CHAPTER 2

Basic Concepts and Preliminaries

In this chapter, we recall some basic and important definitions, lemmas, theorems and known results that are useful for the main results in the later chapter.

2.1 Metric Spaces

Definition 2.1.1. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. Then d is called a *distance function* or *metric* on X if the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$;
2. $d(x, y) = d(y, x)$ for each $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for each $x, y, z \in X$.

The valued of $d(x, y)$ is called a *distance between x and y*, and X together with d is called a *metric space* which will be denoted by (X, d).

Example 2.1.2.

1. The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ is a metric space. The metric d is called the *usual metric* for $\mathbb{R}$.
2. The Euclidian space $\mathbb{R}^n$ with
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2},$$
for each $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, is a metric space. The metric d is called the *Euclidian metric* for $\mathbb{R}^n$. Moreover, the Euclidean space $\mathbb{R}^n$ with
$$\sigma(x, y) = \sum_{k=1}^{n} |x_k - y_k|,$$
for each $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is also an example of metric spaces.

3. Let X be a nonempty set and let $d : X \times X \to \{0, 1\}$ be defined by for $x, y \in X$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \\ 1 & \text{if } x \neq y. \end{cases}$$

   Then (X, d) is a metric space, called a *discrete space*.

4. Let X be the set of all continuous scalar-valued functions on $[a, b]$. Define a metric d by

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

   for all $f, g \in X$. Then (X, d) is a metric space and usually denoted by $C[a, b]$.

Definition 2.1.3. Let C be a subset of a metric space (X, d). The *diameter of set C*, which is denoted by $diam(C)$, is defined

$$diam(C) := \sup\{d(x, y) : x, y \in C\}.$$

If $diam(C) < \infty$, then C is said to be *bounded*, and if not, then C is said to be *unbounded*.

Definition 2.1.4. Let (X, d) be a metric space. A sequence $\{x_n\}$ *converges to* $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$, i.e., for each $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon, \text{ for all } n \geq n_0.$$

We usually symbolize this by writing $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Definition 2.1.5. Let $\{x_n\}$ be a sequence in a metric space (X, d). It is said to be a *Cauchy sequence* if for each $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon, \text{ for all } m, n \geq n_0.$$

The metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges, that is, has a limit which is an element of X.

Theorem 2.1.6. A subspace C of a complete metric space (X, d) is itself complete if and only if the set C is closed in X.

Observations

- In a metric space, every convergent sequence is Cauchy, but the converse need not be true in general.

- Euclidean space $\mathbb{R}^n$ with the Euclidean metric is complete.
- Let X be the space of ordered n-tuples $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ of real numbers and

$$d(x, y) = \max_{1 \le k \le n} |x_k - y_k|.$$

Then (X, d) is complete.

Theorem 2.1.7. Let C be a nonempty subset of a metric space (X, d). Then C is *closed* if and only if for any sequence $\{x_n\}$ in C, if $\{x_n\} \to x$, then $x \in C$.

Theorem 2.1.8. Let (X, d) be a metric space. Then

1. $\emptyset$ and X are closed in X.
2. If $C_1, C_2, ..., C_n$ are closed in X, then $\bigcup_{k=1}^{n} C_k$ is also closed in X.
3. Let A be an arbitrary set. If C_α is closed in X for each $\alpha \in A$, then $\bigcap_{\alpha \in A} C_\alpha$ is also closed in X.

Theorem 2.1.9. Let $\{x_n\}$ be a sequence in a metric space (X, d) and $x \in X$. Then $\{x_n\}$ converges to x if and only if any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x.

Definition 2.1.10. Let X and Y be metric spaces and T a mapping of X into Y. Then T is said to be *continuous* at a in X if $x_n \to a$, then $Tx_n \to Ta$. A mapping T of X into Y is continuous if it is continuous at each x in X.

