CHAPTER 3

Main Results

The purpose of this chapter is to prove our main results and their applications. We also give some examples to support them.

3.1 Fixed Point Theorems for Multivalued Nonself Kannan-Berinde Contraction Mappings in Complete Metric Spaces

We first introduce and study a new type of nonself multivalued contraction, called Kannan-Berinde contraction mapping, which is more general than that of Berinde's contraction and prove its fixed point theorem under some conditions.

Definition 3.1.1. Let (X, d) be a metric space and K a nonempty subset of X. A mapping $T : K \to CB(X)$ is said to be a *multivalued Kannan-Berinde contraction* if there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for any $x, y \in K$.

Example 3.1.2. Let $X = \{0, 1, 2\}$ and $K = \{0, 1\}$. Define a map $T : K \to CB(X)$ by

$$Tx = \begin{cases} \{1, 2\} & \text{if } \ x = 0; \\ \{0, 2\} & \text{if } \ x = 1. \end{cases}$$

Then we see that

$$\begin{aligned} H(T(0), T(1)) = 1 & \\ & = \frac{1}{2} \cdot (1) + \frac{1}{4}[1 + 1] + L \cdot (0) \\ & = \frac{1}{2} d[0, 1) + \frac{1}{4}[D(0, T(0)) + D(1, T(1))] + L \cdot D(1, T(0)), \end{aligned}$$

and for any $0 \leq \delta < 1$ and $L \geq 0$,

$$H(T(0), T(1)) = 1 > \delta \cdot (1) + L \cdot (0) = \delta d[0, 1) + L \cdot D(1, T(0)).$$

Hence T is a multivalued Kannan-Berinde contraction for $\delta = \frac{1}{2}, a = \frac{1}{4}$ and $L \geq 0$ arbitrary but T is not a multivalued almost contraction.

Theorem 3.1.3. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

(i) T satisfies Rothe's type condition, that is, $x \in \partial K$ implies $Tx \subset K$;

(ii) T is a multivalued Kannan-Berinde contraction mapping with

$$\delta(1 + a + L) + a(3 + L) < 1.$$

Then T has a fixed point in K.

Proof. From the assumption (ii), $\delta(1+a+L)+a(3+L) < 1$, there exists $k > 1$ such that

$$\delta(1 + a + L) + a(3 + L) < \frac{1}{k^2} < 1.$$

Then we get

$$k^2[\delta(1 + a + L) + a(3 + L)] < 1.$$

We note that $ka, k\delta < 1$ and

$$\begin{aligned} k^2[\delta(1 + a + L) + a(3 + L)] &= k^2[\delta + 3a + \delta a + \delta L + aL)] \\ &\geq k^2[\frac{\delta + 3a}{k} + \delta a + \delta L + aL)] \\ &= k(\delta + 3a) + k^2(\delta a + \delta L + aL) \\ &= k\delta + 3ka + k^2\delta a + k^2\delta L + k^2 aL. \end{aligned}$$

So we have

$$\begin{aligned} k\delta + 3ka + k^2\delta a + k^2\delta L + k^2 aL &< 1, \\ k\delta + ka + k^2\delta a + k^2\delta L + k^2 aL + k^2a^2 &< 1 - 2ka + k^2a^2, \\ k\delta(1 + ka + kL) + ka(1 + ka + kL) &< (1 - ka)^2, \\ (1 + ka + kL)(k\delta + ka) &< (1 - ka)^2. \end{aligned}$$

Hence

$$\frac{(1 + ka + kL)(k\delta + ka)}{(1 - ka)^2} < 1.$$

Now, we construct two sequences $\{x_n\}$ and $\{y_n\}$ as the following. Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, we denote $x_1 = y_1$. Consider in case $y_1 \notin K$. By Proposition 2.2.1, there exists $x_1 \in P[x_0, y_1]$ such that

$$d(x_0, y_1) = d(x_0, x_1) + d(x_1, y_1).$$

So we have $x_1 \in K$, and, by Lemma 2.2.3, there exists $y_2 \in Tx_1$ such that

$$d(y_1, y_2) \leq kH(Tx_0, Tx_1).$$

If $y_2 \in K$, we denote $x_2 = y_2$. Otherwise, $y_2 \notin K$, then there exists $x_2 \in P[x_1, y_2]$ such that

$$d(x_1, y_2) = d(x_1, x_2) + d(x_2, y_2).$$

Thus $x_2 \in K$, by Lemma 2.2.3, there exists $y_3 \in Tx_2$ such that

$$d(y_2, y_3) \leq kH(Tx_1, Tx_2).$$

Continuing the arguments, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) $y_{n+1} \in Tx_n$;

(2) $d(y_n, y_{n+1}) \leq kH(Tx_{n-1}, Tx_n)$, where

  (a) $x_n = y_n$ if and only if $y_n \in K$;

  (b) $x_n \in P[x_{n-1}, y_n]$ if and only if $y_n \notin K$, i.e., $x_n \neq y_n$ and

$$x_n \in \partial K \text{ such that } d(x_{n-1}, y_n) = d(x_{n-1}, x_n) + d(x_n, y_n).$$

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence.
Now, we put

$$P_1 := \{x_i \in \{x_n\} : x_i = y_i, i = 1, 2, ...\};$$

$$P_2 := \{x_i \in \{x_n\} : x_i \neq y_i, i = 1, 2, ...\}.$$

Note that $\{x_n\} \subset K$.
Now, we will show that we cannot have two consecutive terms of $\{x_n\}$ in the set P_2, that is, if $x_i \in P_2$, then x_{i-1} and x_{i+1} belong to the set P_1. Let $x_i \in P_2$. Then $x_i \neq y_i$. Assume that $x_{i-1} \in P_2$. So $x_{i-1} \in \partial K$. By the assumption (i), we have $Tx_{i-1} \subset K$. Since $y_i \in Tx_{i-1}$, we get that $y_i \in K$. Then $x_i = y_i$. This is a contradiction. Thus $x_{i-1} \in P_1$. On the other hand, since $x_i \in P_2$, we have $x_i \in \partial K$. So, $Tx_i \subset K$. Since $y_{i+1} \in Tx_i \subset K$,

we also get that $x_{i+1} = y_{i+1}$. We conclude that $x_{i+1} \in P_1$. This is done. Now, for $n \geq 2$, we consider the three possibilities as the following.

Case 1. If $x_n, x_{n+1} \in P_1$, then $x_n = y_n$ and $x_{n+1} = y_{n+1}$.

Then we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \\
&\leq kH(Tx_{n-1}, Tx_n) \\
&\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\
&\quad + kL \cdot D(x_n, Tx_{n-1}). \\
&\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, y_{n+1}) \\
&= k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, x_n) + kad(x_n, x_{n+1}),
\end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-1}, x_n).$$

Case 2. If $x_n \in P_1$ and $x_{n+1} \in P_2$, then $x_n = y_n$ and $x_{n+1} \in P[x_n, y_{n+1}]$, i.e.,

$$d(x_n, y_{n+1}) = d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}).$$

From (ii), we have

$$\begin{aligned}
d(y_n, y_{n+1}) &\leq kH(Tx_{n-1}, Tx_n) \\
&\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\
&\quad + kL \cdot D(x_n, Tx_{n-1}). \\
&\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, y_{n+1}) \\
&= k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, x_n) + kad(y_n, y_{n+1}),
\end{aligned}$$

which follows that

$$d(y_n, y_{n+1}) \leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-1}, x_n).$$

So, we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, y_{n+1}) - d(x_{n+1}, y_{n+1}) \\
&\leq d(x_n, y_{n+1}) \\
&= d(y_n, y_{n+1}) \\
&\leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-1}, x_n).
\end{aligned}$$

Case 3. If $x_n \in P_2$ and $x_{n+1} \in P_1$, then $x_{n-1} \in P_1$, that is, $x_{n-1} = y_{n-1}$, $x_{n+1} = y_{n+1}$, and $x_n \in P[x_{n-1}, y_n]$, that is,

$$d(x_{n-1}, y_n) = d(x_{n-1}, x_n) + d(x_n, y_n).$$

Since $y_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and $k\delta < 1$, by (ii), we have

$$\begin{aligned}
d(y_n, y_{n+1}) &\leq kH(Tx_{n-1}, Tx_n) \\
&\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\
&\quad + kL \cdot D(x_n, Tx_{n-1}). \\
&\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, y_{n+1}) + kLd(x_n, y_n) \\
&\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) + kLd(x_n, y_n).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq d(x_n, y_n) + d(y_n, x_{n+1}) \\
&= d(x_n, y_n) + d(y_n, y_{n+1}) \\
&\leq d(x_n, y_n) + k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) \\
&\quad + kLd(x_n, y_n) \\
&= (1 + kL)d(x_n, y_n) + k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) \\
&\leq (1 + kL)d(x_n, y_n) + (1 + kL)d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) \\
&\quad + kad(x_n, x_{n+1}) \\
&= (1 + kL)[d(x_n, y_n) + d(x_{n-1}, x_n)] + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) \\
&= (1 + kL)d(x_{n-1}, y_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}),
\end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq \left(\frac{1 + ka + kL}{1 - ka}\right) d(x_{n-1}, y_n) = \left(\frac{1 + ka + kL}{1 - ka}\right) d(y_{n-1}, y_n).$$

Since $x_{n-1} \in P_1$ and $x_n \in P_2$, it follows from Case 2 that

$$d(y_{n-1}, y_n) \leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-2}, x_{n-1}).$$

Thus

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \left(\frac{1 + ka + kL}{1 - ka}\right)\left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-2}, x_{n-1}) \\
&= \frac{(1 + ka + kL)(k\delta + ka)}{(1 - ka)^2} \cdot d(x_{n-2}, x_{n-1}).
\end{aligned}$$

Since

$$h := \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} < 1,$$

we obtain that

$$d(x_n, x_{n+1}) \leq hd(x_{n-2}, x_{n-1}).$$

We note that

$$\frac{k\delta+ka}{1-ka} \leq \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)} \leq \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} = h.$$

From Case 1, Case 2 and Case 3, we can conclude that for $n \geq 2$,

$$d(x_n, x_{n+1}) = \begin{cases} hd(x_{n-1}, x_n) & \text{if } \ x_n, x_{n+1} \in P_1 \text{ or } x_n \in P_1, x_{n+1} \in P_2; \\ hd(x_{n-2}, x_{n-1}) & \text{if } \ x_n \in P_2, x_{n+1} \in P_1. \end{cases}$$

Following Assad and Kirk [20], we get that for $n \geq 2$,

$$d(x_n, x_{n+1}) \leq r \cdot h^{n/2},$$

where $r := h^{-1/2} \cdot \max\{d(x_0, x_1), d(x_1, x_2)\}$.

