CHAPTER 1

Introduction

This chapter is organized two sections as the following: Section 1.1 presents a brief history of fixed point theory of some generalized contraction mappings in complete metric spaces endowed with graphs and recalls some important results of them. In Section 1.2, we introduce best proximity point theorems for nonexpansive mappings in a Banach space and review some well-known results for mean nonexpansive mappings.

1.1 Fixed Points of Some Generalized Contraction Mappings in Complete Metric Spaces Endowed with Graphs

Fixed point theory plays an important role in the study of theory of equations in nonlinear analysis. They can be applied widely to solve importantly the existence of solutions of various equations. Further, it has various applications in many fields such as optimization, control theory and economics. A fundamental and well-known result is the Banach's contraction principle [1] that has been extended and generalized in many directions both single valued self-map version and multivalued self-map version, for instance, see [2, 3, 4, 5, 6, 7, 8] and for other associated results, see [9, 10, 11, 12, 13, 14, 15, 16].

Theorem 1.1.1. (Banach's Contraction Principle) Let (X, d) be a complete metric space and let $T : X \to X$ be a *contraction mapping*, i.e., there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point in X.

In 1968, Kannan [17] extended the notions of Banach's contraction principle to a new type of mappings which is different from that of contraction as the following:

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *Kannan mapping* if there exists $a \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

It is noted that a contraction mapping is continuous but a Kannan mapping is not. We can found the results as the following in [17].

On the other hand, Chatterjea [2] introduced a new concept of contraction mappings known as *Chatterjea contraction mapping* as follows:

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X,$$

where $a \in \left[0, \frac{1}{2}\right)$.

Zamfirescu [3] proved a fixed point theorem for a new type of contraction mapping by combining the concept of Banach's contraction mapping, Kannan mapping and Chatterjea mapping. This mapping is known as *Zamfirescu operator*.

Theorem 1.1.2. ([3]) Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b, c < \frac{1}{2}$ such that for each pair $x, y \in X$, at least one of the following is true:

(z_1) $d(Tx, Ty) \leq ad(x, y)$;

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T is a Picard operator, that is, T has a unique fixed point $x_0 \in X$ and for each $x \in X$, $T^n x \to x_0$.

In 2004, Berinde [7] introduced and studied the fixed point theorems for weak contraction mapping or almost contraction mapping on a complete metric space which is more general than that of Kannan and Chatterjea mapping and Zamfirescu operator.

Definition 1.1.3. ([7]) Let (X, d) be a metric space. A map $T : X \to X$ is called *weak contraction* or *almost contraction* if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X.$$

Moreover, the Banach's contraction principle was extended to multivalued mappings in a complete metric space. The first well-known fixed point theorem for multivalued contraction mappings using the Pompeiu-Hausdorff metric was studied by Nadler [18].

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let A be a subset of X. The *distance from x to A* is defined by

$$D(x, A) := \inf\{d(x, y) : y \in A\}.$$

For $A, B \in CB(X)$, we define

$$H(A,B) := \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\right\}.$$

It is called a *Pompeiu-Hausdorff distance from A to B*.

Let $T : X \to 2^X$ (collection of all nonempty subsets of X) be a multivalued mapping. A point $x \in X$ is said to be a *fixed point of T* if $x \in Tx$. We denote the set of all fixed points of T by $F(T)$, that is,

$$F(T) := \{x \in X : x \in Tx\}.$$

In 1969, Nadler [18] extended the Banach's contraction principle for a multivalued mapping and proved the Banach's contraction principle in a complete metric space for multivalued version. He proved the following fixed point theorem.

Theorem 1.1.4. (Nadler's fixed point theorem) Let (X, d) be a complete metric space and let T be a map from X into $CB(X)$. Suppose that T is a multivalued contraction mapping, i.e., there exists $k \in [0, 1)$ such that

$$H(Tx, Ty) \le kd(x, y) \quad \text{for all } x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

Nadler's fixed point theorem was extended and generalized in many directions. One of the well-known extensions is a fixed point theorem of multivalued almost contractions introduced by M. Berinde and V. Berinde [19]. They extended Nadler's fixed point theorem to a new class of multivalued self mappings, called multivalued almost contractions, defined as follows:

Let (X, d) be a metric space and let $T : X \to CB(X)$ be a multivalued mapping. Then T is said to be a *multivalued almost contraction* or *multivalued (θ, L)-almost contraction* if there exist two constants $\theta \in (0, 1)$ and $L \ge 0$ such that

$$H(Tx, Ty) \le \theta d(x, y) + L \cdot D(y, Tx) \quad \text{for all } x, y \in X.$$

They proved that in a complete metric space, every multivalued almost contraction $T : X \to CB(X)$ has a fixed point.