2.2 Some Useful Propositions and Lemmas in Metric Spaces

Let (X, d) be a metric space and $CB(X)$ the set of all nonempty closed bounded subsets of X. Let A be a subset of X and any $x \in X$. The *distance from x to A* is defined by

$$D(x, A) := \inf\{d(x, y) : y \in A\}.$$

For $A, B \in CB(X)$, we define

$$H(A, B) := \max\left\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\right\}.$$

It is known that H is called a *Pompeiu-Hausdorff metric* on $CB(X)$ induced by d on X. It is also known that $(CB(X), H)$ is a complete metric space whenever (X, d) is a complete metric space.

A metric space (X, d) is called *metrically convex* or *convex* if for each $x, y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that

$$d(x, y) = d(x, z) + d(z, y).$$

We known that in a convex metric space each two points are the endpoint of at least one metric segment, see [20]. The following proposition and lemmas are useful for our main results.

Proposition 2.2.1. ([20]) Let (X, d) be a complete and convex metric space, K a nonempty closed subset of X. If $x \in K$ and $y \notin K$, then there exists a point z in the boundary of K, denote by ∂K, such that

$$d(x, y) = d(x, z) + d(z, y).$$

For convenience, we denote

$$P[x, y] := \{z \in \partial K : d(x, y) = d(x, z) + d(z, y)\}.$$

The following lemmas are direct consequences of the definition of Pompeiu-Hausdroff metric.

Lemma 2.2.2. For $A, B \in CB(X)$ and $a \in A$, then

$$D(a, B) \leq H(A, B).$$

Lemma 2.2.3. Let $A, B \in CB(X)$ and $k > 1$ be given. Then for $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq kH(A, B).$$

2.3 Normed Spaces and Banach Spaces

The aim of this section is to give definitions and some geometrics properties in Normed spaces and Banach spaces.

Definition 2.3.1. A linear space or vector space X over the field $\mathbb{F}$ (the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{F} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$:

1. $x + y = y + x$;

2. $x + (y + z) = (x + y) + z$;

3. there exists a *zero element* in X, denoted by 0, such that $x + 0 = x$ for all $x \in X$;

4. for each element $x \in X$, there exists an element $-x$, called the *additive inverse* or the *negative of* x, such that $x + (-x) = 0$;

5. $\alpha(x + y) = \alpha x + \alpha y$;

6. $(\alpha + \beta)x = \alpha x + \beta x$;

7. $(\alpha\beta)x = \alpha(\beta x)$;

8. $1 \cdot x = x$;

The element of a vector space X are called *vectors*, and the elements of $\mathbb{F}$ are called *scalars*. X denotes a linear space over filed $\mathbb{F}$.

Definition 2.3.2. Let X be a vector space. A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all elements $x, y \in X$ and each scalar α:

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

2. $\|\alpha x\| = |\alpha|\|x\|$;

3. $\|x + y\| \leq \|x\| + \|y\|$.

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* or *normed vector space*. It is easy to verify that the normed space X is a metric space with respect to the metric d defined by $d(x, y) = \|x - y\|$. A *Banach space* is a complete metric space with respected by normed space.

Definition 2.3.3. Let X be a linear space over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). An *inner product* on X is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$ with the following three properties:

1. $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, where the bar denotes filed conjugation;

3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$.

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space.* $\langle x, y \rangle$ is call inner product of two elements. An inner product on X defines a norm on X given by $||x|| = \sqrt{\langle x, x \rangle}$. A *Hilbert space* is a complete inner product space.

Example 2.3.4. Let $X = \mathbb{R}^n$, $n > 1$ be a linear space. The $\mathbb{R}^n$ is a normed space with the following norms:

(*i*) $||x||_1 = \sum_{k=1}^{n} |x_k|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$;

(*ii*) $||x||_2 = \left[\sum_{k=1}^{n} |x_k|^2\right]^{\frac{1}{2}} = \sqrt{|x_1|^2 + ... + |x_n|^2}$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, which is called that *Euclidean norm*;

(*iii*) $||x||_\infty = \max_{1 \leq k \leq n} |x_k|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition 2.3.5. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* or *convergent* in the norm if there exists a point x in X such that $\lim_{n\to\infty} ||x_n - x|| = 0$. In this case, we write either $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.3.6. A nonempty subset C of a normed space X is said to be *convex* if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

Definition 2.3.7. A Banach space X is said to be *strictly convex* if for $x, y \in X$ with $||x|| = ||y|| = 1$ and $x \neq y$, one has

$$\frac{||x + y||}{2} < 1.$$

Definition 2.3.8. A Banach space X is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ (depending only on ε) such that

$$\frac{||x + y||}{2} \leq 1 - \delta,$$

for all $x, y \in X$ with $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$.