In order to prove this by induction.

For $n = 2$. If $x_2, x_3 \in P_1$ or $x_2 \in P_1, x_3 \in P_2$, then we get that

$$\begin{aligned} d(x_2, x_3) &\leq hd(x_1, x_2) \\ &= h^{-1/2} \cdot h^{3/2} d(x_1, x_2) \\ &\leq r \cdot h^{3/2} \\ &\leq r \cdot h. \end{aligned}$$

On the other hand, if $x_2 \in P_2, x_3 \in P_1$, we also get that

$$\begin{aligned} d(x_2, x_3) &\leq hd(x_0, x_1) \\ &= h^{-1/2} \cdot h^{3/2} d(x_0, x_1) \\ &\leq r \cdot h^{3/2} \\ &\leq r \cdot h. \end{aligned}$$

Now, we assume that $d(x_n, x_{n+1}) \leq r \cdot h^{n/2}$ for all $1 \leq n \leq N$ and for $N \geq 2$. We must prove that

$$d(x_{N+1}, x_{N+2}) \leq r \cdot h^{(N+1)/2}.$$

We consider in two cases as follows:

If $x_N, x_{N+1} \in P_1$ or $x_N \in P_1, x_{N+1} \in P_2$, then we get that

$$\begin{aligned} d(x_{N+1}, x_{N+2}) &\leq hd(x_N, x_{N+1}) \\ &\leq h(r \cdot h^{N/2}) \\ &= r \cdot h^{(N+2)/2} \\ &\leq r \cdot h^{(N+1)/2}. \end{aligned}$$

if $x_N \in P_2, x_{N+1} \in P_1$, then we also get that

$$\begin{aligned} d(x_{N+1}, x_{N+2}) &\leq hd(x_{N-1}, x_N) \\ &\leq h(r \cdot h^{(N-1)/2}) \\ &= r \cdot h^{(N+1)/2}. \end{aligned}$$

This proves the assertion.

Next we consider for $m > n$, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq r \cdot h^{n/2} + r \cdot h^{(n+1)/2} + ... + r \cdot h^{(m-1)/2} \\ &= r \cdot (h^{n/2} + h^{(n+1)/2} + .. + h^{(m-1)/2}). \end{aligned}$$

Since $h < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in K. Since X is complete and K is closed, there exists $x \in K$ such that

$$\lim_{n \to \infty} x_n = x.$$

Further, from the construction of $\{x_n\}$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \subset P_1$. Then $x_{n_j} = y_{n_j} \in Tx_{n_j - 1}$. Finally, we show that x is a fixed point. Since

$$0 \leq D(x, Tx_{n_j-1}) \leq d(x, x_{n_j})$$

for each $j \in \mathbb{N}$, it follows that $D(x, Tx_{n_j-1}) \to 0$ as $j \to \infty$.

For each $j \in \mathbb{N}$, we have

$$\begin{aligned} D(x, Tx) &= \inf_{z \in Tx} d(x, z) \\ &\leq \inf_{z \in Tx} (d(x, x_{n_j}) + d(x_{n_j}, z)) \\ &= d(x, x_{n_j}) + \inf_{z \in Tx} d(x_{n_j}, z) \end{aligned}$$

$$
\begin{aligned}
&= d(x, x_{n_j}) + D(x_{n_j}, Tx) \\
&\leq d(x, x_{n_j}) + H(Tx_{n_j-1}, Tx) \\
&\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + a[D(x_{n_j-1}, Tx_{n_j-1}) + D(x, Tx)] \\
&\quad + L \cdot D(x, Tx_{n_j-1}) \\
&\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + ad(x_{n_j-1}, x_{n_j}) + aD(x, Tx) \\
&\quad + L \cdot D(x, Tx_{n_j-1}).
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
(1-a)D(x, Tx) &\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + ad(x_{n_j-1}, x_{n_j}) + L \cdot D(x, Tx_{n_j-1}) \\
&\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + ad(x_{n_j-1}, x) + ad(x, x_{n_j}) \\
&\quad + L \cdot D(x, Tx_{n_j-1}).
\end{aligned}
$$

Letting $j \to \infty$, we get

$$(1-a)D(x, Tx) = 0.$$

Since $0 \leq a < \frac{1}{3}$, we get that $D(x, Tx) = 0$, hence $x \in Tx$, that is, T has a fixed point in K. This completes the proof. □

As a consequence of Theorem 3.1.3, when we put $a = 0$, we obtain Theorem 9 of [29] as our special case as follows.

Corollary 3.1.4. (Theorem 9 of [29]) Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

(*i*) T has the Rothe's boundary condition;

(*ii*) there exist $\delta \in [0, 1)$ and $L \geq 0$ with $\delta(1 + L) < 1$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + L \cdot D(y, Tx),$$

for any $x, y \in K$.

Then T has a fixed point in K.

If we put $a = 0$ and $L = 0$ in Theorem 3.1.3, then we also obtain Theorem 1 of [20].

Corollary 3.1.5. (Theorem 1 of [20]) Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

(i) T has the Rothe's boundary condition;

(ii) there exists $\delta \in [0, 1)$ such that

$$H(Tx, Ty) \leq \delta d(x, y),$$

for any $x, y \in K$.

Then T has a fixed point in K.

Next, we give an example to illustrate Theorem 3.1.3.

Example 3.1.6. Let $X = \mathbb{R}$ and $K = \left[0, \frac{1}{2}\right]$ endowed with usual metric, that is, $d(x, y) = |x - y|$ for all $x, y \in X$. Define a mapping $T : K \to CB(X)$ by

$$Tx = \begin{cases} \left[0, \frac{x}{10}\right] & \text{if } x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]; \\ \{-\frac{1}{8}\} & \text{if } x = \frac{1}{5}; \\ \{\frac{1}{2}\} & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$$

We see that

$$\partial K = \left\{0, \frac{1}{2}\right\} \Rightarrow T(0) \text{ and } T\left(\frac{1}{2}\right) \text{ are subset of } K,$$

which implies T satisfies Roth's boundary condition. Now we show that T is a multivalued Kannan-Berinde contraction satisfying all conditions of Theorem 3.1.3. We consider the following six cases:

Case 1. If $x = y = \frac{1}{5}$ or $x, y \in \left(\frac{1}{4}, \frac{1}{2}\right]$, then

$$H(Tx, Ty) = 0 \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for any $\delta \in [0, 1), a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$.

Case 2. If $x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$ and $y = \frac{1}{5}$, we note that $\left|\frac{x}{10} + \frac{1}{8}\right| \leq \frac{3}{20}$ and $\left|\frac{1}{5} - \frac{x}{10}\right| \geq \frac{7}{40}$. Then, we have

$$H(Tx, Ty) = H\left(\left[0, \frac{x}{10}\right], \left\{-\frac{1}{8}\right\}\right)$$

$$
\begin{aligned}
&= \left|\frac{x}{10} + \frac{1}{8}\right| \\
&\leq \frac{3}{20} \\
&\leq \delta \left|x - \frac{1}{5}\right| + a\left[\frac{9}{10}x + \frac{13}{40}\right] + L \cdot \left|\frac{1}{5} - \frac{x}{10}\right| \\
&= \delta d(x,y) + a[D(x,Tx) + D(y,Ty)] + L \cdot D(y,Tx),
\end{aligned}
$$

when $L \geq \frac{6}{7}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L) + a(3+L) < 1$.

Case 3. If $x = \frac{1}{5}$ and $y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then $\left|\frac{y}{10} + \frac{1}{8}\right| \leq \frac{3}{20}$ and $\left|y + \frac{1}{8}\right| \geq \frac{1}{8}$. So, we have

$$
\begin{aligned}
H(Tx,Ty) &= \left|\frac{y}{10} + \frac{1}{8}\right| \\
&\leq \frac{3}{20} \\
&\leq \delta \left|\frac{1}{5} - y\right| + a\left[\frac{13}{40} + \frac{9}{10}y\right] + L \cdot \left|y + \frac{1}{8}\right|,
\end{aligned}
$$

when $L \geq \frac{6}{5}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L) + a(3+L) < 1$.

Case 4. If $x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$ and $y \in \left(\frac{1}{4}, \frac{1}{2}\right]$, then $\left|\frac{x}{10} - \frac{1}{2}\right| \leq \frac{1}{2}$ and $\left|y - \frac{x}{10}\right| > \frac{9}{40}$. So, we have

$$
\begin{aligned}
H(Tx,Ty) &= \left|\frac{x}{10} - \frac{1}{2}\right| \\
&\leq \frac{1}{2} \\
&\leq \delta|x - y| + a\left[\frac{9}{10}x + \frac{1}{2} - y\right] + L \cdot \left|y - \frac{x}{10}\right|.
\end{aligned}
$$

when $L \geq \frac{20}{9}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L) + a(3+L) < 1$.

Case 5. If $x \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then $\left|\frac{1}{2} - \frac{y}{10}\right| \leq \frac{1}{2}$ and $\left|y - \frac{1}{2}\right| \geq \frac{1}{4}$. So, we have

$$
\begin{aligned}
H(Tx,Ty) &= \left|\frac{1}{2} - \frac{y}{10}\right| \\
&\leq \frac{1}{2} \\
&\leq \delta\,|x - y| + a\left[\frac{1}{2} - x + \frac{9}{10}y\right] + L \cdot \left|y - \frac{1}{2}\right|,
\end{aligned}
$$

when $L \geq 2, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L) + a(3+L) < 1$.

