}$ is less than 1. This condition may also be expressed in terms of the H_∞-norm:

$$\left\|T_{\tilde{u}d}\right\|_\infty = \left\|\Delta_m T_{ud}\right\|_\infty < 1. \tag{3.5}$$

Hence stability of the actual closed-loop system can be checked by considering the closed-loop transfer matrix T_{ud} calculated from the plant model. This transfer function is referred to as the *input complementary sensitivity function* which can be written as: $T_{ud} = H_c G^m \left(I - H_c G^m\right)^{-1}$. As Δ_m is not usually known exactly we may instead consider a scalar bounding function $\left|W_r(s)\right| \geq \left|\Delta_m\right|$ and apply the robust stability criterion $\left\|W_r T_{ud}\right\|_\infty < 1$. If this holds then we may conclude that the closed-loop system will always be stable. Stability conditions may be similarly derived for the other uncertainty representations.



(a) (b)

Figure 3.5 Application of small gain theorem to establish stability of system in (a) by checking H_∞-norm for system in (b)

2) Modified condition for non-square plant

The typical situation for MIMO systems is that there are more outputs than inputs. To use a state feedback control law, the number of outputs must be the same (or more than) the number of states. In this case, $\Delta_a(s)$ has many elements and so considering each element explicitly may introduce considerable complexity in the controller synthesis problem. One possible alternative approach is to try to map all errors to a multiplicative perturbation at the plant input, as shown in Figure 3.6f (where $H_c(s)$ is a designed controller). By using this representation, however, the implication is that all states are affected by the same multiplicative error (matched error). This is typically the case for model error associated with actuators, sensors and control calculation delays. Suppose again that we have a controller for which the following H_∞-norm specification is satisfied

$$\|T_{\tilde{u}d}\|_\infty = \|\Delta_m T_{ud}\|_\infty < 1 \tag{3.6}$$

where T_{ud} is the input complementary sensitivity function which can be written as: $T_{ud} = H_c G^m \left(I - H_c G^m\right)^{-1}$. In this case, the stability with unmatched errors is also guaranteed if the controller is chosen such that

$$\left\| H_c \Delta_a \left(I - H_c \Delta_a\right)^{-1} T_{ud} \right\|_\infty < 1. \qquad (3.7)$$

This can be shown by basic block diagram manipulation, as the system in Figure 3.6a can be transformed to the form in Figure 3.6f with $\Delta_m = H_c \Delta_a \left(I - H_c \Delta_a\right)^{-1}$. Hence, a controller synthesis may still be based on (3.6) and then robustness to unmatched model error checked following synthesis by evaluating whether condition (3.7) holds for the controller solution H_c. If not, Δ_m can be modified accordingly and a revised controller synthesized.