In many real applicable existence problems, fixed point theorems of self-mappings may not be applied, but those of nonself mappings will be very useful and applicable.

Now, we will focus on the existence of fixed points for nonself multivalued contraction mappings which extended many important results, see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], for example. In 1972, Assad and Kirk [20] obtained a new fixed point theorem for nonself multivalued mappings.

Theorem 1.1.5. (Assad and Kirk's fixed point theorem) Let (X, d) be a complete and metrically convex metric space, K a nonempty closed subset of X, and let $T : K \to CB(X)$ a multivalued contraction mapping. If T satisfies Rothe's type condition, that is, $x \in \partial K$ implies $Tx \subset K$, then T has a fixed point in K.

Recently, Alghamdi *et al.* [29] considered multivalued nonself almost contractions on a convex metric space and proved the existence theorem of this mapping.

Theorem 1.1.6. ([29]) Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that $T : K \to CB(X)$ is a multivalued almost contraction, that is,

$$H(Tx, Ty) \leq \delta d(x, y) + L \cdot D(y, Tx) \quad \text{for all } x, y \in K,$$

with $\delta \in (0, 1)$ and some $L \geq 0$ such that $\delta(1 + L) < 1$. If T satisfies Rothe's type condition, then there exists $z \in K$ such that $z \in Tz$.

Let $G = (V(G), E(G))$ be a directed graph such that $V(G)$ a set of vertices of a graph and $E(G)$ a set of its edges. Let $\Delta := \{(x, x) : x \in V(G)\}$, say that the diagonal of $V(G) \times V(G)$, be a set of all loops in G. Assume that G has no parallel edges, thus one can identify G with the pair $(V(G), E(G))$. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges, that is,

$$E(G^{-1}) := \{(x, y) \in V(G) \times V(G) : (y, x) \in E(G)\}.$$

In 2008, Jachymski [31] combined the concept of fixed point theory and graph theory to study fixed point theory in a metric space endowed with a directed graph. He also introduced a concept of G-contraction self mapping which generalized the regular contraction mapping.

Definition 1.1.7. ([31]) Let (X, d) be a metric space, and $G = (V(G), E(G))$ a directed graph where $V(G) = X$ and $E(G)$ contains all loops, that is, $\Delta \subseteq E(G)$. We say that a mapping $T : X \to X$ is a *Banach G-contraction* or simply *G-contraction* if T preserves edges of G, i.e.,

$$\text{for any } x, y \in X \text{ such that } (x, y) \in E(G) \text{ implies } (Tx, Ty) \in E(G),$$

and there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X \quad \text{with } (x, y) \in E(G).$$

By using this concept and some conditions, he proved the following theorem.

Theorem 1.1.8. ([31]) Let (X, d) be a complete metric space, and let the triple (X, d, G) have the following property:

for any $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$,

then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Let $T : X \to X$ be a G-contraction mapping, and $X_T := \{x \in X : (x, Tx) \in E(G)\}$. Then T has a fixed point if and only if $X_T \neq \emptyset$.

In 2011, Nicolae *et al.* [32] introduced the concept of G-contraction for a multivalued mapping T from X into $CB(X)$ and proved the fixed point results about this mapping in a metric space endowed with a graph G.

Definition 1.1.9. ([32]) Let (X, d) be a metric space. Suppose that $G = (V(G), E(G))$ is a directed graph such that $V(G) = X$ and $E(G)$ contains all loops. A multivalued mapping $T : X \to CB(X)$ is said to be *G-contraction* if there exists $k \in (0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y) \quad \text{for all } (x, y) \in E(G),$$

and for each $(x, y) \in E(G)$, each $u \in Tx$ and $v \in Ty$ satisfying the condition

$$d(u, v) \leq \alpha d(x, y), \quad \text{for some } \alpha \in (0, 1),$$

we have $(u, v) \in E(G)$.

Theorem 1.1.10. ([32]) Let (X, d) be a complete metric space, and G be a directed graph. Let the triple (X, d, G) have the following property:

for any $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$,

then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Let $T : X \to CB(X)$ be a multivalued G-contraction mapping. Then $F(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

In 2013, Dinevari and Frigon [33] introduced the new concept of G-contraction that is weaker than definition of Nicolae.