Observations

- X is a strictly convex space if and only if for $x, y \in S(E) = \{x \in X : ||x|| = 1\}$ with $||x|| = ||y|| = \frac{||x + y||}{2}$, then $x = y$.
- Any Hilbert space is uniformly convex.
- Every uniformly convex Banach space is strictly convex.

- The Euclidean plane $\mathbb{R}^2$ is uniformly convex. But $\mathbb{R}^2$ with $||\cdot||_\infty$ is not uniformly convex.

- The Euclidean plane $\mathbb{R}^n$ for $n \geq 2$ with norm $||\cdot||_1$ is not strictly convex.

Proposition 2.3.9. ([57]) Let A be a nonempty closed convex subset of a reflexive strictly convex Banach space X. Then for $x \in X$, there exists a unique point $z_x \in C$ such that $||x - z_x|| = D(x, A)$.

Let $(X, ||\cdot||)$ be a normed space. We denoted by X^* the dual space of X, i.e., the space of all bounded linear functionals on a normed space X, and by X^{**} the second dual space of X. For each $x \in X$, we define a functional $g_x : X^* \to \mathbb{F}$ by

$$g_x(f) = f(x) \text{ for all } f \in X^*. \tag{2.1}$$

Theorem 2.3.10. For each $x \in X$, the functional g_x defined by (2.1) is a bounded linear functional on X^*, so that $g_x \in X^{**}$ and has the norm $||g_x|| = ||x||$.

Now we define a mapping $C : X \to X^{**}$ by $x \mapsto g_x$ and call it the *canonical mapping* of X into X^{**}.

Definition 2.3.11. A normed space X is said to be *reflexive* if $C(X) = X^{**}$ where $C : X \to X^{**}$ is the canonical mapping.

Observations

- Every uniformly convex Banach space is reflexive.

- Every finite-dimensional Banach space is reflexive.

Definition 2.3.12. A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there exists a point x in X such that $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in X^*$.

Definition 2.3.13. Let X be a Banach space. We say that X satisfies *Opial's condition* if for each $x \in X$ and each sequence $\{x_n\}$ weakly converging to x, then for all $y \neq x$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Example 2.3.14. Every Hilbert space satisfies the Opial's condition.

2.4 Graph Theory and Some Multivalued Nonself Generalized G-contraction Mappings

We now move on some basic and definitions in graph theory. Let $G = (V(G), E(G))$ be a directed graph such that $V(G)$ a set of vertices of a graph and $E(G)$ a set of its edges. Let $\Delta := \{(x, x) : x \in V(G)\}$, say that the diagonal of $V(G) \times V(G)$, be a set of all loops in G. Assume that G has no parallel edges, thus one can identify G with the pair $(V(G), E(G))$. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges, that is,

$$E(G^{-1}) := \{(x, y) \in V(G) \times V(G) : (y, x) \in E(G)\}.$$

Property A. ([63]) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for any $n \in \mathbb{N}$.

In order to obtain our main results, we need to know the following definition of domination in a graph, see [64, 65]:

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is said to be a *dominating set* if for all $v \in V(G) \setminus X$, there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x *dominates* v or v is *dominated by* x. For each $v \in V(G)$, a set $X \subseteq V(G)$ is dominated by v if $(v, x) \in E(G)$ for all $x \in X$ and we call that v dominates X if $(v, x) \in E(G)$ for all $x \in X$.

Definition 2.4.1. ([10]) Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$, and let $T : X \to CB(X)$. Then T is said to be *edge-preserving* if for any $x, y \in X$,

$$(x, y) \in E(G) \text{ implies } (u, v) \in E(G),$$

for all $u \in Tx$ and $v \in Ty$.

Definition 2.4.2. Let (X, d) be a metric space, K a nonempty subset of X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$.