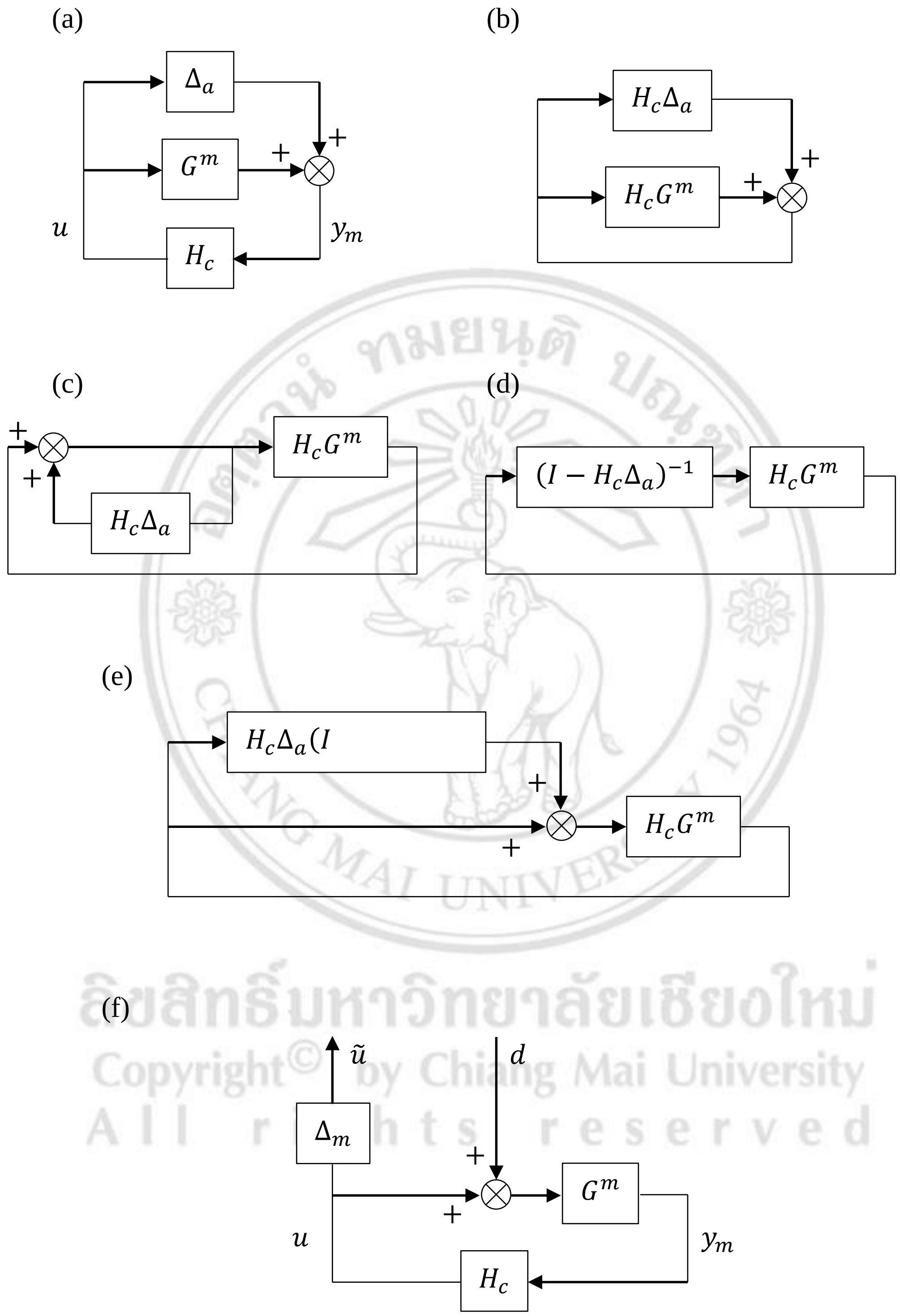


Figure 3.6 Steps for constructing equivalent input multiplicative perturbation from the additive perturbation

For both cases described, the set of design constraints necessary to account for the multiplicative uncertainty can be defined in terms of the weighting function $W_r(s)$ that satisfies $|\Delta_m(j\omega)| < |W_r(j\omega)|$ for all frequencies of ω. Then, if

$$\|\Delta_m T_{ud}\|_\infty < \|W_r T_{ud}\|_\infty < 1 \tag{3.8}$$

the actual closed-loop system is always stable. The weighting function can be defined in the form of Laplace domain, $W_r(s)$ which also can be described by the state-space model as:

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r u \\ \tilde{u} &= C_r x_r + D_r u. \end{aligned} \tag{3.9}$$

3.2 Linear Matrix Inequalities (LMIs) and Lyapunov Stability Theory

3.2.1 LMI's Definition [43]

A linear matrix inequality (LMI) is any set of constraints in the form

$$F(x) = F_0 + x_1F_1 + x_2F_2 + \dots + x_{n-1}F_{n-1} + x_nF_n = F_0 + \sum_{i=1}^{n} x_iF_i < 0 \tag{3.10}$$

where $(x_1, \dots, x_n)$ is a vector of unknown scalars (the optimization variable) and $F_0, \dots, F_n$ are real symmetric matrices. The inequality (<0) means that "*negative definite*", i.e., $F(x) < 0$ means the biggest eigenvalue of $F(x)$ is negative. This is the general definition of an LMI.

For most applications, LMIs do not naturally arise in the canonical form (3.10), but rather in the form

$$L(X_1,\dots,X_n) < R(X_1,\dots,X_n) \tag{3.11}$$

where $L(\bullet)$ and $R(\bullet)$ are affine functions of some constructed matrix variable $X_1,\dots,X_n$. Consider, as an example, the Lyapunov inequality

$$A^T P + PA < 0 \tag{3.12}$$

where P is an unknown symmetric matrix. Defining $x_1, \ldots, x_n$ as the dependent scalar entries of P, this LMI could be rewritten in the form (3.10). Yet it is more convenient and efficient to describe it in its natural form in (3.12).

An LMI defines a convex constraint on the optimization variables. That is the set of $\Phi = \{x : F(x) < 0\}$ is convex. Indeed, if $x_1, x_2 \in \Phi$ and $\alpha \in (0,1)$ then

$$F\left(\alpha x_1 + (1-\alpha)x_2\right) = \alpha F(x_1) + (1-\alpha)F(x_2) < 0$$

where in the first equality we used the fact that F is affine. The last inequality follows from the fact that $\alpha \geq 0$ and $1-\alpha \geq 0$. This is an important property since powerful numerical solution techniques are available for problems involving convex solution sets.

3.2.2 Generic problem

There are three generic problems that arise involving LMIs.

i) Feasibility problem: finding a solution x to the LMI by the form

$$F(x) < 0$$

ii) Minimization problem: minimizing a convex objective function under some LMI constraint is also a convex problem. In particular, the linear objective minimization problem is

$$\text{Minimize } \; c^T x \; \text{ subject to } \; F(x) < 0$$

iii) The generalized eigenvalue minimization problem: This amounts to minimize a scalar $\lambda \in R$ with constraints in the form

$$\begin{aligned} &\text{Minimize } \; \lambda \\ &\text{subject to } \; A(x) > 0, \; B(x) > 0 \\ &\qquad\qquad\quad F(x) < \lambda B(x) \end{aligned} \tag{3.13}$$

which is quasi-convex and can be solved by similar techniques. It owes its name to the face that λ is related to the largest generalized eigenvalue of the pencil $A(x)$ and $B(x)$.

3.2.3 Lyapunov stability [87]

Lyapunov stability theory has an importantly role in dynamic systems and control theory. The most relevant aspects relate to system stability with regard to an equilibrium point which this can be established using Lyapunov theory. It can be briefly stated that if the solutions starting out near an equilibrium point x_0 stays near x_0 forever then x_0 is Lyapunov stable. If not then it is unstable. For a stronger stability condition we may ask if the solutions starting out near x_0 converge to x_0. If so then x_0 is asymptotically stable.

For autonomous systems, the basic Lyapunov stability theorem is applied to a system in the form

$$\dot{x} = f(x) \tag{3.14}$$

where $f : \mathrm{D} \to \mathbb{R}^n$ is a local Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\bar{x} \in \mathrm{D}$ is an equilibrium point of (3.14) that is $f(\bar{x}) = 0$. Without loss of generality, all definitions and theorems for the case when the equilibrium point is that the origin of $\mathbb{R}^n$, that is $\bar{x} \in 0$, because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and consider a change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) = g(y), \text{ where } g(0) = 0. \tag{3.15}$$

In the new variable y, the system has equilibrium at the origin. Therefore, without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and study the stability of the origin $x = 0$.