Definition 1.1.11. ([33]) Let $T : X \to 2^X$ be a map with nonempty values. We say that T is a multivalued G-contraction (in the sense of Dinevari and Frigon) if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ and all $u \in Tx$, there exists $v \in Ty$ such that

$$(u, v) \in E(G) \text{ and } d(u, v) \leq \alpha d(x, y).$$

They proved that under some properties on a metric space, a multivalued G-contraction with closed values has a fixed point (see [33], Theorem 2.10 and Corollary 2.11).

One year later, Tiammee and Suantai [10] introduced the concepts of edge-preserving multivalued mapping and a new type of multivalued almost G-contraction and then they proved the existence fixed point theorem on a metric space with a directed graph. From their main result (see [10], Theorem 3.3), if we put $g(x) = x$ and $\alpha(x) = \delta$ for some nonnegative real number δ such that $0 \leq \delta < 1$ for all $x \in X$, we obtain the following result.

Corollary 1.1.12. ([10]) Let (X, d) be a complete metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and let $T : X \to CB(X)$ be a multivalued mapping. Suppose that

(1) T is a edge-preserving mapping , i.e., if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$;

(2) there exists $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in Tx_0$;

(3) X has Property A;

(4) there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + L \cdot D(y, Tx) \text{ for all } (x, y) \in E(G).$$

Then there exists $u \in X$ such that $u \in Tu$.

Recently, Tiammee *et al.* [34] introduced and proved the fixed point theorems for multivalued nonself G-almost contractions in Banach spaces endowed with graphs, for more details on this, see [34].

Many existence theorems of fixed points can be applied to obtain fixed point theorems for cyclic mappings and coupled fixed point theorems.

In 2003, Kirk *et al.* [6] introduced the definition of cyclic representations as the following:

Let X be a nonempty set, m a positive integer, $\{A_i\}_{i=1}^{m}$ nonempty closed subsets of X and $T : X \to X$ an operator. We call that $X = \bigcup_{i=1}^{m} A_i$ is a *cyclic representation of* X *with respect to* T if

$$T(A_1) \subset A_2, ..., T(A_{m-1}) \subset A_m, T(A_m) \subset A_1,$$

and operator T is known as a *cyclic operator*.

A partial ordering is a binary relation $\preceq$ over the set X which satisfies the following conditions:

1. $x \preceq x$ (reflexivity);

2. $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry);

3. $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity),

for all $x, y, z \in X$. A set with a partial ordering $\preceq$ is called *a partially ordered set*. We write $x \prec y$ if $x \preceq y$ and $x \neq y$.

Definition 1.1.13. ([35]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. The mapping F is said to have mixed monotone property if it is monotone nondecreasing in x and monotone nonincreasing in y, that is,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

The Fixed point theorems for monotone single valued mappings have been investigated and studied in partially ordered metric spaces by many authors, see [35, 36, 37, 38], for examples. They can apply to study an existence problem of ordinary differential equations.

In 2006, Bhasker and Lakshmikantham [35] introduced the concept of coupled fixed points for mixed monotone mappings in partially ordered metric spaces. Coupled fixed point theorems and their applications were investigated many directions by several authors, see [11, 15, 39, 40, 41, 42], for examples.

1.2 Best Proximity Point Theorems and Mean Nonexpansive Mappings

Let (X, d) be a metric space, A a nonempty subset of X and let $T : A \to X$ be a mapping. T is said to have a fixed point in A if the fixed point equation $Tx = x$ has at least one solution, that is, there exists $x \in A$ such that $d(x, Tx) = 0$. Some parts of this thesis, we consider in case that the equation does not have a solution, i.e., $d(x, Tx) > 0$ for all $x \in A$. Our aim is to find an element $x \in A$ such that the error $d(x, Tx)$ is minimum. The point x is said to be the best approximation of the fixed point of T. This is the idea behind best approximation theory.

In 1961, Fan [43] proved the following well-known best approximation theorem.

Theorem 1.2.1. ([43]) Let A be a nonempty compact convex set in a normed linear space X. If T is a continuous map from A into X, then there exists a point x in A such that $\|x - Tx\| = D(Tx, A)$.

An element x in the previous theorem is called a best approximation point of T in A. Note that if $x \in A$ is a best approximation point, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that, the minimization problem $\min\limits_{z \in A} \|z - Tz\|$ has at least one solution.

