CHAPTER 4

Conclusion

In this chapter, we conclude all main results obtained in this thesis. We divide our main results into three sections as the following:

4.1 Fixed Point Theorems

We start to talk about the fixed point theorems for multivalued generalized contractions. First of all, we extend a multivalued nonself almost contraction mapping to a new class of multivalued nonself mappings, called a multivalued Kannan-Berinde contraction, defined as follows:

Let (X, d) be a metric space and K a nonempty subset of X. A mapping $T : K \to CB(X)$ is said to be a *multivalued Kannan-Berinde contraction* if there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for any $x, y \in K$.

Then we prove the existence of fixed points of this mapping in complete convex metric spaces, as the following theorem.

1. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

   (i) T satisfies Rothe's type condition, that is, $x \in \partial K$ implies $Tx \subset K$;

   (ii) T is a multivalued Kannan-Berinde contraction mapping with

   $$\delta(1 + a + L) + a(3 + L) < 1.$$

   Then T has a fixed point in K.

After that, we introduce a multivalued Kannan-Berinde G-contraction mapping and prove the fixed point theorems for this mapping in complete convex metric spaces endowed with graphs which is more general than that of previous result.

Let (X, d) be a metric space, K a nonempty subset of X and $G := (V(G), E(G))$ be a directed graph such that $V(G) = K$. A mapping $T : K \to CB(X)$ is said to be a *multivalued Kannan-Berinde G-contraction* if there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for all $x, y \in K$ with $(x, y) \in E(G)$.

2. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Let $G := (V(G), E(G))$ be a directed graph such that $V(G) = K$. Suppose that K has Property A. If a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:

   (i) there exists $x_0 \in K$ such that $(x_0, y) \in E(G)$ for some $y \in Tx_0$;

   (ii) T is an edge-preserving mapping, that is, if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$;

   (iii) for each $x \in K$ and $y \in Tx$ with $y \notin K$,

      (a) $P[x, y]$ is dominated by x and

      (b) for each $z \in P[x, y]$, z dominates Tz;

   (iv) T has Rothe's boundary condition;

   (v) T is a multivalued Kannan-Berinde G-contraction mapping with

   $$\delta(1 + a + L) + a(3 + L) < 1.$$

   Then T has a fixed point in K.

4.2 Best Proximity Points

Next, we introduce a nonself (a, b)-mean nonexpansive mapping in a Banach space which is more general that of a nonexpansive mapping, defined as follows:

Let A and B be nonempty subsets of a Banach space X and let a and b be non-negative real numbers such that $a + b \leq 1$. A mapping $T : A \to B$ is said to be a *nonself (a, b)-mean nonexpansive* on a subset C of A, if

$$||Tx - Ty|| \leq a||x - y|| + b||Px - Ty||,$$

for all $x, y \in C$.

By using some strictly convexity properties of Banach spaces, we prove the existence of best proximity points of this mapping.

1. Let X be a reflexive strictly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Suppose that $T : A \to B$ is a nonself (a, b)-mean nonexpansive mapping on A_0 for some nonnegative real numbers a and b such that $a + b \leq 1$ and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A, i.e., there exists $x^* \in A$ such that

$$||x^* - Tx^*|| = D(A, B).$$

   Moreover,

   (i) If $a + b < 1$, then T has a unique best proximity point in A_0.

   (ii) If T is continuous and $a < 1$, then $\{(PT)^n(x)\}$ converges to a proximity point for all $x \in A_0$.

2. Let X be a uniformly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X, and B a nonempty closed convex subset of X. Suppose that $T : A \to B$ is a nonself nonexpansive mapping on A_0 and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A.

4.3 Applications

Finally, we apply all of obtained results for a coupled fixed point and fixed point theorem for some cyclic mappings and mean nonexpansive mappings, as the followings.

1. Let (X, d) be a complete metric space, m a positive integer and $\{A_i\}_{i=1}^m$ nonempty closed subsets of X. Suppose that $W = \bigcup_{i=1}^m A_i$ and an operator $T : W \to W$. If $\bigcup_{i=1}^m A_i$ is a cyclic representation of W with respect to T and there exist $\delta \in [0, 1)$, $a \in [0, \frac{1}{3})$ and $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + a[d(x, Tx) + d(y, Ty)] + Ld(y, Tx)$$

   for any $x \in A_i, y \in A_{i+1}, i = 1, 2, ..., m$ where $A_{m+1} = A_1$, then T has at least one fixed point $t \in \bigcap_{i=1}^m A_i$.

2. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Let $G = (V(G), E(G))$ a directed graph such that $V(G) = K$. Let $F : Y = K \times K \to X$ be an edge-preserving mapping such that $F(\partial Y) \subset K$. Suppose the following properties hold:

   (i) there exist $x_0, y_0 \in K$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$;

   (ii) K has the following properties:

      (a) if any sequence $\{x_n\}$ in K such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;

      (b) if any sequence $\{y_n\}$ in K such that $y_n \to y$ and $(y_n, y_{n+1}) \in E(G^{-1})$ for $n \in \mathbb{N}$, then $(y_n, y) \in E(G^{-1})$ for all $n \in \mathbb{N}$.

   (iii) there exist $\delta \in [0, 1)$, $a \in [0, \frac{1}{3})$ and some $L \geq 0$ with $\delta(1 + a + L) + a(3 + L) < 1$ such that

$$
\begin{aligned}
&d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
&\quad \leq \delta[d(x, u) + d(y, v)] \\
&\qquad + a[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))] \\
&\qquad + L[d(u, F(x, y)) + d(v, F(y, x))]
\end{aligned}
$$

   for all $x, y, u, v \in X$ with $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$.

   If for each $(x, y) \in Y$ with $T_F(x, y) \notin Y$ such that $P_{T_F}(x, y)$ is dominated by (x, y) and for each $(u, v) \in P_{T_F}(x, y)$ dominates $T_F(u, v)$, then F has a coupled fixed point.

3. Let X be a reflexive strictly convex Banach space which satisfies Opial's condition and A a nonempty closed bounded convex subset of X. Suppose that $T : A \to A$ is a nonself (a, b)-mean nonexpansive mapping on A for some nonnegative real numbers a and b such that $a + b \leq 1$. Then T has at least fixed point in A.