Definition 3.2

The equilibrium point $x = 0$ of (3.14) is

- stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \forall t \geq 0$$

- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) < 0 \qquad \square$$

Theorem 3.3

Let $x = 0$ be an equilibrium point for (3.14) is and $\mathrm{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathrm{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

$$\dot{V}(x) \le 0 \text{ in D}$$

then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) \le 0 \text{ in } \mathrm{D} - \{0\}$$

then, $x = 0$ is asymptotically stable. $\square$

Theorem 3.4

Let $x = 0$ be an equilibrium point for (3.14). Let $V : \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \ne 0$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

$$\dot{V}(x) < 0, \quad \forall x \ne 0$$

then, $x = 0$ is globally asymptotically stable. $\square$

3.2.4 Lyapunov stability via LMI

Consider a linear time invariant dynamic system described by

$$\dot{x} = Ax\,. \tag{3.16}$$

A quadratic Lyapunov function may then be defined having the form

$$V(x) = x^T P x > 0 \tag{3.17}$$

where P is positive symmetry definite matrix, $P = P^T > 0$. The time derivative of the Lyapunov function $\left(\dot{V}(x) = \frac{d}{dt}V(x)\right)$ is

$$\dot{V}(x) = 2x^T P\dot{x} = x^T\left(PA + A^T P\right)x\,. \tag{3.18}$$

From Theorem 3.4 the system in (3.16) is globally asymptotically stable if $\dot{V}(x) < 0$, $\forall x \neq 0$. That means the symmetric matrix in (3.18) must be negative definite:

$$PA + A^T P < 0\,. \tag{3.19}$$

The LMI in (3.19) is also called *Lyapunov inequality* [43]. The feasible solution for P must be obtain in order to confirm that the dynamic system in (3.16) is globally asymptotically stable.

3.2.5 System norm-bound via LMIs

Given, a dynamics system has the form:

$$\begin{aligned} \dot{x} &= Ax + B_d d \\ z &= C_z x \end{aligned} \tag{3.20}$$

where d is the exogenous perturbation and z is the output signals. An upper bound on the H_∞ norm for this system may be determined using a quadratic Lyapunov function as stated in the following Theorem.

Theorem 3.5

For the dynamic system in (3.20), if there exists a quadratic Lyapunov function $V(x) = x^T Px,\ P > 0$ and $\gamma > 0$, such that for all t and all admissible x and d,

$$\frac{d}{dt}V(x)+\frac{1}{\gamma}z^T z-\gamma d^T d\le 0 \tag{3.21}$$

then, the induced L_2 gain for this system is less that γ. □

Note that if the induced L_2 gain is less than γ then, by definition, $\|z\|_{L_2}\le\gamma\|d\|_{L_2}$ $\forall d\in L_2[0,\infty)$ where the L_2 norm is $\|\varsigma\|_{L_2}=\sqrt{\int_0^\infty \varsigma^T(t)\varsigma(t)\,dt}$ as previously defined.

For linear systems the peak L_2 gain is given by the H_∞ norm and so the quadratic constraint is equivalent to $\|T_{zd}\|_\infty<\gamma$.

For the Lyapunov function in (3.17), the time derivative is $\dot V(x)=2x^T P\dot x$. Then the inequality in Theorem 3.5 becomes

$$2x^T P\dot x+\frac{1}{\gamma}z^T z-\gamma d^T d\le 0. \tag{3.22}$$

This leads to the LMI constraint

$$\begin{bmatrix} PA+A^T P+\frac{1}{\gamma}C_z^T C_z & PB_d \\ sym & -\gamma I \end{bmatrix}<0. \tag{3.23}$$

So, if there exist positive definite matrices $P>0$ and $Q>0$ such that (3.23) holds then the system in (3.20) is asymptotically stable (for $d=0$) and the H_∞ norm from d to z is less than γ. In order to prove that (3.22) implies that the H_∞ norm-bound holds, we can integrate (3.22) from $t\in[0,\infty)$. This yields

$$\int_0^\infty\left(\dot V(x)+\frac{1}{\gamma}z^T z-\gamma d^T d\right)dt\le 0$$

so

$$V(\infty)-V(0)+\int_0^\infty\frac{1}{\gamma}z^T z\,dt+\int_0^\infty\gamma d^T d\,dt\le 0$$

for the zero initial condition $V(0)=0$, hence

$$V(\infty)+\int_0^\infty \frac{1}{\gamma} z^T z \; dt \le \int_0^\infty \gamma d^T d \; dt .$$

Since $V(\infty)\ge 0$, therefore

$$\int_0^\infty \frac{1}{\gamma} z^T z \; dt \le V(\infty)+\int_0^\infty \frac{1}{\gamma} z^T z \; dt \le \int_0^\infty \gamma d^T d \; dt .$$

That means,

$$\sqrt{\int_0^\infty \frac{1}{\gamma} z^T z \; dt} \;\le\; \gamma \sqrt{\int_0^\infty d^T d \; dt} .$$

According to L_2 norm definition, the H_∞ norm-bounded criterion of

$$\|z\|_{L_2} \le \gamma \|d\|_{L_2}, \;\; \forall d \in L_2[0,\infty)$$

is satisfied.