Case 6. If $x, y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then we have

$$\begin{aligned}
H(Tx, Ty) &= H\left(\left[0, \frac{x}{10}\right], \left[0, \frac{y}{10}\right]\right) \\
&= \left|\frac{x}{10} - \frac{y}{10}\right| \\
&\leq \frac{x}{10} + \frac{y}{10} \\
&= \frac{1}{10}(x+y) \\
&= \frac{1}{9} \cdot \frac{9}{10}(x+y) \\
&= \frac{1}{9}\left(\frac{9}{10}x + \frac{9}{10}y\right) \\
&= \frac{1}{9}\left[\left(x - \frac{x}{10}\right) + \left(y - \frac{y}{10}\right)\right] \\
&= \frac{1}{9}[D(x, Tx) + D(y, Ty)] \\
&\leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx).
\end{aligned}$$

We choose that $a = \frac{1}{9}$, $0 \leq \delta < 1$ and $L \geq 0$ such that $\delta(1 + a + L) + a(3 + L) < 1$. Now, by summarizing all cases, we conclude that T is a multivalued Kannan-Berinde contraction with $a = \frac{1}{9}$, $L = \frac{20}{9}$ and

$$0 \leq \delta < \frac{1 - \frac{1}{9}(3 + \frac{20}{9})}{1 + \frac{1}{9} + \frac{20}{9}} = \frac{17}{135},$$

which the condition $\delta(1+a+L)+a(3+L) < 1$ is also satisfied. Therefore, T is a multivalued Kannan-Berinde contraction that all assumptions in Theorem 3.1.3, and there exist $z \in K$ such that $z \in Tz$. Notice that

$$F(T) = \left\{0, \frac{1}{2}\right\}.$$

However, we see that T is not multivalued contraction mapping. If we put $x = \frac{1}{2}$ and $y = \frac{1}{5}$, then

$$H\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{5}\right)\right) = H\left(\left\{\frac{1}{2}\right\}, \left\{-\frac{1}{8}\right\}\right) = \frac{5}{8} > k \cdot \frac{3}{10} = kd\left(\frac{1}{2}, \frac{1}{5}\right),$$

for all $0 \leq k < 1$.

3.2 Fixed Point Theorems for Multivalued Nonself Kannan-Berinde G-contraction Mappings in Complete Metric Spaces Endowed with Graphs

In this section, we introduce a multivalued Kannan-Berinde G-contraction mapping in a metric space endowed with a directed graph and prove some fixed point theorems.

Definition 3.2.1. Let (X, d) be a metric space, K a nonempty subset of X and $G := (V(G), E(G))$ be a directed graph such that $V(G) = K$. A mapping $T : K \to CB(X)$ is said to be a *multivalued Kannan-Berinde G-contraction* if there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for all $x, y \in K$ with $(x, y) \in E(G)$.

Remark 3.2.2. Let (X, d) be a metric space. Consider the graph G_1 defined by $G_1 = X \times X$. Then any multivalued Kannan-Berinde contraction mapping is multivalued Kannan-Berinde G-contraction mapping.

Theorem 3.2.3. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Let $G := (V(G), E(G))$ be a directed graph such that $V(G) = K$. Suppose that K has Property A. If a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

(i) there exists $x_0 \in K$ such that $(x_0, y) \in E(G)$ for some $y \in Tx_0$;

(ii) T is an edge-preserving mapping, that is, if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$;

(iii) for each $x \in K$ and $y \in Tx$ with $y \notin K$,

  (a) $P[x, y]$ is dominated by x and

  (b) for each $z \in P[x, y]$, z dominates Tz;

(iv) T has Rothe's boundary condition;

(v) T is a multivalued Kannan-Berinde G-contraction mapping with

$$\delta(1 + a + L) + a(3 + L) < 1.$$

Then T has a fixed point in K.

Proof. From the assumption (v), as in Theorem 3.1.3, there exists $k > 1$ such that

$$\frac{(1 + ka + kL)(k\delta + ka)}{(1 - ka)^2} < 1.$$

Now, we construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Let $x_0 \in K$ be such that $(x_0, y_1) \in E(G)$ for some $y_1 \in Tx_0$. If $y_1 \in K$, we denote $x_1 = y_1$. Consider in case $y_1 \notin K$. By Proposition 2.2.1, there exists $x_1 \in P[x_0, y_1]$ such that

$$d(x_0, y_1) = d(x_0, x_1) + d(x_1, y_1).$$

By $(iii) - (a)$, that is, $P[x_0, y_1]$ is dominated by x_0, we obtain $(x_0, x_1) \in E(G)$. Moreover, we have $x_1 \in K$, and, by Lemma 2.2.3, there exists $y_2 \in Tx_1$ such that

$$d(y_1, y_2) \leq kH(Tx_0, Tx_1).$$

If $y_2 \in K$, we denote $x_2 = y_2$. When $y_1 \in K$, that is, $x_1 = y_1 \in Tx_0$ and $x_2 = y_2 \in Tx_1$, using T is edge-preserving, we get that $(x_1, x_2) \in E(G)$. When $y_1 \notin K$, that is, $x_1 \in P[x_0, y_1]$, by $(iii) - (b)$, we also get $(x_1, x_2) \in E(G)$. Otherwise, if $y_2 \notin K$, then there exists $x_2 \in P[x_1, y_2]$ such that

$$d(x_1, y_2) = d(x_1, x_2) + d(x_2, y_2).$$

Thus $(x_1, x_2) \in E(G)$ and $x_2 \in K$, by Lemma 2.2.3, there exists $y_3 \in Tx_2$ such that

$$d(y_2, y_3) \leq kH(Tx_1, Tx_2).$$

Continuing the arguments, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) $y_{n+1} \in Tx_n$;

(2) $d(y_n, y_{n+1}) \leq kH(Tx_{n-1}, Tx_n)$, where

- $x_n = y_n$ if and only if $y_n \in K$;
- $x_n \in P[x_{n-1}, y_n]$ if and only if $y_n \notin K$, i.e., $x_n \neq y_n$ and

$$x_n \in \partial K \text{ such that } d(x_{n-1}, y_n) = d(x_{n-1}, x_n) + d(x_n, y_n).$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that

$$P_1 := \{x_i \in \{x_n\} : x_i = y_i, i = 1, 2, ...\};$$

$$P_2 := \{x_i \in \{x_n\} : x_i \neq y_i, i = 1, 2, ...\}.$$

Note that $\{x_n\} \subset K$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Moreover, if $x_i \in P_2$, then x_{i-1} and x_{i+1} belong to the set P_1. By virtue of (iv), we cannot have two consecutive terms of $\{x_n\}$ in the set P_2. Now, for $n \geq 2$, we consider the distance $d(x_n, x_{n+1})$. The three possibilities as follows.

Case 1. $x_n \in P_1$ and $x_{n+1} \in P_1$

Case 2. $x_n \in P_1$ and $x_{n+1} \in P_2$

Case 3. $x_n \in P_2$ and $x_{n+1} \in P_1$

Since $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, using the same proof as in Theorem 3.1.3, we obtain that $\{x_n\}$ is a Cauchy sequence in K. Since X is complete and K is closed, there exists $x \in K$ such that $\lim_{n\to\infty} x_n = x$. Moreover, from $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and by Property A, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$. From the construction of $\{x_n\}$, there is an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \subset P_1$. Then $x_{n_j} = y_{n_j} \in Tx_{n_j-1}$. Now, we show that $x \in Tx$. Since $\lim_{j\to\infty} x_{n_j} = x$, we have $0 \leq D(x, Tx_{n_j-1}) \leq d(x, x_{n_j})$. So, $D(x, Tx_{n_j-1}) \to 0$ as $j \to \infty$. Then for each $j \in \mathbb{N}$,

$$\begin{aligned}
D(x,Tx) &\leq d(x,x_{n_j}) + D(x_{n_j},Tx)\\
&\leq d(x,x_{n_j}) + H(Tx_{n_j-1},Tx)\\
&\leq d(x,x_{n_j}) + \delta d(x_{n_j-1},x) + a[D(x_{n_j-1},Tx_{n_j-1}) + D(x,Tx)]\\
&\quad + L\cdot D(x,Tx_{n_j-1})\\
&\leq d(x,x_{n_j}) + \delta d(x_{n_j-1},x) + ad(x_{n_j-1},x_{n_j}) + aD(x,Tx) + L\cdot D(x,Tx_{n_j-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
(1-a)D(x,Tx) &\leq d(x,x_{n_j}) + \delta d(x_{n_j-1},x) + ad(x_{n_j-1},x_{n_j}) + L\cdot D(x,Tx_{n_j-1})\\
&\leq d(x,x_{n_j}) + \delta d(x_{n_j-1},x) + ad(x_{n_j-1},x) + ad(x,x_{n_j})\\
&\quad + L\cdot D(x,Tx_{n_j-1}).
\end{aligned}$$

Letting $j \to \infty$, we obtain $(1-a)D(x,Tx) = 0$. Since $a \in \left[0, \frac{1}{3}\right)$, we get that $D(x,Tx) = 0$. Hence $x \in Tx$, that is, T has a fixed point in K. This completes the proof. □

Remark 3.2.4. As a consequence of Theorem 3.2.3, we can conclude the following results as our special cases:

(*i*) If we put $E(G) = K \times K$, i.e., G is a complete graph, in Theorem 3.2.3, we obtain Theorem 3.1.3.

(*ii*) If we put $a = 0$ in Theorem 3.2.3, we obtain Theorem 5 of [34].

(*iii*) If we put $a = 0$ and $E(G) = K \times K$ in Theorem 3.2.3, we obtain Theorem 9 of [29].

(*iv*) If we put $a = L = 0$ and $E(G) = K \times K$ in Theorem 3.2.3, we also obtain Theorem 1 of [20].

Next, we give an example to illustrate Theorem 3.2.3.