(1) A mapping $T : K \to CB(X)$ is said to be a *multivalued almost G-contraction* if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that, for any $x, y \in K$,

$$H(Tx, Ty) \leq \delta d(x, y) + L \cdot D(y, Tx)$$

whenever $(x, y) \in E(G)$.

(2) A mapping $T : K \to CB(X)$ is said to be a *multivalued G-contraction* if there

exists $k \in (0,1)$ such that, for any $x, y \in K$,

$$H(Tx, Ty) \leq kd(x, y)$$

whenever $(x, y) \in E(G)$.

2.5 Fixed Point Theorem for Mean Nonexpansive Mappings

Definition 2.5.1. Let X be a Banach space. A mapping $T : X \to X$ is said to be *nonexpansive* if for $x, y \in X$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Example 2.5.2. Let $X = \mathbb{R}$ and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = 1 - x, \text{ for all } x \in [0, 1].$$

Then T is a nonexpansive mapping.

Definition 2.5.3. ([61]) Let A be a nonempty subset of Banach space X and T a mapping from A into A. A map T is called *mean nonexpansive* if there exist two nonnegative real numbers a and b with $a + b \leq 1$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|, \text{ for each } x, y \in A.$$

Example 2.5.4. ([62]) Let T be the unit interval defined by

$$Tx = \begin{cases} \frac{x}{5} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{6} & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

and the norm is the usual norm on the line. Here, T is a mean nonexpansive mapping on unit interval by taking $a = \frac{1}{3}$ and $b = \frac{2}{3}$ but it is not nonexpansive.

Theorem 2.5.5. ([62]) Let X be a real reflexive Banach space which satisfies Opial's condition, A a nonempty bounded closed convex subset of X, and $T : A \to A$ a mean nonexpansive mapping. Then T has a fixed point.

2.6 A Characterization of Strictly Convex Spaces and Useful Properties

Now, we recall the d-property in a normed linear space and its characterization related to strictly convexity of Banach spaces which was introduced and studied by Raj and Eldred [56].

Definition 2.6.1. ([56]) A pair (A, B) of nonempty subsets of a normed linear space X is said to have the d-property if and only if

$$\left.\begin{aligned} ||x_1 - y_1|| = D(A, B) \\ ||x_2 - y_2|| = D(A, B) \end{aligned}\right\} \Longrightarrow ||x_1 - x_2|| = ||y_1 - y_2||,$$

whenever $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.6.2. ([56]) A normed linear space X is said to have the d-property if and only if every pair (A, B) of nonempty closed convex subsets of X has the d-property.

Theorem 2.6.3. ([56]) X is strictly convex if and only if X has the d-property.

The following result is obtained by Theorem 2.6.3 which is useful for our main results.

Corollary 2.6.4. ([56]) Let A and B be nonempty closed convex subsets of a strictly convex space X such that A_0 is nonempty. Then the restriction of the metric projection mapping P_{A_0} to B_0 is an isometry, i.e., $P_{A_0} : B_0 \to A_0$ is an isometry, that is,

$$||P_{A_0}(x) - P_{A_0}(y)|| = ||x - y|| \text{ for all } x, y \in B_0.$$

Notice that, from Corollary 2.6.4, we also have that

$$||P_{B_0}(x) - P_{B_0}(y)|| = ||x - y|| \text{ for all } x, y \in A_0.$$

Let A and B be nonempty closed convex subsets of a strictly convex space X such that A_0 is nonempty. Consider the mapping $P : A \cup B \to A \cup B$ defined by

$$Px = \begin{cases} P_B(x) & \text{if } x \in A; \\ P_A(x) & \text{if } x \in B. \end{cases}$$

We give some properties of the mapping P as in the following propositions.

Proposition 2.6.5. Let X be a reflexive strictly convex Banach space and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Then $Px = P_B(x) = P_{B_0}(x)$ and $Py = P_A(y) = P_{A_0}(y)$ for all $x \in A_0$ and $y \in B_0$, respectively.