3.3 Lyapunov-Krasovskii Functional

Suppose that the time-delay system has the form

$$\dot{x} = A_0 x + A_\tau x(t-\tau) . \tag{3.24}$$

For this system, a Lyapunov-Krasovskii functional (LKF) can be used in the same way as a quadratic Lyapunov function can be used with non-time-delay systems in order to establish stability and input-output properties. See the review papers [42] and [47] for further background. Furthermore, the approach can be used to synthesize controllers for time-delay systems. Here we choose (among other possibilities) an LKF involving a time-delay-dependent integral quadratic form:

$$V(x,t) = x^T(t) P x(t) + \int_{t-\tau}^{t} x^T(\alpha) Q x(\alpha) d\alpha \tag{3.25}$$

where $P = P^T > 0$ and $Q = Q^T > 0$. The time derivative of (3.25) with a constant time-delay τ is

$$\dot{V}(x,t) = 2x^T P\dot{x} + x^T Qx - x^T(t-\tau)Qx(t-\tau) .$$

Therefore,

$$\dot{V}(x,t) = \begin{bmatrix} x \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} PA_0 + A_0^T P + Q & PA_D \\ sym & -Q \end{bmatrix} \begin{bmatrix} x \\ x(t-\tau) \end{bmatrix}. \tag{3.26}$$

If there exist positive matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} PA_0 + A_0^T P + Q & PA_\tau \\ sym & -Q \end{bmatrix} < 0 \tag{3.27}$$

then the time-delay system is asymptotically stable. Note that although the LKF (3.25) is delay-dependent, the final stability condition (3.27) is not. Therefore, LKF is suitable for establishing a delay-independent stability limit. For an application to the machining vibration problem, this will allow us to determine stable conditions which are independent of tool speed (rotational frequency).

3.3.1 Application of LKF to chatter stability limits

For milling operations, the cutting force involves a time-delayed feedback effect from the previous tooth-pass, as described in Chapter 2. The cutting force model (2.5) may be combined with the passive structure dynamics in (2.1) leading to the time-delay system in the form

$$\dot{x} = A_0 x + A_\tau x(t-\tau) + B_w w_0 \tag{3.28}$$

where $A_0 = A - b_K B_w C_t$, $A_\tau = b_K B_w C_t$, and the zero-vibration cutting force is $w_0 = b_K h_m$.

The LMI stability criterion (3.27) can be used to calculate the stability limit in terms of the value of b_K. If the LMI problem is feasible for a given value of b_K, which can be checked using a standard LMI solver (e.g. "*feasp*" in MATLAB®), then the system is asymptotically stable and chatter will not occur. To find the maximum depth-of-cut, $b_{K,\max}$, we can repeatedly solve the feasibility problem with iteration over b_K to find the maximum stable value, e.g. using a bisection algorithm.

Consider as a numerical example, the system given in Section 2.4 with transfer function (2.11). This system can also be described in the state space form (2.1) for which the values of constant matrices are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -135526 & 50792 & -149 & 0.33 \\ 322230 & -212343 & -25.3 & 46.3 \end{bmatrix},\ B_w = \begin{bmatrix} 0 \\ 0 \\ 1.55 \\ 54.5 \end{bmatrix},\ C_t = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

Constructing the time delay system model in the form (3.24) and checking for feasibility of (3.27), as previously described, allows us to determine the stability limit in terms of b_K. For this example, the obtained value was $b_{K,\max} \approx 184.18$ N/mm which is in close agreement with the value obtained by the frequency response method used to construct the SLD in Figure 2.8.

Consider now the case of a time-delay system with exogenous input and output variables:

$$\begin{aligned} \dot{x} &= A_0 x + A_\tau x(t-\tau) + B_d d \\ z &= C_z x + D_z d \end{aligned} \tag{3.29}$$

where z is the output and d is the input signals. To establish whether the system satisfies a H_∞ norm-bound criterion, the following equivalent condition is considered: $\|z\|_{L_2} \le \gamma \|d\|_{L_2}$, $\forall d \in L_2[0,\infty)$ with $\gamma > 0$. According to Theorem 3.5, this condition may be proved by the existence of an LKF $V(x,t)$ such that

$$\dot{V}_D = \dot{V}(x,t) + \frac{1}{\gamma} z^T z - \gamma d^T d \le 0. \tag{3.30}$$

Adopting the form of LKF given by (3.25), if there exist positive matrices $P > 0$ and $Q > 0$ then the time-delay system (3.29) is stable and the H_∞ norm from d to z is less than γ.

For the nominal time-delay system in (3.29), equation (3.30) can be written in the quadratic form

$$\dot{V}_D = \begin{bmatrix} x \\ x(t-\tau) \\ d \end{bmatrix}^T \begin{bmatrix} PA_0 + A_0^T P + Q + \frac{1}{\gamma} C_z^T C_z & PA_\tau & PB_d \\ & -Q & 0 \\ sym & & -\gamma I \end{bmatrix} \begin{bmatrix} x \\ x(t-\tau) \\ d \end{bmatrix} \leq 0 .$$

Consequently $\dot{V}_D = \xi^T \Psi \xi$ where $\xi^T = \begin{bmatrix} x^T & x^T(t-\tau) & d^T \end{bmatrix}$ and

$$\Psi = \begin{bmatrix} PA_0 + A_0^T P + Q + \frac{1}{\gamma} C_z^T C_z & PA_\tau & PB_d \\ & -Q & 0 \\ sym & & -\gamma I \end{bmatrix} < 0 . \tag{3.31}$$

The term $\frac{1}{\gamma} C_z^T C_z$ here is quadratic in C_z. Therefore, if C_z, is an optimization variable then (3.31) is not in the LMI form. To obtain an equivalent condition in LMI form, Schur complements may be used, as given in the following Lemma.