Example 3.2.5. Let $X = \mathbb{R}$ and $K = [0,1] \cup \{2\}$ endowed with usual metric $d(x,y) = |x-y|$. Let $G = (V(G), E(G))$ be a graph consisting of $V(G) := K$ and

$$E(G) := \left\{(0,0), \left(\frac{1}{9}, 0\right)\right\} \cup \left\{\left(0, \frac{1}{3^n}\right), \left(\frac{1}{3^n}, 0\right), \left(\frac{1}{3^m}, \frac{1}{3^n}\right) : m, n \in \mathbb{N}\backslash\{2\}\right\}.$$

Notice that K has Property A. Define a mapping $T : K \to CB(X)$ by

$$Tx = \begin{cases} \{1,2\} & \text{if } x = 2; \\ \{-\frac{1}{27}\} & \text{if } x = \frac{1}{9}; \\ \left\{0, \frac{x}{9}\right\} & \text{otherwise.} \end{cases}$$

We see that

$$\partial K = \{0,1,2\} \Rightarrow T(0), T(1) \text{ and } T(2) \text{ are subset of } K,$$

which implies T satisfies Roth's boundary condition. We choose $x_0 = 0 \in K$ and $y' = 0 \in \{0\} = Tx_0$, and so $(x_0, y') = (0,0) \in E(G)$. Since the only $x \in K$ and $y \in Tx$ with $y \notin K$ are $x = \frac{1}{9}$ and $y = -\frac{1}{27}$, we get that $P\left[\frac{1}{9}, -\frac{1}{27}\right] = \{0\}$. Further, we have $\left(\frac{1}{9}, 0\right), (0,0) \in E(G)$. Hence $P[x,y]$ is dominated by x and 0 dominates $T(0)$. Next, we prove that T is edge-preserving. Let $(x,y) \in E(G)$. Then we get that $x, y \in \{0\} \cup \left\{\frac{1}{3^k} : k \in \mathbb{N}\backslash\{2\}\right\}$. We obtain that $Tx, Ty \subset \{0\} \cup \left\{\frac{1}{3^k} : k \in \mathbb{N}\backslash\{1,2,4\}\right\}$. Then for all $u \in Tx$ and $v \in Ty$, we get that $(u,v) \in E(G)$, which show that T is edge-preserving. Now we claim that T is a multivalued Kannan-Berinde G-contraction mapping. Let $(x,y) \in E(G)$. We will discuss the following three possible cases.

Case 1. If $(x,y) = (0,0)$, then $H(T(0), T(0)) = 0$.

Case 2. If $(x,y) = \left(\frac{1}{9}, 0\right)$, we have

$$\begin{aligned} H(Tx,Ty) &= H\left(\left\{-\frac{1}{27}\right\}, \{0\}\right) \\ &= \frac{1}{27} \\ &\leq \delta \cdot \frac{1}{9} + a \cdot \frac{4}{27} + L \cdot \frac{1}{27} \\ &= \delta\left|\frac{1}{9} - 0\right| + a\left[D\left(\frac{1}{9}, \left\{-\frac{1}{27}\right\}\right) + D(0,\{0\})\right] + L \cdot D\left(0, \left\{-\frac{1}{27}\right\}\right) \\ &= \delta d(x,y) + a[D(x,Tx) + D(y,Ty)] + L \cdot D(y,Tx), \end{aligned}$$

where $L \geq 1, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L) + a(3+L) < 1$.

Case 3. If $(x, y) = \left(0, \frac{1}{3^n}\right)$ or $\left(\frac{1}{3^n}, 0\right)$ or $\left(\frac{1}{3^m}, \frac{1}{3^n}\right)$ for all $m, n \in \mathbb{N}\backslash\{2\}$, we have

$$\begin{aligned}
H(Tx, Ty) &= H\left(\left\{0, \frac{x}{9}\right\}, \left\{0, \frac{y}{9}\right\}\right) \\
&= \left|\frac{x}{9} - \frac{y}{9}\right| \\
&\leq \frac{x}{9} + \frac{y}{9} \\
&= \frac{1}{8}\left(\frac{8}{9}x + \frac{8}{9}y\right) \\
&= \frac{1}{8}\left[\left(x - \frac{x}{9}\right) + \left(y - \frac{y}{9}\right)\right] \\
&= \frac{1}{8}\left[D\left(x, \left\{0, \frac{x}{9}\right\}\right) + D\left(y, \left\{0, \frac{y}{9}\right\}\right)\right] \\
&\leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx),
\end{aligned}$$

where $a = \frac{1}{8}$, $0 \leq \delta < 1$ and $L \geq 0$ such that $\delta(1+a+L) + a(3+L) < 1$.

Now, by summarizing all cases, we can conclude that T is a multivalued Kannan-Berinde G-contraction with $a = \frac{1}{8}$, $L = 1$ and

$$0 \leq \delta < \frac{1 - \frac{1}{8}(3+1)}{1 + \frac{1}{8} + 1} = \frac{4}{17},$$

which the condition $\delta(1+a+L)+a(3+L) < 1$ is also satisfied. Therefore, T is a multivalued Kannan-Berinde G-contraction mapping that satisfies all assumptions in Theorem 3.2.3. Then there exist $z \in K$ such that $z \in Tz$. Notice that $F(T) = \{0, 2\}$.

However, T is not multivalued Kannan-Berinde contraction mapping because if we take $x = 2 \in K$ and $y = 1 \in K$, then

$$\begin{aligned}
H(T(2), T(1)) &= H\left(\{1, 2\}, \left\{0, \frac{1}{9}\right\}\right) \\
&= \frac{17}{9} \\
&> \delta \cdot (1) + a \cdot \left(\frac{8}{9}\right) + L \cdot (0) \\
&= \delta \cdot |2-1| + a\left[D(2, \{1, 2\}) + D\left(1, \left\{0, \frac{1}{9}\right\}\right)\right] + L \cdot D(1, \{1, 2\}) \\
&= \delta d(2, 1) + a[D(2, T(2)) + D(1, T(1))] + L \cdot D(1, T(2))
\end{aligned}$$

for all $0 \leq \delta < 1$, $0 \leq a < \frac{1}{3}$ and $L \geq 0$.

The following results are obtained directly from Theorem 3.2.3 in case that T is a single valued mapping.

Corollary 3.2.6. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Let $G := (V(G), E(G))$ be a directed graph such that $V(G) = K$. Suppose that K has Property A. If a map $T : K \to X$ is a single valued mapping satisfying the following properties:

(i) there exists $x_0 \in K$ such that $(x_0, Tx_0) \in E(G)$;

(ii) T is edge-preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$;

(iii) for each $x \in K$ with $Tx \notin K$,

- (a) $P[x, Tx]$ is dominated by x and
- (b) $(z, Tz) \in E(G)$ for all $z \in P[x, Tx]$;

(iv) T has Rothe's boundary condition, i.e., $T(\partial K) \subset K$;

(v) there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that for any $x, y \in K$ with $(x, y) \in E(G)$, we have

$$d(Tx, Ty) \leq \delta d(x, y) + a[d(x, Tx) + d(y, Ty)] + Ld(y, Tx).$$

Then T has a fixed point in K.

If we put $K = X$ in Corollary 3.2.6, we have the following Corollary.

Corollary 3.2.7. Let (X, d) be a complete convex metric space and let $G := (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that X has Property A. If a self-map $T : X \to X$ is a single valued mapping satisfying the following properties:

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;

(ii) T is edge-preserving;

(iii) there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that for any $x, y \in X$ with $(x, y) \in E(G)$, we have

$$d(Tx, Ty) \leq \delta d(x, y) + a[d(x, Tx) + d(y, Ty)] + Ld(y, Tx).$$

Then T has a fixed point.

3.3 Applications for Single Valued Nonself Kannan-Berinde G-contraction Mappings in Complete Metric Spaces Endowed with Graphs

We apply our main results to obtain a fixed point theorem for some cyclic mappings and prove the existence of a coupled fixed point for a single valued nonself Kannan-Berinde G-contraction mapping in a complete metric space endowed with graphs.

Theorem 3.3.1. Let (X,d) be a complete metric space, m a positive integer and $\{A_i\}_{i=1}^m$ nonempty closed subsets of X. Suppose that $W = \bigcup\limits_{i=1}^m A_i$ and an operator $T : W \to W$. If $\bigcup\limits_{i=1}^m A_i$ is a cyclic representation of W with respect to T and there exist $\delta \in [0,1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ with $\delta(1+a+L) + a(3+L) < 1$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + a[d(x,Tx) + d(y,Ty)] + Ld(y,Tx)$$

for any $x \in A_i, y \in A_{i+1}, i = 1,2,...,m$ where $A_{m+1} = A_1$, then T has at least one fixed point $t \in \bigcap\limits_{i=1}^m A_i$.

Proof. Since for each A_i where $i = 1,2,...,m$ are closed, we obtain that $W = \bigcup\limits_{i=1}^m A_i$ is closed. Then (W,d) is a complete metric space. Defined a directed graph $G = (V(G),E(G))$ by $V(G) := W$ and

$$E(G) := \Delta \cup \{(x,y) \in W \times W : x \in A_i, y \in A_{i+1}, i = 1,...,m \text{ where } A_{m+1} = A_1\}.$$

We show that T is edge-preserving. Let $(x,y) \in E(G)$. If $(x,y) \in \Delta$, then $(Tx,Ty) \in \Delta$. If $(x,y) \notin \Delta$, then $x \in A_i, y \in A_{i+1}$ for each $i = 1,2,...,m$. Since $\bigcup\limits_{i=1}^m A_i$ is a cyclic representation of W with respect to T, we obtain that $Tx \in A_{i+1}$ and $Ty \in A_{i+2}$, that is, $(Tx,Ty) \in E(G)$. So T is edge-preserving. By definition of the graph G, we have T is a Kannan-Berinde G-contraction mapping for all $(x,y) \in E(G)$ and there exists $x_0 \in W$ such that $(x_0,Tx_0) \in E(G)$. Finally, we claim that W has a Property A. Let $\{x_n\}$ be a sequence in W such that $x_n \to x^* \in W$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \geq 1$. Now we will consider two cases as the following:

Case 1. If $\{x_n : n \in \mathbb{N}\}$ is a finite set. Since $x_n \to x^*$, there exists $n_0 \in \mathbb{N}$ such that $x_n = x^*$ for all $n \geq n_0$. Since $(x_n,x_{n+1}) \in E(G)$ for all $n \geq 1$, i.e., if $x_n \in A_j$ for some $j \in \mathbb{N}$, then $x_{n+1} \in A_{j+1}$, so that $x^* \in A_i$ for all $i = 1,2,...,m$. So, $(x_n,x^*) \in E(G)$ for all $n \in \mathbb{N}$. Then W has a Property A.