Proof. By Theorem 1.2.2, we have that A_0 is nonempty. Let $x \in A_0$ be given. From $P_B(A_0) \subseteq B_0$, we get

$$||x - P_B(x)|| = D(x, B_0) = ||x - P_{B_0}(x)||.$$

By Proposition 2.3.9, we obtain that $Px = P_B(x) = P_{B_0}(x)$ for all $x \in A_0$.
Similarly, for each $y \in B_0$, by $P_A(B_0) \subseteq A_0$, we get

$$||y - P_A(y)|| = D(y, A_0) = ||y - P_{A_0}(y)||.$$

Again, by Proposition 2.3.9, we obtain that $Py = P_A(y) = P_{A_0}(y)$ for all $y \in B_0$. □

Proposition 2.6.6. Let A and B be two nonempty subsets of a normed linear space X such that A_0 is nonempty. Let $T : A \to B$ be a nonself mapping such that $T(A_0) \subseteq B_0$. Then, a point $x \in A_0$ is the fixed point of PT if and only if $||x - Tx|| = D(A, B)$.

Proof. Suppose that $x \in A_0$. Since $T(A_0) \subseteq B_0$, we get that $Tx \in B_0$. We also note that

$$\begin{aligned} x = (PT)(x) = P(Tx) &\Leftrightarrow ||x - Tx|| = ||P(Tx) - Tx|| \\ &\Leftrightarrow ||x - Tx|| = ||P_{A_0}(Tx) - Tx|| \\ &\Leftrightarrow ||x - Tx|| = D(Tx, A_0) \\ &\Leftrightarrow ||x - Tx|| = \inf_{z \in A_0} ||Tx - z|| \\ &\Leftrightarrow ||x - Tx|| = D(A, B). \end{aligned}$$

Hence the result is obtained. □

Lemma 2.6.7. ([60]) Let A and B be nonempty closed convex subsets of a strictly convex space X such that A_0 is nonempty. Then $P^2(x) = x$ for all $x \in A_0 \cup B_0$.

2.7 Cyclic Mappings and Couple Fixed Points

Definition 2.7.1. ([6]) Let X be a nonempty set, m a positive integer, $\{A_i\}_{i=1}^m$ nonempty closed subsets of X and $T : X \to X$ an operator. We call that $X = \bigcup_{i=1}^{m} A_i$ is a *cyclic representation of* X *with respect to* T if

$$T(A_1) \subset A_2, ..., T(A_{m-1}) \subset A_m, T(A_m) \subset A_1,$$

and operator T is known as a *cyclic operator.*

Example 2.7.2. ([66]) Let $X = \mathbb{R}$ with the usual metric. Consider the closed nonempty subset of X as follows:

$$A_1 = [0, 2], A_2 = \left[\frac{1}{3}, 1\right], A_3 = \left[\frac{2}{3}, 1\right], A_4 = \left[1, \frac{4}{3}\right], A_5 = \left[1, \frac{5}{3}\right] \quad \text{and} \quad A_6 = [1, 2].$$

Then $W = \bigcup_{i=1}^{6} A_i = [0, 2] = A_1$

Let $f : W \to W$ be defined by

$$f(A_1) = \begin{cases} \frac{1}{2} & ; 0 \leq x < \frac{1}{3} \\ 1 & ; \text{otherwise}, \end{cases} \qquad f(A_2) = \begin{cases} \frac{5}{6} & ; \frac{1}{3} \leq x < \frac{2}{3} \\ 1 & ; \text{otherwise}, \end{cases}$$

$$f(A_3) = \begin{cases} \frac{7}{6} & ; \frac{2}{3} \leq x < \frac{5}{6} \\ 1 & ; \text{otherwise}, \end{cases} \qquad f(A_4) = \begin{cases} \frac{3}{2} & ; 1 \leq x < \frac{7}{6} \\ 1 & ; \text{otherwise}, \end{cases}$$

$$f(A_5) = \begin{cases} \frac{11}{6} & ; \frac{4}{3} \leq x < \frac{3}{2} \\ 1 & ; \text{otherwise}, \end{cases} \qquad f(A_6) = \begin{cases} 2 & ; \frac{5}{3} \leq x < \frac{11}{6} \\ 1 & ; \text{otherwise}. \end{cases}$$

Then $f(A_1) \subset A_2, f(A_2) \subset A_3, f(A_3) \subset A_4, f(A_4) \subset A_5, f(A_5) \subset A_6$ and $f(A_6) \subset A_1$. Thus, $\bigcup_{i=1}^{6} A_i$ is a cyclic representation of W with respect to f.