Lemma 3.1. Schur complement **[43]**

Suppose there are positive definite matrices, $Q = Q^T > 0$ and $R = R^T > 0$ for which a bilinear matrix inequality (BMI) is defined $Q - S^T R^{-1} S > 0$. We can transform BMI to equivalent LMI as:

$$Q - S^T R^{-1} S > 0 \Leftrightarrow \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} > 0. \qquad \square$$

To apply Lemma 3.1, we note that an equivalent condition to equation (3.31) is

$$\begin{bmatrix} PA_0 + A_0^T P + Q & PA_\tau & PB_d \\ & -Q & 0 \\ sym & & -\gamma I \end{bmatrix} - \begin{bmatrix} C_z^T \\ 0 \\ 0 \end{bmatrix} (-\gamma I)^{-1} \begin{bmatrix} C_z & 0 & 0 \end{bmatrix} < 0 . \tag{3.32}$$

Hence, from Lemma 3.1, an equivalent condition to (3.31) may be obtained as

$$\Psi = \begin{bmatrix} PA_0 + A_0^T P & PA_\tau & PB_d & C_z^T \\ & -Q & 0 & 0 \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0 . \quad (3.33)$$

The final LMI form for the H_∞ norm-bound in (3.33) is suitable for various controller synthesis problems will be shown in Section 3.4.

3.4 LKF Controller Designs

For milling operations, a time delay system with active control and exogenous inputs can be formulated as,

$$\begin{aligned} \dot{x} &= A_0 x + A_\tau x(t-\tau) + B_d d + B_u u \\ z &= C_z x + D_z u \\ y &= Cx. \end{aligned} \quad (3.34)$$

For this system, the exogenous (disturbance) input is d, the control input is u, and the output y is measured signals that can be used for feedback. The output z is defined for use in H_∞ norm-bound criterion for controller design. For the robust control approach described in Section 3.1.3, robust stability is achieved if we can synthesize controllers such that the closed-loop system satisfies an H_∞ norm-bound criterion in the form $\|T_{zd}\|_\infty < \gamma$.

3.4.1 State feedback control

If all the system states can be measured continuously, the state vector, x, in (3.29) can be used by the feedback controller. In this case, a standard linear state feedback controller would take the form $u = K_0 x$ where K_0 is the constant controller gain matrix which must be suitably chosen to match the system. For a time-delay system, there are already feedback effects involving the delayed states. As an extension to the basic state feedback approach, it is natural to consider additional delayed state feedback in the general form:

$$u = K_0 x + K_\tau x(t-\tau)\,. \tag{3.35}$$

Using the control law (3.35) for the nominal time-delay system (3.34), yields

$$\begin{aligned} \dot{x} &= \left(A_0 + B_u K_0\right)x + \left(A_\tau + B_u K_\tau\right)x(t-\tau) + B_d d \\ z &= \left(C_z + D_z K_0\right)x + D_z K_\tau x(t-\tau)\,. \end{aligned} \tag{3.36}$$

In order to establish an H_∞ norm-bound for this system, the steps described in Section 3.3.1 that led to the LMI condition (3.31) may be applied. By comparing (3.29) and (3.36) we see that $A_0 \Rightarrow A_0 + B_u K_0$, $A_\tau \Rightarrow A_\tau + B_u K_\tau$ and so the required condition is

$$\begin{bmatrix} P\left(A_0 + B_u K_0\right) + \left(A_0 + B_u K_0\right)^T P + Q & P\left(A_\tau + B_u K_\tau\right) & PB_d \\ & -Q & 0 \\ sym & & -\gamma I \end{bmatrix}$$

$$-\begin{bmatrix} \left(C_z + D_z K_0\right)^T \\ \left(D_z K_\tau\right)^T \\ 0 \end{bmatrix} \left(-\gamma I\right)^{-1} \left[\left(C_z + D_z K_0\right) \quad D_z K_\tau \quad 0\right] < 0\,.$$

So, from Lemma 3.1, the equivalent condition is

$$\Psi = \begin{bmatrix} PA_0 + A_0^T P + PB_u K_0 + K_0^T B_u^T P + Q & PA_\tau + PB_u K_\tau & PB_d & C_z^T + K_0^T D_z^T \\ & -Q & 0 & K_\tau^T D_z^T \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0\,. \tag{3.37}$$

The matrix Ψ in (3.37) has terms that are bilinear and involve P, K_0, and K_τ. In order to recover an LMI that can be solved for both control gains. We can apply a change of variables by first using the following Lemma.

Lemma 3.2. Equivalent condition

For any symmetric positive definite matrix $N = N^T > 0$ the following conditions are equivalent

$$\mathrm{X} < 0 \quad \Leftrightarrow \quad N^T \mathrm{X} N < 0\,. \qquad \square$$

Thus, we can apply the congruence transformation in Lemma 3.2 to (3.36) with the matrix

$$\Theta = \mathrm{diag}\begin{bmatrix} P^{-1} & P^{-1} & I & I \end{bmatrix}.$$

So we may consider the equivalent condition $\Psi^* = \Theta^T \Psi \Theta < 0$. Using a substitution of variables $L = P^{-1}$, $S = L^T Q L$, $K_0^* = K_0 L$ and $K_\tau^* = K_\tau L$, leads to the LMI feasibility problem:

If there exist positive definite matrices $L > 0$, $S > 0$ and matrices K_0^* and K_τ^* such that the following LMI holds:

$$\Psi^* = \begin{bmatrix} A_0 L + L A_0^T + B_u K_0^* + K_0^{*T} B_u^T + S & A_\tau L + B_u K_\tau^* & B_d & L C_z^T + K_0^{*T} D_z^T \\ & -S & 0 & K_\tau^{*T} D_z^T \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0 \qquad (3.38)$$

then, the time-delay system with controller in (3.36) is stable and the H_∞ norm from the d to z is less than γ.