Case 2. If $\{x_n : n \in \mathbb{N}\}$ is an infinite set. We will show that the sequence $\{x_n\}$ has infinitely many terms in each A_i where $i = 1, 2, ..., m$. Assume that $\{x_n\}$ has finite terms in A_{j_0} for some $j_0 \in \mathbb{N}$. Since $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, we have A_i has the finite terms of $\{x_n\}$ for all $i = 1, 2, ..., m$. So the set $\{x_n : n \in \mathbb{N}\}$ is finite, which is a contradiction. Then the sequence $\{x_n\}$ has infinitely many terms in each A_i. So that for each A_i, there exists a subsequence of $\{x_n\}$ such that converge to some $x^* \in W$. Since A_i is closed, we obtain that $x^* \in A_i$ for all $i = 1, 2, ..., m$. Then $x^* \in \bigcap_{i=1}^{m} A_i$. Moreover, by the defined graph G, we get that $(x_n, x^*) \in E(G)$ for all $n \in \mathbb{N}$. So W has a Property A. Thus, by Corollary 3.2.7, we can conclude that T has a fixed point $t \in \bigcap_{i=1}^{m} A_i$. □

Now, we move on the next topic that we prove the existence for a coupled fixed point for a single valued mapping in a complete metric space endowed with a directed graph.

Theorem 3.3.2. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Let $G = (V(G), E(G))$ a directed graph such that $V(G) = K$. Let $F : Y = K \times K \to X$ be an edge-preserving mapping such that $F(\partial Y) \subset K$. Suppose the following properties hold:

(i) there exist $x_0, y_0 \in K$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$;

(ii) K has the following properties:

(a) if any sequence $\{x_n\}$ in K such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;

(b) if any sequence $\{y_n\}$ in K such that $y_n \to y$ and $(y_n, y_{n+1}) \in E(G^{-1})$ for $n \in \mathbb{N}$, then $(y_n, y) \in E(G^{-1})$ for all $n \in \mathbb{N}$.

(iii) there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and some $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that

$$\begin{aligned}
&d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
&\quad \leq \delta[d(x, u) + d(y, v)] \\
&\qquad + a[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))] \\
&\qquad + L[d(u, F(x, y)) + d(v, F(y, x))]
\end{aligned}$$

for all $x, y, u, v \in X$ with $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$.

If for each $(x, y) \in Y$ with $T_F(x, y) \notin Y$ such that $P_{T_F}(x, y)$ is dominated by (x, y) and for each $(u, v) \in P_{T_F}(x, y)$ dominates $T_F(u, v)$, then F has a coupled fixed point.

Proof. Now, we shall show that the mapping T_F satisfies all conditions of Corollary 3.2.6. Since there exists $(x_0, y_0) \in Y$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$, we obtain that $((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \in E(G_Y)$. So we get that $((x_0, y_0), T_F(x_0, y_0)) \in E(G_Y)$. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset Y$ such that $(x_n, y_n) \to (x, y)$ and $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G_Y)$ for all $n \in \mathbb{N}$. Then $x_n \to x, y_n \to y, (x_n, x_{n+1}) \in E(G)$, and $(y_n, y_{n+1}) \in E(G^{-1})$. Using (ii), we get that $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G^{-1})$. Thus $((x_n, y_n), (x, y)) \in E(G_Y)$ for all $n \in \mathbb{N}$. This shows that Y has the property A. Let $(x, y), (u, v) \in Y$ such that $((x, y), (u, v)) \in E(G_Y)$. We have $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. Since F is edge-preserving, we have $(F(x, y), F(u, v)) \in E(G)$ and $(F(y, x), F(v, u)) \in E(G^{-1})$, which implies $((F(x, y), F(y, x)), (F(u, v), F(v, u))) \in E(G_Y)$. Hence $(T_F(x, y), T_F(u, v)) \in E(G_Y)$. So T_F is edge-preserving. Next, we will prove that T_F is a Kannan-Berinde G-contraction mapping. Let $((x, y), (u, v)) \in E(G_Y)$. Then $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. By condition (iii), we consider as the following:

$$
\begin{aligned}
&\eta(T_F(x, y), T_F(u, v)) \\
&\quad = \eta((F(x, y), F(y, x)), (F(u, v), F(v, u))) \\
&\quad = d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
&\quad \leq \delta[d(x, u) + d(y, v)] \\
&\qquad + a[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))] \\
&\qquad + L[d(u, F(x, y)) + d(v, F(y, x))] \\
&\quad = \delta\eta((x, y), (u, v)) \\
&\qquad + a[\eta((x, y), (F(x, y), F(y, x))) + \eta((u, v), (F(u, v), F(v, u)))] \\
&\qquad + L\eta((u, v), (F(x, y), F(y, x))) \\
&\quad = \delta\eta((x, y), (u, v)) + a[\eta((x, y), T_F(x, y)) + \eta((u, v), T_F(u, v))] \\
&\qquad + L\eta((u, v), T_F(x, y)),
\end{aligned}
$$

it is satisfied. For each $(x, y) \in \partial Y = \partial(K \times K) = (\partial K \times K) \cup (K \times \partial K)$. Then $(y, x) \in \partial Y$. Since $F(\partial Y) \subset K$, we get that $F(x, y), F(y, x) \in K$. Thus $(F(x, y), F(y, x)) \in Y$, that is, $T_F(x, y) \in Y$. We obtain that $T_F(\partial Y) \subset Y$. By condition of T_F, assumption (iii) of Corollary 3.2.6 is satisfied. Thus all conditions of Corollary 3.2.6 are satisfied. Hence there exists $(x, y) \in Y$ such that $(x, y) = T_F(x, y)$ which implies that F has a coupled fixed point. $\square$

As a consequence of Theorem 3.3.2, we consider on a partially ordered set, the obtained result as the following:

Corollary 3.3.3. Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and K a nonempty closed subset of X. Suppose that $F : K \times K \to X$ is a mapping such that $F(\partial(K \times K)) \subset K$ and satisfies the following properties:

(i) for $x, y, u, v \in K$, if $x \preceq u$ and $y \succeq v$ imply $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$;

(ii) there exist $x_0, y_0 \in K$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$;

(iii) K has the following properties:

  (a) if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,

  (b) if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$,

(iv) there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and some $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\quad \leq \delta[d(x, u) + d(y, v)] \\ &\qquad + a[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))] \\ &\qquad + L[d(u, F(x, y)) + d(v, F(y, x))] \end{aligned}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$.

Suppose that for each $(u, v) \in P_{T_F}(x, y)$, we have $x \preceq u, u \preceq F(u, v), y \succeq v$ and $v \succeq F(v, u)$ for all $(x, y) \in K \times K$ with $T_F(x, y) \notin K \times K$. Then F has a coupled fixed point.

Proof. Define a directed graph $G = (V(G), E(G))$ where $V(G) := K$, $E(G) := \{(x, y) \mid x \preceq y\}$. First, we show that F is edge-preserving. Let $x, y, u, v \in K$ such that $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. Then $x \preceq u$ and $y \succeq v$. From (i), we get that $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. So $(F(x, y), F(u, v)) \in E(G)$ and $(F(y, x), F(v, u)) \in E(G^{-1})$. Thus F is edge-preserving. From (ii), we see that there exists $(x_0, y_0) \in K \times K$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$. Let $\{x_n\}$ and $\{y_n\}$ be sequence in K such that $x_n \to x$, $y_n \to y$, $(x_n, x_{n+1}) \in E(G)$ and $(y_n, y_{n+1}) \in E(G^{-1})$. Then $x_n \preceq x_{n+1}$

and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$. So $\{x_n\}$ is nondecreasing and $\{y_n\}$ is nonincreasing, by (iii), we have $x_n \preceq x$ and $y_n \succeq y$. Thus $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G^{-1})$ for all $n \in \mathbb{N}$. From (iv), there exist $\delta \in [0,1)$, $a \in \left[0, \frac{1}{3}\right)$ and some $L \geq 0$ with $\delta(1+a+L)+a(3+L) < 1$ such that

$$\begin{aligned}
&d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\
&\quad \leq \delta[d(x,u) + d(y,v)] \\
&\qquad + a[d(x, F(x,y)) + d(y, F(y,x)) + d(u, F(u,v)) + d(v, F(v,u))] \\
&\qquad + L[d(u, F(x,y)) + d(v, F(y,x))]
\end{aligned}$$

for all $x, y, u, v \in X$ with $(x,u) \in E(G)$ and $(y,v) \in E(G^{-1})$. Moreover, for each $(x,y) \in K \times K$ with $T_F(x,y) \notin K \in K$, by our assumption, we have $(x,u), (u, F(u,v)) \in E(G)$ and $(y,v), (v, F(v,u)) \in E(G^{-1})$. We obtain that $((x,y),(u,v)), ((u,v),(F(u,v),F(v,u)) \in E(G_Y)$. Therefore all conditions of Theorem 3.3.2 are satisfied, so that F has a coupled fixed point. $\square$

Example 3.3.4. Let $X = \mathbb{R}, K = [0,1]$ and $d(x,y) = |x-y|$ for all $x, y \in X$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$ and

$$E(G) = \{(1,1)\} \cup \left\{(x,y) : x, y \in \left[0, \frac{1}{8}\right]\right\}.$$

Define the mapping $F : K \times K \to X$ by

$$F(x,y) = \begin{cases} 1 & \text{if } \ x = y = 1; \\ -\frac{1}{8} & \text{if } \ x = y = \frac{1}{8}; \\ \frac{x}{8} & \text{if } \ \text{otherwise.} \end{cases}$$

Notice that F is edge-preserving. Let $(x,u) \in E(G)$ and $(y,v) \in E(G^{-1})$. Then $F(x,y) = F(u,v) = F(y,x) = F(v,u) = 1$ if $(x,u) = (y,v) = (1,1)$, otherwise we have $F(x,y), F(u,v), F(y,x), F(v,u) \in \left[0, \frac{1}{8}\right]$. So we get that $(F(x,y), F(u,v)) \in E(G)$ and $(F(y,x), F(v,u)) \in E(G^{-1})$. Hence F is edge-preserving. Moreover, we also see that

$$\partial([0,1] \times [0,1]) = \{(0,x), (x,0), (1,x), (x,1) : x \in [0,1]\},$$

which implies $F(\partial([0,1] \times [0,1])) = \left[0, \frac{1}{8}\right] \cup \{1\} \subset [0,1]$. Since there is only $(1,1) \in K \times K$ and $(1, F(1,1)) = (1,1) \in E(G) \cap E(G^{-1})$. So (i) of Theorem 3.3.2 is satisfied. From the definition of the graph G, we obtain that K satisfies properties $(ii)-(a)$ and $(ii)-(b)$

of Theorem 3.3.2. Now, we will show that F satisfies condition of inequality in (iii). Let $(x,u) \in E(G)$ and $(y,v) \in E(G^{-1})$. We want to show that there exist $\delta \in [0,1)$, $a \in \left[0, \frac{1}{3}\right)$ and some $L \geq 0$ with $\delta(1+a+L)+a(3+L)<1$ such that

$$\begin{aligned} & d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \\ & \quad \leq \delta[d(x,u)+d(y,v)] \\ & \qquad + a[d(x,F(x,y)) + d(y,F(y,x)) + d(u,F(u,v)) + d(v,F(v,u))] \\ & \qquad + L[d(u,F(x,y)) + d(v,F(y,x))] \end{aligned}$$

which considers six possible cases depending on the valued of x, u, y and v.