To have a concrete lower bound, let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. The natural problem is whether we can find an element $x_0 \in A$ such that

$$d(x_0, Tx_0) = \min_{x \in A} d(x, Tx).$$

Now, we denote that $D(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. Since $d(x, Tx) \geq D(A, B)$ for any $x \in A$, the interesting problem is to find a point $x \in A$ such that

$$d(x, Tx) = D(A, B).$$

It is called a *best proximity point of* T. In particular case, if $D(A, B) = 0$, then best proximity points of T are exactly fixed points of T. Many interesting results about best proximity points can be found in the following works [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55], for examples.

In 2014, Raj and Eldred [56] established some geometric properties on a strictly convex Banach space.

We will recall some notations for convenience. Let A, B be two nonempty subsets of a normed linear space X. Let

$$A_0 := \{x \in A : \|x - y\| = D(A, B), \quad \text{for some } y \in B\},$$

$$B_0 := \{y \in B : \|x - y\| = D(A, B), \quad \text{for some } x \in A\}.$$

The metric projection operator $P_A : X \to A$ is defined by

$$P_A(x) := \{y \in A : \|x - y\| = D(x, A)\} \text{ for all } x \in X,$$

where $D(x, A) := \inf\{d(x, a) : a \in A\}$. It is known that if A is assumed to be a closed and convex subset of a strictly convex and reflexive Banach space X, then $P_A(x)$ is nonempty and single valued for all $x \in X$, see [57].

In 2003, Kirk *et al.* [58] proved the following lemma to guarantee that A_0 and B_0 are nonempty.

Lemma 1.2.2. ([58]) Let X be a reflexive Banach space and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Then A_0 and B_0 are nonempty and satisfy $P_B(A_0) \subseteq B_0$ and $P_A(B_0) \subseteq A_0$.

Moreover, in 2000, Basha and Veeramani [59] showed that $A_0 \subseteq \partial A$ and $B_0 \subseteq \partial B$ where ∂E denotes the boundary of E for any $E \subseteq X$. It is easy to see that A_0 and B_0 are closed convex subsets of A and B, respectively, if A and B are closed and convex.

Recently, Kong *et al.* [60] proved the following best proximity point theorem of some mappings by using the results of Raj and Eldred [56].

Let A and B be two nonempty closed convex subsets of a strictly convex space X such that A_0 is nonempty. Let us define a mapping $P : A \cup B \to A \cup B$ as the following:

$$Px = \begin{cases} P_B(x) & \text{if } x \in A; \\ P_A(x) & \text{if } x \in B. \end{cases}$$

Corollary 1.2.3. ([60]) Let X be a uniformly convex Banach space and A, B be nonempty closed convex subsets of X such that A_0 is nonempty. Suppose that $T(A_0) \subseteq B_0$ and $T : A \to B$ is a nonexpansive mapping on A_0, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in A_0$. Then T has at least one best proximity point if and only if there exists $x \in A_0$ such that $\{(PT)^n(x)\}$ is bounded.

In 1975, Zhang [61] introduced a new mapping which is more general than that of nonexpansive mappings and proved the existence of a fixed point of this mapping in a Banach space, defined as follows:

Let A be a nonempty subset of Banach space X and T a mapping from A into A. A map T is called *mean nonexpansive* if there exist two nonnegative real numbers a and b with $a + b \leq 1$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|, \quad \text{for each } x, y \in A.$$

In 2014, Zuo [62] proved the following existence theorem for mean nonexpansive mappings in a reflexive Banach space which satisfies Opial's condition.

Motivated and inspired by all of these works mentioned above. First of all, we aim to study and prove the existence theorems for multivalued nonself Kannan-Berinde contraction mappings, which is a new class of multivalued nonself contractions and more general than that of Alghamdi, Berinde and Shahzad [29], in a complete metric space. Secondly, we will introduce multivalued nonself Kannan-Berinde G-contractions mappings and prove the existence theorems for this mapping in complete metric spaces with directed graphs, which extended those results in Assad and Kirk's fixed point theorem [20] and Tiammee, Charoensawan and Suantai [34]. Finally, by using the idea given by Sanker and Anthony [56] and Kong, Liu and Mu [60], we also aim to study and find some sufficient conditions for the existence of a best proximity point for mean nonexpansive mappings in a strictly convex Banach space. Moreover, we will apply all of obtained results for a coupled fixed point and fixed point theorem for some cyclic mappings and mean nonexpansive mappings. We also give some examples to illustrate all our main results.