An optimized solution to (3.38) can be obtained via a generalized eigenvalue problem (GEVP) defined as

$$\begin{aligned} \text{Minimize} \quad & \gamma \text{ over } L, S, X, K_0^*, K_\tau^* \\ \text{subject to} \quad & L > 0,\ S > 0,\ X, 0 \\ & \Psi_X^* < 0 \\ & X < \gamma I \end{aligned} \qquad (3.39)$$

where

$$\Psi_X^* = \begin{bmatrix} A_0 L + L A_0^T + B_u K_0^* + K_0^{*T} B_u^T + S & A_\tau L + B_u K_\tau^* & B_d & L C_z^T + K_0^{*T} D_z^T \\ & -S & 0 & K_\tau^{*T} D_z^T \\ & & -X & 0 \\ sym & & & -X \end{bmatrix}.$$

Note that A_0 and A_τ depend on b_K. Therefore, for controller synthesis based on (3.39), the value of the depth-of-cut parameter b_K is fixed and must be selected first. For improved stability limits the chosen value should exceed the stability limit without control. The state feedback control law is then given by $u(t) = K_0^* L^{-1} x(t) + K_\tau^* L^{-1} x(t-\tau)$.

3.4.2 Output feedback control

Suppose that the system state x cannot be measured completely, but there is some set of measured output variables y that may be used for feedback. The control approach in this case is referred to as output feedback control. A LTI dynamic output feedback controller will have a state space realization:

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y \end{aligned}$$

where A_K, B_K, C_K and D_K are chosen controller matrices and x_k is the controller states. For the time-delay system (3.29), we may consider a more general controller structure involving delayed feedback terms in the form

$$\begin{aligned} \dot{x}_K &= A_K x_K + A_{K\tau} x(t-\tau) + B_K y + B_{K\tau} y(t-\tau) \\ u &= C_K x_K + C_{K\tau} x(t-\tau) + D_K y + D_{K\tau} y(t-\tau). \end{aligned} \tag{3.40}$$

The closed-loop system may be defined in terms of the combined state vector $x_{cl} = \begin{bmatrix} x & x_k \end{bmatrix}^T$ as

$$\begin{aligned} \dot{x}_{cl} &= A_{cl,0} x_{cl} + A_{cl,\tau} x(t-\tau) + B_{cl,d} d + B_{cl,w} w_0 \\ z &= C_{cl,z} x_{cl} + C_{cl,z\tau} x_{cl}(t-\tau) \\ y &= C_{cl} x_{cl} \end{aligned} \tag{3.41}$$

where $A_{cl,0} = \begin{bmatrix} A_0 + B_u D_K C & B_u C_K \\ B_K C & A_K \end{bmatrix}$, $A_{cl,\tau} = \begin{bmatrix} A_\tau + B_u D_{K\tau} C & B_u C_{K\tau} \\ B_{K\tau} C & A_{K\tau} \end{bmatrix}$, $B_{cl,d} = \begin{bmatrix} B_d \\ 0 \end{bmatrix}$,

$B_{cl,w} = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$, $C_{cl,z} = \begin{bmatrix} C_z + D_z D_K C & D_z C_K \end{bmatrix}$, $C_{cl,z\tau} = \begin{bmatrix} D_z D_{K\tau} C & D_z C_{K\tau} \end{bmatrix}$, $C_{cl} = \begin{bmatrix} C & 0 \end{bmatrix}$

In order to synthesize an output feedback controller for the closed-loop system we consider the robust stability condition based on the LKF in the same form as (3.25), as defined by

$$V(x_{cl},t) = x_{cl}^T(t)Px_{cl}(t) + \int_{t-\tau}^{t} x_{cl}^T(\alpha)Qx_{cl}(\alpha)d\alpha \,. \tag{3.42}$$

Then, the robust stability is established as follows:

If there exist positive matrices $P > 0$ and $Q > 0$ such that

$$\dot{V}_D = \dot{V}(x_{cl},t) + \frac{1}{\gamma} z^T z - \gamma d^T d \le 0 \tag{3.43}$$

then the time-delay system is stable and the H_∞ norm from d to z is less than γ. The constraint (3.43) leads to the matrix inequality

$$\Phi = \begin{bmatrix} PA_{cl,0} + A_{cl,0}^T P + Q & PA_{cl,\tau} & PB_{cl,d} & C_{cl,z}^T \\ & -Q & 0 & C_{cl,z\tau}^T \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0. \tag{3.44}$$

The matrix Φ is in a bilinear form involving many optimization matrix variables (P, A_K, $A_{K\tau}$, B_K, $B_{K\tau}$, C_K, $C_{K\tau}$, D_K and $D_{K\tau}$) which must be solved for to obtain the output feedback controller solution. To obtain an LMI condition in all the optimization variables, the equivalent condition in Lemma 3.3 is applied together with a change of free variables. This technique was first proposed in [44] for LMI-based controller synthesis and has been used extensively for multi-objective controller design. The application of this approach to time-delay systems based on LKFs is a novel aspect of the work in this thesis.