Case 1. If $(x,u)=(y,v)=(1,1)$ or $(x,u)=(y,v)=\left(\frac{1}{8},\frac{1}{8}\right)$ or $(x,u)=\left(\frac{1}{8},\frac{1}{8}\right),(y,v)=(1,1)$, then $d(F(x,y),F(u,v))+d(F(y,x),F(v,u))=0$. So that it satisfied with $0 \leq \delta < 1, 0 \leq a < \frac{1}{3}$ and $L \geq 0$ with $\delta(1+a+L)+a(3+L)<1$.

Case 2. If $(x,u)=(1,1)$ and $(y,v) \in \left[0,\frac{1}{8}\right] \times \left[0,\frac{1}{8}\right]$, then

$$\frac{1}{8}|y-v| \leq \delta \cdot |y-v| + a\left[\frac{7}{4}+\frac{7}{8}y+\frac{7}{8}v\right] + L\left[\frac{7}{8}+\left|v-\frac{y}{8}\right|\right].$$

Since $y,v \in \left[0,\frac{1}{8}\right]$, we have $\frac{1}{8}|y-v| \leq \frac{1}{64}$ and $\frac{7}{8}+\left|v-\frac{y}{8}\right| \geq \frac{7}{8}$. Then we take $L \geq \frac{1}{56}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L)+a(3+L)<1$.

Case 3. If $(x,u)=\left(\frac{1}{8},\frac{1}{8}\right)$ and $(y,v) \in \left\{\frac{1}{8}\right\} \times \left[0,\frac{1}{8}\right)$, then

$$\frac{9}{64}+\frac{1}{8}|v+1| \leq \delta \cdot \left|\frac{1}{8}-v\right| + a\left[\frac{39}{64}+\frac{7}{8}v\right] + L\left[\frac{1}{4}+\left|v+\frac{1}{8}\right|\right].$$

Since $v \in \left[0,\frac{1}{8}\right)$, we have $\frac{9}{64}+\frac{1}{8}|v+1| < \frac{9}{32}$ and $\frac{1}{4}+\left|v+\frac{1}{8}\right| \geq \frac{3}{8}$. Then we take $L \geq \frac{3}{4}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L)+a(3+L)<1$.

Case 4. If $(x,u)=\left(\frac{1}{8},\frac{1}{8}\right)$ and $(y,v) \in \left[0,\frac{1}{8}\right) \times \left\{\frac{1}{8}\right\}$, then

$$\frac{9}{64}+\frac{1}{8}|y+1| \leq \delta \cdot \left|y-\frac{1}{8}\right| + a\left[\frac{39}{64}+\frac{7}{8}y\right] + L\left[\frac{7}{64}+\frac{1}{8}|1-y|\right].$$

Since $y \in \left[0,\frac{1}{8}\right)$, we have $\frac{9}{64}+\frac{1}{8}|y+1| < \frac{9}{32}$ and $\frac{7}{64}+\frac{1}{8}|1-y| \geq \frac{15}{64}$. Then we take $L \geq \frac{6}{5}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L)+a(3+L)<1$.

Case 5. If $(x,u)=\left(\frac{1}{8},\frac{1}{8}\right)$ and $(y,v) \in \left[0,\frac{1}{8}\right) \times \left[0,\frac{1}{8}\right)$, then

$$\frac{1}{8}|y-v| \leq \delta \cdot |y-v| + a\left[\frac{7}{32}+\frac{7}{8}y+\frac{7}{8}v\right] + L\left[\frac{7}{64}+\left|v-\frac{y}{8}\right|\right].$$

Since $y,v \in \left[0,\frac{1}{8}\right)$, we have $\frac{1}{8}|y-v| < \frac{1}{64}$ and $\frac{7}{64}+\left|v-\frac{y}{8}\right| \geq \frac{7}{64}$. Then we take $L \geq \frac{1}{7}, 0 \leq \delta < 1$ and $0 \leq a < \frac{1}{3}$ such that $\delta(1+a+L)+a(3+L)<1$.

Case 6. If $(x,u) \in \left[0,\frac{1}{8}\right) \times \left[0,\frac{1}{8}\right)$ and $(y,v) \in \left[0,\frac{1}{8}\right] \times \left[0,\frac{1}{8}\right] \cup \{(1,1)\}$, then

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))$$

$$
\begin{aligned}
&= \left|\frac{x}{8} - \frac{u}{8}\right| + \left|\frac{y}{8} - \frac{v}{8}\right| \\
&\leq \left(\frac{x}{8} + \frac{u}{8} + \frac{y}{8} + \frac{v}{8}\right) \\
&= \frac{1}{8}[(x+y) + (u+v)] \\
&= \frac{1}{7} \cdot \frac{7}{8}[(x+y) + (u+v)] \\
&= \frac{1}{7}\left[\left(\frac{7}{8}x + \frac{7}{8}y\right) + \left(\frac{7}{8}u + \frac{7}{8}v\right)\right] \\
&= \frac{1}{7}\left[\left(\left|x - \frac{1}{8}x\right| + \left|y - \frac{1}{8}y\right|\right) + \left(\left|u - \frac{1}{8}u\right| + \left|v - \frac{1}{8}v\right|\right)\right] \\
&= \frac{1}{7}[d(x, F(x,y)) + d(y, F(y,x)) + d(u, F(u,v)) + d(v, F(v,u))] \\
&\leq \delta[d(x,u) + d(y,v)] \\
&\quad + a[d(x, F(x,y)) + d(y, F(y,x)) + d(u, F(u,v)) + d(v, F(v,u))] \\
&\quad + L[d(u, F(x,y)) + d(v, F(y,x))].
\end{aligned}
$$

Then we take $a = \frac{1}{7}$, $L \geq 0$ and $0 \leq \delta < 1$ such that $\delta(1 + a + L) + a(3 + L) < 1$. Hence we conclude that condition is satisfied with $L = \frac{6}{5}$, $a = \frac{1}{7}$ and

$$0 \leq \delta < \frac{1 - \frac{1}{7}(3 + \frac{6}{5})}{1 + \frac{1}{7} + \frac{6}{5}} = \frac{7}{41},$$

which is $\delta(1 + a + L) + a(3 + L) < 1$.
Finally, we consider that there is only $\left(\frac{1}{8}, \frac{1}{8}\right) \in K \times K$ such that $F(\frac{1}{8}, \frac{1}{8}) \notin K$. Notice that

$$P_{T_F}\left(\frac{1}{8}, \frac{1}{8}\right) = \left\{(0,x), (x,0) : x \in \left[0, \frac{1}{8}\right]\right\},$$

and $T_F(0,x) = \left(0, \frac{x}{8}\right), T_F(x,0) = \left(\frac{x}{8}, 0\right)$ for all $x \in \left[0, \frac{1}{8}\right]$. Then, we see that $(0,0)$, $\left(\frac{1}{8}, x\right)$, $\left(\frac{x}{8}, x\right) \in E(G) \cap E(G^{-1})$ for all $x \in \left[0, \frac{1}{8}\right]$, by definition of G_Y, we can conclude that

$$\left(\left(\frac{1}{8}, \frac{1}{8}\right), (0,x)\right), \left(\left(\frac{1}{8}, \frac{1}{8}\right), (x,0)\right), \left((0,x), \left(0, \frac{x}{8}\right)\right), \left((x,0), \left(\frac{x}{8}, 0\right)\right) \in E(G_Y).$$

Therefore all conditions of Theorem 3.3.2 are satisfied, so F has a coupled fixed point and we see that $CFix(F) = \{(0,0), (1,1)\}$.

3.4 Best Proximity Point Theorems for Mean Nonexpansive mappings in Banach Spaces

In this section, we introduce a nonself (a,b)-mean nonexpansive mapping in a Banach space and prove some best proximity point theorems by using strictly convexity property of a Banach space.

Definition 3.4.1. Let A and B be nonempty subsets of a Banach space X and let a and b be nonnegative real numbers such that $a+b \leq 1$. A mapping $T : A \to B$ is said to be a *nonself* (a,b)*-mean nonexpansive* on a subset C of A, if

$$||Tx - Ty|| \leq a||x-y|| + b||Px - Ty||,$$

for all $x, y \in C$.

Notice that a nonexpansive mapping $T : A \to B$ is a nonself $(1,0)$-mean nonexpansive mapping.

Now, we prove our main result.

Theorem 3.4.2. Let X be a reflexive strictly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Suppose that $T : A \to B$ is a nonself (a,b)-mean nonexpansive mapping on A_0 for some nonnegative real numbers a and b such that $a+b \leq 1$ and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A, i.e., there exists $x^* \in A$ such that

$$||x^* - Tx^*|| = D(A,B).$$

Moreover,

(*i*) If $a+b<1$, then T has a unique best proximity point in A_0.

(*ii*) If T is continuous and $a<1$, then $\{(PT)^n(x)\}$ converges to a proximity point for all $x \in A_0$.