Let K be a nonempty subset of X and $F : K \times K \to X$ a single valued mapping. An element $(x, y) \in K \times K$ is said to be *a coupled fixed point* of F if $x = F(x, y)$ and $y = F(y, x)$. We denote $CFix(F)$ is the set of all coupled fixed points of the mapping F, that is,

$$CFix(F) = \{(x, y) \in K \times K \ : \ F(x, y) = x \text{ and } F(y, x) = y\}.$$

Definition 2.7.3. We say that $F : K \times K \to X$ is *edge-preserving* if for each $x, y, u, v \in K$ with $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$, then we have that $(F(x, y), F(u, v)) \in E(G)$ and $(F(y, x), F(v, u)) \in E(G^{-1})$.

Let (X, d) be a metric space, K a nonempty closed subset of X and let $Z = X \times X$ and $Y = K \times K$. Then the mapping $\eta : Z \times Z \to [0, \infty)$ defined by

$$\eta((x, y), (u, v)) = d(x, u) + d(y, v) \text{ for all } (x, y), (u, v) \in Z.$$

It is easy to show that η is a metric on Z and (X, d) is complete if and only if (Z, η) is complete.

Proposition 2.7.4. If (X, d) is convex, then (Z, η) is convex.

Proof. Let (X, d) be convex and $(x, y), (u, v) \in Z$ such that $(x, y) \neq (u, v)$. If $x = u$ and $y \neq v$, using X is convex, then there exists $b \in X$ such that $y \neq b \neq v$ and

$$d(y, v) = d(y, b) + d(b, v).$$

Since $d(x, u) = d(x, x) + d(x, u)$, there exists $(x, b) \in Z$ such that

$$\eta((x, y), (u, v)) = \eta((x, y), (x, b)) + \eta((x, b), (u, v)).$$

Similarly in case $x \neq u$ and $y = v$. There exists $(a, y) \in Z$ such that

$$\eta((x, y), (u, v)) = \eta((x, y), (a, y)) + \eta((a, y), (u, v)).$$

Now, we consider that if $x \neq u$ and $y \neq v$. Since X is convex, there exist $c, d \in X$ such that $x \neq c \neq u$ and $y \neq d \neq v$ and $d(x, u) = d(x, c) + d(c, u)$, $d(y, v) = d(y, d) + d(d, v)$. Then $\eta((x, y), (u, v)) = \eta((x, y), (c, d)) + \eta((c, d), (u, v))$. Hence, Z is convex. □

Let G_Y be a directed graph defined by $G_Y = (V(G_Y), E(G_Y))$ where $V(G_Y) := Y$ and

$$E(G_Y) := \{((x, y), (u, v)) : (x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1})\}.$$

For a mapping $F : Y \to X$, we define the mapping $T_F : Y \to Z$ by

$$T_F(x, y) = (F(x, y), F(y, x)) \text{ for all } (x, y) \in Y.$$

If $(x, y) \in Y$ with $T_F(x, y) \notin Y$, by Proposition 2.2.1, then we can always choose an $(u, v) \in \partial Y$ such that

$$\eta((x, y), T_F(x, y)) = \eta((x, y), (u, v)) + \eta((u, v), T_F(x, y)),$$

we denote that

$$P_{T_F}(x, y) := \{(u, v) \in \partial Y : \eta((x, y), T_F(x, y)) = \eta((x, y), (u, v)) + \eta((u, v), T_F(x, y))\}.$$

Let us recall the definition of domination, a set $Y \subset Z$ is dominated by (u, v) if $((u, v), (x, y)) \in E(G_Y)$ for any $(x, y) \in Y$ and we say that Y dominates (u, v) if $((x, y), (u, v)) \in E(G_Y)$ for all $(x, y) \in Y$.

Note that an element $(x, y) \in Y$ is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F.