Lemma 3.3. *Equivalent condition for output feedback controller synthesis* **[44]**

Rewriting P using sub-matrix variables $P = \begin{bmatrix} Y & N \\ N^T & \Omega \end{bmatrix}$ then the inverse of P is also defined in sub-matrix form as $P^{-1} = \begin{bmatrix} X & M \\ M^T & \theta \end{bmatrix}$ where X and Y are symmetric positive definite matrices. The multiplication $PP^{-1} = I$ leads to the following constraints:

$$YX + NM^T = I \text{ and } N^T X + \Omega M^T = 0.$$

Then, we define the matrix $\psi_1 = \begin{bmatrix} X & I \\ M^T & \theta \end{bmatrix}$ so that

$$P\psi_1 = \begin{bmatrix} Y & N \\ N^T & \Omega \end{bmatrix}\begin{bmatrix} X & I \\ M^T & \theta \end{bmatrix} = \begin{bmatrix} YX + NM^T & Y \\ N^T X + \Omega M^T & N^T \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$$

Also $\psi_2 = P\psi_1 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$.

Now we may apply a congruence transformation to the matrix inequality in (3.44) using

$$\Gamma = \text{diag}\begin{bmatrix} \psi_1 & \psi_1 & I & I \end{bmatrix}$$

From Lemma 3.2, $\Gamma\Phi\Gamma < 0 \Leftrightarrow \Phi < 0$ and so the equivalent condition of (3.44) is

$$\Phi_\Gamma = \begin{bmatrix} \psi_1^T PA_{cl,0}\psi_1 + \psi_1^T A_{cl,0}^T P\psi_1 + \psi_1^T Q\psi_1 & \psi_1^T PA_{cl,\tau}\psi_1 & \psi_1^T PB_{cl,d} & \psi_1^T C_{cl,z}^T \\ & -\psi_1^T Q\psi_1 & 0 & \psi_1^T C_{cl,z\tau}^T \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0.$$

From the definition of ψ_1 and ψ_2,

$$\Phi_\Gamma = \begin{bmatrix} \psi_2^T A_{cl,0}\psi_1 + \psi_1^T A_{cl,0}^T \psi_2 + \psi_1^T Q\psi_1 & \psi_2^T A_{cl,\tau}\psi_1 & \psi_2^T B_{cl,d} & \psi_1^T C_{cl,z}^T \\ & -\psi_1^T Q\psi_1 & 0 & \psi_1^T C_{cl,z\tau}^T \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0$$

where

$$\psi_2^T A_{cl,0}\psi_1 = \begin{bmatrix} I & 0 \\ Y & N \end{bmatrix}\begin{bmatrix} A_0 + B_u D_K C & B_u C_K \\ B_K C & A_K \end{bmatrix}\begin{bmatrix} X & I \\ M^T & \theta \end{bmatrix}$$

$$= \begin{bmatrix} A_0 X + B_u\left(D_K CX + C_K M^T\right) & A_0 + B_u D_K C \\ Y\left(A_0 + B_u D_K C\right)X + NB_K CX + YB_u C_K M^T + NA_K M^T & YA_0 + \left(YB_u D_K + NB_K\right)C \end{bmatrix}$$

$$\psi_2^T A_{cl,\tau}\psi_1 = \begin{bmatrix} I & 0 \\ Y & N \end{bmatrix}\begin{bmatrix} A_\tau + B_u D_{K\tau} C & B_u C_{K\tau} \\ B_{K\tau} C & A_{K\tau} \end{bmatrix}\begin{bmatrix} X & I \\ M^T & \theta \end{bmatrix}$$

$$= \begin{bmatrix} A_\tau X + B_u\left(D_{K\tau} CX + C_{K\tau} M^T\right) & A_\tau + B_u D_{K\tau} C \\ Y\left(A_\tau + B_u D_{K\tau} C\right)X + NB_{K\tau}CX + YB_u C_{K\tau} M^T + NA_{K\tau} M^T & YA_\tau + \left(YB_u D_{K\tau} + NB_{K\tau}\right)C \end{bmatrix}$$

$$\psi_1^T C_{cl,z}^T = \begin{bmatrix} X & M \\ I & 0 \end{bmatrix}\left[C_z + D_z D_K C \quad D_z C_K\right]^T = \begin{bmatrix} XC_z^T + \left(XC^T D_K^T + MC_K^T\right)D_z^T \\ C_z^T + C^T D_K^T D_z^T \end{bmatrix}$$

$$\psi_1^T C_{cl,z\tau}^T = \begin{bmatrix} X & M \\ I & 0 \end{bmatrix}\left[D_z D_{K\tau} C \quad D_z C_{K\tau}\right]^T = \begin{bmatrix} \left(XC^T D_{K\tau}^T + MC_{K\tau}^T\right)D_z^T \\ C_z^T + C^T D_{K\tau}^T D_z^T \end{bmatrix}.$$

We now define new variables as

$$\begin{aligned}
\widehat{A}_K &= Y\left(A_0 + B_u D_K C\right)X + NB_K CX + YB_u C_K M^T + NA_K M^T \\
\widehat{B}_K &= YB_u D_K + NB_K \\
\widehat{C}_K &= D_K CX + C_K M^T \\
\widehat{D}_K &= D_K \\
\widehat{A}_{K\tau} &= Y\left(A_\tau + B_u D_{K\tau} C\right)X + NB_{K\tau} CX + YB_u C_{K\tau} M^T + NA_{K\tau} M^T \\
\widehat{B}_{K\tau} &= YB_u D_{K\tau} + NB_{K\tau}^T \\
\widehat{C}_{K\tau} &= D_{K\tau} CX + C_{K\tau} M^T \\
\widehat{D}_{K\tau} &= D_{K\tau} \\
\psi_1^T Q \psi_1 &= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}.
\end{aligned}$$