Proof. We know from Lemma 1.2.2 that A_0 is nonempty. Since $T(A_0) \subseteq B_0$, by Proposition 2.6.5, we also see that $P_{A_0}T(A_0) \subseteq A_0$. Now, we will show that $P_{A_0}T : A_0 \to A_0$ is an (a,b)-mean nonexpansive mapping. Let $x, y \in A_0$. By Corollary 2.6.4, Proposition 2.6.5, and Lemma 2.6.7, we have

$$\begin{aligned}
||(P_{A_0}T)(x) - (P_{A_0}T)(y)|| &= ||P_{A_0}(Tx) - P_{A_0}(Ty)|| \\
&= ||Tx - Ty|| \\
&\leq a||x-y|| + b||Px - Ty|| \\
&= a||x-y|| + b||Px - P^2(Ty)|| \\
&= a||x-y|| + b||P_{B_0}(x) - P_{B_0}(P_{A_0}T(y))|| \\
&= a||x-y|| + b||x - P_{A_0}T(y)||.
\end{aligned}$$

So $P_{A_0}T : A_0 \to A_0$ is (a,b)-mean nonexpansive. Since X has Opial's property, by Theorem 2.5.5, we obtain that PT has a fixed point, say that $x^* \in A_0$. This implies by Proposition 2.6.6 that x^* is a best proximity point of T in A.

(i) Now suppose $a+b<1$. Assume that $x, y \in A_0$ are best proximity points of T. Then x and y are fixed points of PT. So we have

$$\begin{aligned}
||x-y|| &= ||(PT)(x)-(PT)(y)|| \\
&= ||(P_{A_0}T)(x)-(P_{A_0}T)(y)|| \\
&\leq a||x-y|| + b||x-(P_{A_0}T)(y)|| \\
&\leq a||x-y|| + b||x-y|| + b||y-(P_{A_0}T)(y)|| \\
&= (a+b)||x-y|| + b||y-(PT)(y)|| \\
&= (a+b)||x-y||.
\end{aligned}$$

Since $a+b<1$, we obtain that $||x-y||=0$, that is, $x=y$. Hence T has a unique best proximity point in A_0.

(ii) Suppose that T is continuous and $a<1$. Let $x \in A_0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&||(PT)^{n+1}(x)-(PT)^n(x)|| \\
&= ||(P_{A_0}T)^{n+1}(x)-(P_{A_0}T)^n(x)|| \\
&= ||P_{A_0}T(P_{A_0}T)^n(x)-P_{A_0}T(P_{A_0}T)^{n-1}(x)|| \\
&= ||T(P_{A_0}T)^n(x)-T(P_{A_0}T)^{n-1}(x)|| \\
&\leq a||(P_{A_0}T)^n(x)-(P_{A_0}T)^{n-1}(x)|| + b||P(P_{A_0}T)^n(x)-T(P_{A_0}T)^{n-1}(x)|| \\
&= a||(P_{A_0}T)^n(x)-(P_{A_0}T)^{n-1}(x)|| \\
&\quad + b||P(P_{A_0}T)(P_{A_0}T)^{n-1}(x)-T(P_{A_0}T)^{n-1}(x)|| \\
&= a||(P_{A_0}T)^n(x)-(P_{A_0}T)^{n-1}(x)|| \\
&\quad + b||P^2T(P_{A_0}T)^{n-1}(x)-T(P_{A_0}T)^{n-1}(x)|| \\
&= a||(P_{A_0}T)^n(x)-(P_{A_0}T)^{n-1}(x)|| + b||T(P_{A_0}T)^{n-1}(x)-T(P_{A_0}T)^{n-1}(x)|| \\
&= a||(P_{A_0}T)^n(x)-(P_{A_0}T)^{n-1}(x)|| \\
&= a||(PT)^n(x)-(PT)^{n-1}(x)|| \\
&\leq a^2||(PT)^{n-1}(x)-(PT)^{n-2}(x)|| \\
&\leq a^3||(PT)^{n-2}(x)-(PT)^{n-3}(x)|| \\
&\vdots
\end{aligned}$$

$$\leq a^n||(PT)(x) - x||.$$

For $m > n$, we get

$$\begin{aligned}
&||(PT)^m(x) - (PT)^n(x)|| \\
&\quad \leq ||(PT)^m(x) - (PT)^{m-1}(x)|| + ||(PT)^{m-1}(x) - (PT)^{m-2}(x)|| \\
&\qquad + ... + ||(PT)^{n+1}(x) - (PT)^n(x)|| \\
&\quad \leq a^{m-1}||(PT)(x) - x|| + a^{m-2}||(PT)(x) - x|| + ... + a^n||(PT)(x) - x|| \\
&\quad = (a^n + a^{n+1} + ... + a^{m-1})||(PT)(x) - x|| \\
&\quad \leq a^n(1 + a + ... + a^{m-n-1} + ...)||(PT)(x) - x||.
\end{aligned}$$

Because $a < 1$, it follows that $\{(PT)^n(x)\}$ is a Cauchy sequence in A_0. Hence, there exists $y^* \in A_0$ such that $(PT)^n(x) \to y^*$. Since T is continuous, we have $T(PT)^n(x) \to Ty^*$. Further, we note that $(PT)^{n+1}(x) \in A_0$ for $n \in \mathbb{N}$ and

$$\begin{aligned}
D(A,B) &= ||(PT)^{n+1}(x) - P_{B_0}(PT)^{n+1}(x)|| \\
&= ||(PT)^{n+1}(x) - P(PT)(PT)^n(x)|| \\
&= ||(PT)^{n+1}(x) - P^2T(PT)^n(x)|| \\
&= ||(PT)^{n+1}(x) - T(PT)^n(x)||.
\end{aligned}$$

Thus

$$||y^* - Ty^*|| = \lim_{n\to\infty} ||(PT)^{n+1}(x) - T(PT)^n(x)|| = D(A,B).$$

Therefore, we can conclude that $\{(PT)^n(x)\}$ converges to a proximity point y^* for all $x \in A_0$. This proof is now completed. □

If we take $A = B$ in Theorem 3.4.2, then we obtain similarly the fixed point theorem of Zuo (see [62], Theorem 9) as the following Corollary:

Corollary 3.4.3. Let X be a reflexive strictly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X. Suppose that $T : A \to A$ is a nonself (a,b)-mean nonexpansive mapping on A for some nonnegative real numbers a and b such that $a + b \leq 1$. Then T has at least fixed point in A.

As a consequence of Theorem 3.4.2, we obtain the following results.

Theorem 3.4.4. Let X be a uniformly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Suppose that $T : A \to B$ is a nonself (a, b)-mean nonexpansive mapping on A_0 for some nonnegative real numbers a and b such that $a + b \leq 1$ and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A.

Corollary 3.4.5. Let X be a uniformly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Suppose that $T : A \to B$ is a nonself nonexpansive mapping on A_0 and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A.

Next, we give an example to illustrate Theorem 3.4.2.

Example 3.4.6. Consider the uniformly convex Banach space $(\mathbb{R}^2, ||\cdot||_2)$ where $||(x,y)||_2 = \sqrt{x^2 + y^2}$. Let

$$A := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\},$$

$$B := \{(x, y) : 2 \leq x \leq 3, 0 \leq y \leq 1\}.$$

Then A and B are nonempty closed bounded convex subsets of $\mathbb{R}^2$ with $D(A, B) = 1$. We see that $A_0 = \{(1, y) : 0 \leq y \leq 1\}$, and $B_0 = \{(2, y) : 0 \leq y \leq 1\}$. Define a map $T : A \to B$ by

$$T(x,y) = \begin{cases} \left(x+1, \frac{y}{8}\right) & \text{if } (x,y) \in A_0 \text{ and } y > \frac{1}{2}; \\ \left(3-x, \frac{y}{7}\right) & \text{if } (x,y) \in A_0 \text{ and } y \leq \frac{1}{2}; \\ (x+2, y) & \text{if } (x,y) \in A \backslash A_0. \end{cases}$$

Note that $T(A_0) = \{(2, y) : 0 \leq y \leq \frac{1}{8}\} \subset B_0$. Next, we will show that T is a $\left(\frac{1}{5}, \frac{2}{5}\right)$-mean nonexpansive mapping on A_0. Let $(1, u), (1, v) \in A_0$. Then $u, v \in [0, 1]$. We consider the following four cases.

Case 1. If $u, v \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} &||T(1,u) - T(1,v)|| \\ &\quad = \left|\left|\left(2, \frac{u}{7}\right) - \left(2, \frac{v}{7}\right)\right|\right| \\ &\quad = \frac{1}{7}|u - v| \\ &\quad = \frac{1}{6}\left|\frac{6}{7}u - \frac{6}{7}v\right| \\ &\quad = \frac{1}{6}\left|u - \frac{u}{7} - v + \frac{v}{7}\right| \\ &\quad = \frac{1}{6}\left|u - \frac{v}{7} + \frac{v}{7} - \frac{u}{7} - v + u - u + \frac{v}{7}\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{6}\left|u-\frac{v}{7}\right|+\frac{1}{6}\left|\frac{v}{7}-\frac{u}{7}\right|+\frac{1}{6}|u-v|+\frac{1}{6}\left|u-\frac{v}{7}\right| \\
&= \frac{1}{6}|u-v|+\frac{1}{3}\left|u-\frac{v}{7}\right|+\frac{1}{6}\left|\frac{v}{7}-\frac{u}{7}\right| \\
&= \frac{1}{6}||(1,u)-(1,v)||+\frac{1}{3}||(2,u)-T(1,v)||+\frac{1}{6}||T(1,u)-T(1,v)||.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{5}{6}||T(1,u)-T(1,v)|| &\leq \frac{1}{6}||(1,u)-(1,v)||+\frac{1}{3}||(2,u)-T(1,v)|| \\
&= \frac{1}{6}||(1,u)-(1,v)||+\frac{1}{3}||P(1,u)-T(1,v)||.
\end{aligned}
$$

Hence

$$
||T(1,u)-T(1,v)|| \leq \frac{1}{5}||(1,u)-(1,v)||+\frac{2}{5}||P(1,u)-T(1,v)||.
$$