When we rewrite Φ_Γ in terms of the new variables it becomes a linear matrix inequality:

$$\Phi_\Gamma = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ & \Phi_{22} & 0 & \Phi_{24} \\ & & -\gamma I & 0 \\ sym & & & -\gamma I \end{bmatrix} < 0 \tag{3.45}$$

where

$$\Phi_{11} = \begin{bmatrix} A_0 X + X A_0^T + B_u \hat{C}_K + \hat{C}_K^T B_u^T + Q_{11} & \hat{A}_K + A_0 + B_u \hat{D}_K C + Q_{12} \\ sym & Y A_0 + A_0^T Y + \hat{B}_K C + C^T \hat{B}_K^T + Q_{22} \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} A_\tau X + B_u \hat{C}_{K\tau} & A_\tau + B_u \hat{D}_{K\tau} C \\ \hat{A}_{K\tau} & Y A_\tau + \hat{B}_{K\tau} C \end{bmatrix}, \ \Phi_{13} = \begin{bmatrix} B_d \\ Y B_d \end{bmatrix}, \ \Phi_{14} = \begin{bmatrix} X C_z^T + \hat{C}_K^T D_z^T \\ C_z^T + C \hat{D}_K^T D_z^T \end{bmatrix},$$

$$\Phi_{22} = \begin{bmatrix} Q_{11} & Q_{12} \\ sym & Q_{22} \end{bmatrix} \text{ and } \Phi_{24} = \begin{bmatrix} \hat{C}_{K\tau}^T D_z^T \\ C \hat{D}_{K\tau}^T D_z^T \end{bmatrix}.$$

Therefore, if there exists positive matrices $X > 0$, $Y > 0$ and $\begin{bmatrix} Q_{11} & Q_{12} \\ sym & Q_{22} \end{bmatrix} > 0$ satisfying (3.45) then the time-delay system in (3.29) is stable and the H_∞ norm from d to z less than γ. This is an LMI problem in the new optimization variables.

The controller matrices in (3.40) can be calculated by reconstructing from the solution to (3.45) as follows:

$$\begin{aligned} D_K &= \hat{D}_K \\ C_K &= \left(\hat{C}_K - D_K C X\right) M^{-T} \\ B_K &= N^{-1}\left(\hat{B}_K - Y B_u D_K\right) \\ A_K &= N^{-1}\left(\hat{A}_K - Y\left(A_0 + B_u D_K C\right) X - N B_K C X - Y B_u C_K M^T\right) M^{-T} \\ D_{K\tau} &= \hat{D}_{K\tau} \\ C_{K\tau} &= \left(\hat{C}_{K\tau} - D_{K\tau} C X\right) M^{-T} \\ B_{K\tau} &= N^{-1}\left(\hat{B}_{K\tau} - Y B_u D_{K\tau}\right) \\ A_{K\tau} &= N^{-1}\left(\hat{A}_{K\tau} - Y\left(A_\tau + B_u D_{K\tau} C\right) X - N B_{K\tau} C X - Y B_u C_{K\tau} M^T\right) M^{-T}. \end{aligned}$$

The unknown matrices of N and M can be solved using a singular value decomposition approach. Considering the relationship of the matrices N, M, X and Y given by $YX + NM^T = I$, then

$$NM^T = I - YX = \left(\Lambda_N \Sigma^{1/2}\right)\left(\Sigma^{1/2} \Lambda_M^T\right).$$

Therefore, $N = \Lambda_N \Sigma^{1/2}$ and $M^T = \Sigma^{1/2} \Lambda_M^T$.

An optimized solution to (3.45) can be obtained via a generalized eigenvalue problem (GEVP) defined as

$$
\begin{aligned}
\text{Minimize} \quad & \gamma \quad \text{over } X, Y, Z, \begin{bmatrix} Q_{11} & Q_{12} \\ sym & Q_{22} \end{bmatrix} \\
& \hat{A}_K, \hat{B}_K, \hat{C}_K, \hat{D}_K, \hat{A}_{K\tau}, \hat{B}_{K\tau}, \hat{C}_{K\tau}, \hat{D}_{K\tau} \\
\text{subject to} \quad & X > 0, Y > 0, \ Z > 0, \begin{bmatrix} Q_{11} & Q_{12} \\ sym & Q_{22} \end{bmatrix} > 0 \\
& \Phi_\Gamma^* < 0 \\
& Z < \gamma I
\end{aligned}
\tag{3.46}
$$

where

$$
\Phi_\Gamma^* = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ & \Phi_{22} & 0 & \Phi_{24} \\ & & -Z & 0 \\ sym & & & -Z \end{bmatrix} < 0 .
$$

As for the state feedback synthesis, the value of the depth-of-cut parameter is embedded in (3.46) and a fixed value must be chosen before solving the LMI synthesis problem.

Note that, for the controller in (3.40), we can consider a more simple controller form by omitting the delayed input/output terms, i.e. setting $B_{K\tau}$, $C_{K\tau}$ and $D_{K\tau}$ to zero. In this case, the output feedback controller has the form

$$
\begin{aligned}
\dot{x}_K &= A_K x_K + A_{K\tau} x(t-\tau) + B_K y \\
u &= C_K x_K + D_K y .
\end{aligned}
$$

Note that the controller state delay matrix $A_{K\tau}$ cannot be eliminated as it is required to transfer the BMI form to the LMI form by allowing the bilinear term of $YA_\tau X$ to be eliminated using the additional free variable. □