Case 2. If $u, v \in \left(\frac{1}{2}, 1\right]$, then we get

$$
\begin{aligned}
&||T(1,u)-T(1,v)|| \\
&= \left|\left|\left(2,\frac{u}{8}\right)-\left(2,\frac{v}{8}\right)\right|\right| \\
&= \frac{1}{8}|u-v| \\
&= \frac{1}{7}\left|\frac{7}{8}u-\frac{7}{8}v\right| \\
&= \frac{1}{7}\left|u-\frac{u}{8}-v+\frac{v}{8}\right| \\
&= \frac{1}{7}\left|u-\frac{v}{8}+\frac{v}{8}-\frac{u}{8}-v+u-u+\frac{v}{8}\right| \\
&\leq \frac{1}{7}\left|u-\frac{v}{8}\right|+\frac{1}{7}\left|\frac{v}{8}-\frac{u}{8}\right|+\frac{1}{7}|u-v|+\frac{1}{7}\left|u-\frac{v}{8}\right| \\
&= \frac{1}{7}|u-v|+\frac{2}{7}\left|u-\frac{v}{8}\right|+\frac{1}{7}\left|\frac{v}{8}-\frac{u}{8}\right| \\
&= \frac{1}{7}||(1,u)-(1,v)||+\frac{2}{7}||(2,u)-T(1,v)||+\frac{1}{7}||T(1,u)-T(1,v)||.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{6}{7}||T(1,u)-T(1,v)|| &\leq \frac{1}{7}||(1,u)-(1,v)||+\frac{2}{7}||(2,u)-T(1,v)|| \\
&= \frac{1}{7}||(1,u)-(1,v)||+\frac{2}{7}||P(1,u)-T(1,v)||.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||T(1,u)-T(1,v)|| &\leq \frac{1}{6}||(1,u)-(1,v)||+\frac{1}{3}||P(1,u)-T(1,v)|| \\
&\leq \frac{1}{5}||(1,u)-(1,v)||+\frac{2}{5}||P(1,u)-T(1,v)||.
\end{aligned}
$$

Case 3. If $u \in [0, \frac{1}{2}]$ and $v \in (\frac{1}{2}, 1]$, then

$$
\begin{aligned}
&||T(1,u) - T(1,v)|| \\
&= \left|\left|\left(2, \frac{u}{7}\right) - \left(2, \frac{v}{8}\right)\right|\right| \\
&= \left|\frac{u}{7} - \frac{v}{8}\right| \\
&= \left|\frac{u}{7} - \frac{u}{7^2} + \frac{u}{7^2} - \frac{v}{7\cdot 8} + \frac{v}{7\cdot 8} - \frac{v}{8}\right| \\
&\leq \left|\frac{u}{7} - \frac{u}{7^2}\right| + \left|\frac{u}{7^2} - \frac{v}{7\cdot 8}\right| + \left|\frac{v}{7\cdot 8} - \frac{v}{8}\right| \\
&\leq \left|\frac{u}{7} - \frac{u}{7^2}\right| + \left|\frac{u}{7^2} - \frac{v}{7\cdot 8}\right| + \left|\frac{v}{7\cdot 8} - \frac{v}{7}\right| \\
&= \frac{1}{7}\left|u - \frac{u}{7}\right| + \frac{1}{7}\left|\frac{u}{7} - \frac{v}{8}\right| + \frac{1}{7}\left|\frac{v}{8} - v\right| \\
&= \frac{1}{7}\left|\left|(2,u) - \left(2, \frac{u}{7}\right)\right|\right| + \frac{1}{7}\left|\left|\left(2, \frac{u}{7}\right) - \left(2, \frac{v}{8}\right)\right|\right| + \frac{1}{7}\left|\left|\left(2, \frac{v}{8}\right) - (2,v)\right|\right| \\
&= \frac{1}{7}||(2,u) - T(1,u)|| + \frac{1}{7}||T(1,u) - T(1,v)|| + \frac{1}{7}||T(1,v) - (2,v)|| \\
&\leq \frac{1}{7}||(2,u) - T(1,v)|| + \frac{1}{7}||T(1,v) - T(1,u)|| + \frac{1}{7}||T(1,u) - T(1,v)|| \\
&\quad + \frac{1}{7}||T(1,v) - (2,u)|| + \frac{1}{7}||(2,u) - (2,v)|| \\
&= \frac{2}{7}||(2,u) - T(1,v)|| + \frac{2}{7}||T(1,v) - T(1,u)|| + \frac{1}{7}||(1,u) - (1,v)||.
\end{aligned}
$$

This implies

$$
\begin{aligned}
||T(1,u) - T(1,v)|| &\leq \frac{1}{5}||(1,u) - (1,v)|| + \frac{2}{5}||(2,u) - T(1,v)|| \\
&= \frac{1}{5}||(1,u) - (1,v)|| + \frac{2}{5}||P(1,u) - T(1,v)||.
\end{aligned}
$$

Case 4. If $u \in (\frac{1}{2}, 1]$ and $v \in [0, \frac{1}{2}]$, then

$$
\begin{aligned}
&||T(1,u) - T(1,v)|| \\
&= \left|\left|\left(2, \frac{u}{8}\right) - \left(2, \frac{v}{7}\right)\right|\right| \\
&= \left|\frac{u}{8} - \frac{v}{7}\right| \\
&= \left|\frac{u}{8} - \frac{u}{7\cdot 8} + \frac{u}{7\cdot 8} - \frac{v}{7^2} + \frac{v}{7^2} - \frac{v}{7}\right| \\
&\leq \left|\frac{u}{8} - \frac{u}{7\cdot 8}\right| + \left|\frac{u}{7\cdot 8} - \frac{v}{7^2}\right| + \left|\frac{v}{7^2} - \frac{v}{7}\right| \\
&\leq \left|\frac{u}{7} - \frac{u}{7\cdot 8}\right| + \left|\frac{u}{7\cdot 8} - \frac{v}{7^2}\right| + \left|\frac{v}{7^2} - \frac{v}{7}\right| \\
&= \frac{1}{7}\left|u - \frac{u}{8}\right| + \frac{1}{7}\left|\frac{u}{8} - \frac{v}{7}\right| + \frac{1}{7}\left|\frac{v}{7} - v\right| \\
&= \frac{1}{7}\left|\left|(2,u) - \left(2, \frac{u}{8}\right)\right|\right| + \frac{1}{7}\left|\left|\left(2, \frac{u}{8}\right) - \left(2, \frac{v}{7}\right)\right|\right| + \frac{1}{7}\left|\left|\left(2, \frac{v}{7}\right) - (2,v)\right|\right| \\
&= \frac{1}{7}||(2,u) - T(1,u)|| + \frac{1}{7}||T(1,u) - T(1,v)|| + \frac{1}{7}||T(1,v) - (2,v)||
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{7}||(2,u)-T(1,v)|| + \frac{1}{7}||T(1,v)-T(1,u)|| + \frac{1}{7}||T(1,u)-T(1,v)|| \\
&\quad + \frac{1}{7}||T(1,v)-(2,u)|| + \frac{1}{7}||(2,u)-(2,v)|| \\
&= \frac{2}{7}||(2,u)-T(1,v)|| + \frac{2}{7}||T(1,v)-T(1,u)|| + \frac{1}{7}||(1,u)-(1,v)||.
\end{aligned}
$$

This implies

$$
\begin{aligned}
||T(1,u)-T(1,v)|| &\le \frac{1}{5}||(1,u)-(1,v)|| + \frac{2}{5}||(2,u)-T(1,v)|| \\
&= \frac{1}{5}||(1,u)-(1,v)|| + \frac{2}{5}||P(1,u)-T(1,v)||.
\end{aligned}
$$

Now, by summarizing all cases, we conclude that T is a $\left(\frac{1}{5}, \frac{2}{5}\right)$-mean nonexpansive mapping on A_0. By Theorem 3.4.2, there exists $x^* \in A$ such that

$$||x^* - Tx^*|| = D(A,B).$$

Note that $a+b = \frac{3}{5} < 1$, so $x^* = (1,0) \in A$ is a unique best proximity point of T, i.e.,

$$||(1,0) - T(1,0)|| = ||(1,0)-(2,0)|| = 1 = D(A,B).$$

However, if we put $(1,u) = \left(1, \frac{1}{2}\right)$ and $(1,v) = \left(1, \frac{127}{252}\right)$, we note that

$$
\begin{aligned}
\left|\left|T\left(1,\frac{1}{2}\right) - T\left(1,\frac{127}{252}\right)\right|\right| &= \left|\left|\left(2,\frac{1}{14}\right) - \left(2,\frac{127}{2016}\right)\right|\right| \\
&= \frac{17}{2016} \\
&> \frac{8}{2016} \\
&= \left|\left|\left(1,\frac{1}{2}\right) - \left(1,\frac{127}{252}\right)\right|\right|,
\end{aligned}
$$

so T is not nonexpansive on A_0.

Example 3.4.7. Consider the uniformly convex Banach space $(\mathbb{R}^2, ||\cdot||_2)$, let

$$A := \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\},$$

$$B := \{(x,y) : 2 \le x \le 3, 0 \le y \le 1\}.$$

Define a map $T : A \to B$ by

$$
T(x,y) = \begin{cases}
\left(x+1, \frac{y}{6}\right) & \text{if } (x,y) \in A_0 \text{ and } y > \frac{1}{2}; \\
\left(3-x, \frac{y}{5}\right) & \text{if } (x,y) \in A_0 \text{ and } y \le \frac{1}{2}; \\
(x+2, y) & \text{if } (x,y) \in A \backslash A_0.
\end{cases}
$$

Using the same proof as in Example 3.4.6, we can show that T is a $\left(\frac{1}{3}, \frac{2}{3}\right)$-mean nonexpansive mapping on A_0. By Theorem 3.4.2, there exists at least $x^* \in A$ such that $||x^* - Tx^*|| = D(A, B)$. We see that $(1, 0) \in A$ is a unique best proximity point of T.

Example 3.4.8. Consider the uniformly convex Banach space $(\mathbb{R}^2, ||\cdot||_2)$, let

$$A := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\},$$

$$B := \{(x, y) : 2 \leq x \leq 3, 0 \leq y \leq 1\}.$$

Define a map $T : A \to B$ by $T(x, y) := (x + 2, y)$ for all $(x, y) \in A$. It is easy to prove that T is a $(1, 0)$-mean nonexpansive mapping on A_0. By Theorem 3.4.2, there exists at least $x^* \in A$ such that $||x^* - Tx^*|| = D(A, B)$. We see that any $(1, y) \in A$ is a best proximity point of T.